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On a new class of rigid Coxeter groups

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Abstract. In this paper, we give a new class of rigid Coxeter groups, which is an extension of [9] and a result of D. Radcliffe in [10].

Key words: rigidity of Coxeter groups.

1. Introduction and preliminaries

The purpose of this paper is to give a new class of rigid Coxeter groups. A *Coxeter group* is a group W having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(i) $m(s,t) = m(t,s)$ for any $s, t \in S$,
(ii) $m(s,s) = 1$ for any $s \in S$, and
(iii) $m(s,t) \geq 2$ for any $s, t \in S$ such that $s \neq t$.

The pair (W,S) is called a *Coxeter system*. For a Coxeter group W, a generating set S' of W is called a *Coxeter generating set for W* if (W,S') is a Coxeter system. Let (W,S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. A subset $T \subset S$ is called a *spherical subset of S*, if the parabolic subgroup W_T is finite.

Let (W,S) and (W',S') be Coxeter systems. Two Coxeter systems (W,S) and (W',S') are said to be *isomorphic*, if there exists a bijection $\psi : S \to S'$ such that

$$m(s,t) = m'(\psi(s), \psi(t))$$

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for every $s, t \in S$, where $m(s,t)$ and $m'(s',t')$ are the values appeared in the Coxeter presentations of (W,S) and (W',S'), and we note that it is known that $m(s,t)$ and $m'(s',t')$ are the orders of st in W and $s't'$ in W', respectively.

A *diagram* is an undirected graph Γ without loops or multiple edges with a map $\mathrm{Edges}(\Gamma) \to \{2,3,4,\ldots\}$ which assigns an integer greater than 1 to each of its edges. Since such diagrams are used to define Coxeter systems, they are called *Coxeter diagrams*.

In general, a Coxeter group does not always determine its Coxeter system up to isomorphism. Indeed some counter-examples are known (cf. [4], [5]). Here there exists the following natural problem.

Problem ([5]) When does a Coxeter group determine its Coxeter system up to isomorphism?

A Coxeter group W is said to be *rigid*, if the Coxeter group W determines its Coxeter system up to isomorphism (i.e., for each Coxeter generating sets S and S' for W the Coxeter systems (W,S) and (W,S') are isomorphic).

We can find some research on rigidity of Coxeter groups in [1], [2], [5], [6], [7], [8], [9] and [10].

A Coxeter system (W,S) is said to be *even*, if $m(s,t)$ is even or ∞ for all $s \neq t$ in S. Also a Coxeter system (W,S) is said to be *strongly even*, if $m(s,t) \in \{2\} \cup 4\mathbb{N} \cup \{\infty\}$ for all $s \neq t$ in S. In [1], [2] and [3], P. Bahls and M. Mihalik have investigated even Coxeter systems. Concerning strongly even Coxeter systems, the following theorem was proved by D. Radcliffe in [10] (in [10], strongly even Coxeter systems are called "even" Coxeter systems).

Theorem 1.1 ([10]) *If (W,S) is a strongly even Coxeter system, then the Coxeter group W is rigid.*

In this paper, we say that a Coxeter system (W,S) satisfies the condition $(*)$, if (W,S) satisfies the following conditions:

(0) for each $s, t \in S$ such that $m(s,t)$ is even, $m(s,t) \in \{2\} \cup 4\mathbb{N}$,
(1) for each $s \neq t \in S$ such that $m(s,t)$ is odd, $\{s,t\}$ is a maximal spherical subset of S,
(2) there does not exist a three-points subset $\{s,t,u\} \subset S$ such that $m(s,t)$

and $m(t, u)$ are odd, and

(3) for each $s \neq t \in S$ such that $m(s, t)$ is odd, there exists at most one maximal spherical subset of S that is different from $\{s, t\}$ and intersecting with $\{s, t\}$.

The purpose of this paper is to prove the following theorem which is an extension of Theorem 1.1 and [9, Theorem 1.2]. (In [9, Theorem 1.2], we needed the condition that for each $s, t \in S$ such that $m(s, t)$ is even, $m(s, t) = 2$.)

Theorem 1.2 *Let (W, S) be a Coxeter system which satisfies the condition $(*)$. Then the Coxeter group W is rigid.*

The condition $(*)$ is somewhat technical. However the class of Coxeter systems satisfying the condition $(*)$ is large.

Example The Coxeter groups defined by the diagrams in Figure 1 are rigid by Theorem 1.2.

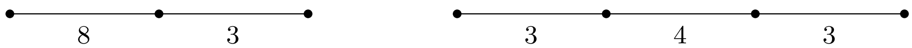

Figure 1. Coxeter diagrams for rigid Coxeter groups

Now, we introduce that we can not omit some conditions in the condition $(*)$.

Example ([4, p. 38 Exercise 8], [5]) It is known that for an odd number $k \geq 3$, the Coxeter groups defined by the diagrams in Figure 2 are isomorphic and D_{2k}.

Hence, we can not omit the conditions (0) and (1) in the condition $(*)$.

Figure 2. Two distinct Coxeter diagrams for D_{2k}

Example ([5]) It is known that the Coxeter groups defined by the diagrams in Figure 3 are isomorphic by the *diagram twisting* ([5, Definition

4.4]).

Hence, we can not omit the condition (3) in the condition $(*)$.

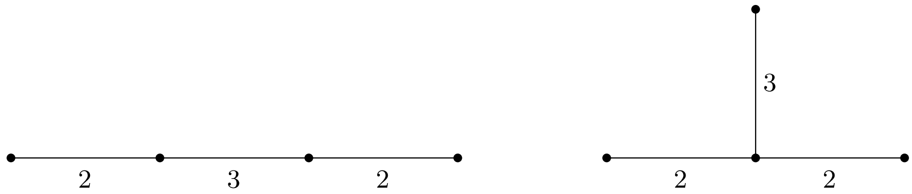

Figure 3. Coxeter diagrams for isomorphic Coxeter groups

2. Proof of the theorem

Let (W, S) be a Coxeter system which satisfies the condition $(*)$. Let (W', S') be a Coxeter system. We suppose that there exists an isomorphism $\phi : W \to W'$. To prove Theorem 1.2, we show that the Coxeter systems (W, S) and (W', S') are isomorphic.

The following lemma is known.

Lemma 2.1 (cf. [5]) *For each maximal spherical subset $T \subset S$, there exists a unique maximal spherical subset $T' \subset S'$ such that $\phi(W_T) = w'W'_{T'}w'^{-1}$ for some $w' \in W'$, i.e., $\phi(W_T) \sim W'_{T'}$. Here we denote $A \sim B$ if A and B are conjugate.*

We first prove the following lemma.

Lemma 2.2 *The Coxeter system (W', S') satisfies the condition $(*)$, i.e.,*

(0′) *for each $s', t' \in S'$ such that $m'(s', t')$ is even, $m'(s', t') \in \{2\} \cup 4\mathbb{N}$,*
(1′) *for each $s' \neq t' \in S'$ such that $m'(s', t')$ is odd, $\{s', t'\}$ is a maximal spherical subset of S',*
(2′) *there does not exist a three-points subset $\{s', t', u'\} \subset S'$ such that $m'(s', t')$ and $m'(t', u')$ are odd, and*
(3′) *for each $s' \neq t' \in S'$ such that $m'(s', t')$ is odd, there exists at most one maximal spherical subset of S' that is different from $\{s', t'\}$ and intersecting with $\{s', t'\}$.*

Proof. Let $s' \neq t' \in S'$ with $m'(s', t') < \infty$. There exists a maximal spherical subset T' of S' such that $\{s', t'\} \subset T'$. By Lemma 2.1, $\phi^{-1}(W'_{T'}) \sim W_T$ for some maximal spherical subset T of S. By (0) and (1), either

(i) (W_T, T) is a strongly even Coxeter system, or
(ii) $|T| = 2$ and if $T = \{s, t\}$ then $m(s, t)$ is odd.

Hence W_T is a rigid Coxeter group by Theorem 1.1 and [8], and (W_T, T) and $(W'_{T'}, T')$ are isomorphic. Thus if $m'(s', t')$ is even then $m'(s', t') \in \{2\} \cup 4\mathbb{N}$, and if $m'(s', t')$ is odd then $\{s', t'\}$ is a maximal spherical subset of S'. Hence $(0')$ and $(1')$ hold. We can show $(2')$ and $(3')$ by the same argument as the proof of [9, Lemma 3.1] $\square$

Let $\mathcal{A}$ and $\mathcal{A}'$ be the sets of all maximal spherical subsets of S and S', respectively. For each $T \in \mathcal{A}$, there exists a unique element $T' \in \mathcal{A}'$ such that $\phi(W_T) \sim W'_{T'}$ by Lemma 2.1.

We define

$$\bar{S} = \bigcup \{T \in \mathcal{A} \mid (W_T, T) \text{ is strongly even}\}$$

$$\bar{S'} = \bigcup \{T' \in \mathcal{A}' \mid (W'_{T'}, T') \text{ is strongly even}\}.$$

We note that $(W_{\bar{S}}, \bar{S})$ and $(W_{\bar{S'}}, \bar{S'})$ are strongly even. Also we note that for each $s \in S \setminus \bar{S}$, there exists a unique element $t \in S \setminus \{s\}$ such that $m(s, t)$ is odd. Then $m(s, u) = \infty$ for any $u \in S \setminus \{s, t\}$ by the condition $(*)$.

Let W^{ab} and W'^{ab} be the abelianizations of W and W' respectively, and let $\pi : W \to W^{\mathrm{ab}}$ and $\pi' : W' \to W'^{\mathrm{ab}}$ be the abelianization maps.

We can obtain the following lemma by the same argument as the proof of [10, Theorem 4.4], since $(W_{\bar{S}}, \bar{S})$ is strongly even.

Lemma 2.3 *If A and B are subsets of $\bar{S}$ and $\pi(W_A) = \pi(W_B)$, then $A = B$.*

For $A \subset \bar{S}$ and $A' \subset \bar{S'}$, we denote $A\tau A'$ if $\pi'(\phi(W_A)) = \pi'(W'_{A'})$.

We can obtain the following lemma by the same argument as the proof of [10, Theorem 4.5].

Lemma 2.4 *Let A and B be subsets of $\bar{S}$ and let A' and B' be subsets of $\bar{S'}$.*

(i) *If $A\tau A'$ and $B\tau A'$ then $A = B$.*
(ii) *If $A\tau A'$ and $A\tau B'$ then $A' = B'$.*
(iii) *If $A\tau A'$ and $B\tau B'$ then $(A \cap B)\tau(A' \cap B')$.*

We obtain the following lemma from Lemmas 2.3 and 2.4.

Lemma 2.5 *Let A and B be subsets of $\bar{S}$ and let A' and B' be subsets of $\bar{S}'$. If $A\tau A'$, $B\tau B'$ and $A \subset B$, then $A' \subset B'$.*

Proof. Suppose that $A\tau A'$, $B\tau B'$ and $A \subset B$. By Lemma 2.4 (iii), $(A \cap B)\tau(A' \cap B')$. Since $A \subset B$, $A\tau(A' \cap B')$. Now $A\tau A'$. By Lemma 2.4 (ii), $A' = A' \cap B'$, i.e., $A' \subset B'$. □

A subset T of S is said to be *independent*, if $m(s,t) = 2$ for all $s \neq t$ in T. We note that if T is an independent subset of S then $W_T \cong \mathbb{Z}_2^{|T|}$. Let $\mathcal{B}$ and $\mathcal{B}'$ be the sets of all maximal independent subsets of $\bar{S}$ and $\bar{S}'$, respectively.

We show the following lemma which corresponds to [10, Theorem 4.7].

Lemma 2.6 *For each $T \in \mathcal{B}$, there exists a unique $T' \in \mathcal{B}'$ such that $T\tau T'$.*

Proof. Let $T \in \mathcal{B}$. Then there exists $U \in \mathcal{A}$ such that $T \subset U \subset \bar{S}$. By Lemma 2.1, $\phi(W_U) = w'W'_{U'}w'^{-1}$ for some $U' \in \mathcal{A}'$ and $w' \in W'$. Here $\phi : W_U \to w'W'_{U'}w'^{-1}$ is an isomorphism and (W_U, U) and $(W'_{U'}, U')$ are strongly even. By the proof of [10, Theorem 4.7], there exists a unique independent subset T' of U' such that $T\tau T'$. We show that T' is a *maximal* independent subset of $\bar{S}'$. Suppose that $T' \subset T'_0$ and T'_0 is an independent subset of $\bar{S}'$. Then by the above argument, there exists an independent subset T_0 of $\bar{S}$ such that $T_0\tau T'_0$. Since $T' \subset T'_0$, $T \subset T_0$ by Lemma 2.5. Hence $T = T_0$ because T is a maximal independent subset of $\bar{S}$. By Lemma 2.4 (ii), $T' = T'_0$. Thus T' is a maximal independent subset of $\bar{S}'$, i.e., $T' \in \mathcal{B}'$ which is a unique element such that $T\tau T'$. □

We can obtain the following lemma from Lemmas 2.4 (iii) and 2.6 and the proof of [10, Theorem 4.8].

Lemma 2.7 *Let $T_1, \dots, T_k \in \mathcal{A} \cup \mathcal{B}$ and $T'_1, \dots, T'_k \in \mathcal{A}' \cup \mathcal{B}'$ such that $T_i \subset \bar{S}$ and $T_i\tau T'_i$ for each $i = 1, \dots, k$. Then $|T_1 \cap \cdots \cap T_k| = |T'_1 \cap \cdots \cap T'_k|$.*

Lemma 2.7 implies that there exists a bijection $\bar{\psi} : \bar{S} \to \bar{S}'$ such that for each $s \in \bar{S}$ and $T \in \mathcal{A} \cup \mathcal{B}$ with $T \subset \bar{S}$, $s \in T$ if and only if $\bar{\psi}(s) \in T'$, where T' is the element of $\mathcal{A}' \cup \mathcal{B}'$ such that $T\tau T'$ (cf. [10]). By the proof of [10, Theorem 4.11], the bijection $\bar{\psi} : \bar{S} \to \bar{S}'$ induces an isomorphism between the Coxeter systems $(W_{\bar{S}}, \bar{S})$ and $(W_{\bar{S}'}, \bar{S}')$.

Here we note that we can construct $\bar{\psi}: \bar{S} \to \bar{S}'$ so that $\bar{\psi}(t) = t'$ for each $t \in \bar{S}$ and $t' \in \bar{S}'$ such that $\{t\}\tau\{t'\}$. Indeed, suppose that $\{t\}\tau\{t'\}$ (such t' is unique, since $(W_{\bar{S}}, \bar{S})$ and $(W'_{\bar{S}'}, \bar{S}')$ are even). Then for $T \in \mathcal{A} \cup \mathcal{B}$ with $T \subset \bar{S}$ and $T' \in \mathcal{A}' \cup \mathcal{B}'$ such that $T\tau T'$, $t \in T$ if and only if $t' \in T'$ by Lemma 2.5.

Using the above argument, we show the following.

Theorem 2.8 *The Coxeter systems (W, S) and (W', S') are isomorphic.*

Proof. We define a bijection $\psi : S \to S'$ as follows: Let $s \in S$. If $s \in \bar{S}$ then we define $\psi(s) = \bar{\psi}(s)$. Suppose that $s \in S \setminus \bar{S}$. Then there exists a unique element $t \in S \setminus \{s\}$ such that $m(s,t)$ is odd. Here we note that $m(s,u) = \infty$ for any $u \in S \setminus \{s,t\}$. Now either $t \in \bar{S}$ or $t \notin \bar{S}$. We first suppose that $t \notin \bar{S}$, i.e., $\{s,t\} \subset S \setminus \bar{S}$. Then $\{T \in \mathcal{A} \mid T \cap \{s,t\} \neq \emptyset\} = \{\{s,t\}\}$. There exists a unique $\{s',t'\} \in \mathcal{A}'$ such that $\phi(W_{\{s,t\}}) \sim W'_{\{s',t'\}}$ by Lemma 2.1. Here $\{s',t'\} \subset S' \setminus \bar{S}'$ by [9, Lemma 2.6]. We define $\psi(s) = s'$ and $\psi(t) = t'$. Next we suppose that $t \in \bar{S}$. Then $|\{T \in \mathcal{A} \mid T \cap \{s,t\} \neq \emptyset\}| = 2$, and there exists a unique $T \in \mathcal{A}$ such that $t \in T \subset \bar{S}$. By Lemma 2.1, there exist unique $\{s',t'\}, T' \in \mathcal{A}'$ such that $\phi(W_{\{s,t\}}) \sim W'_{\{s',t'\}}$ and $\phi(W_T) \sim W'_{T'}$. The proof of [9, Lemma 2.6] implies that $\{s',t'\} \cap T' \neq \emptyset$ and $\phi(t) \sim s' \sim t'$. We may suppose that $t' \in T'$. Then $\bar{\psi}(t) = t'$ because $\{t\}\tau\{t'\}$ by Lemma 2.4 (iii). We define $\psi(s) = s'$.

Then the bijection $\psi : S \to S'$ induces an isomorphism between the Coxeter systems (W, S) and (W', S') by the construction of ψ. $\square$

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