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# A characterization of dense vector fields in $\mathcal{G}^1(M)$ on 3-manifolds

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**Abstract.** Recently Morales-Pacfico-Pujals introduced the new concept of singular hyperbolicity and showed that $C^1$ robust transitive sets of 3-flows are singular hyperbolic sets ([8], [9]). Based on their papers, we shall characterize a dense subset of $\mathcal{G}^1(M)$ with $\dim M = 3$.

*Key words*: $\mathcal{G}^1(M)$, singular hyperbolic set, Axiom A.

## 1. Introduction

The purpose of this paper is to study the space of vector fields known as $\mathcal{G}^1(M)$. Let $M$ be a compact smooth manifold without boundary. We denote by $\chi^1(M)$ the set of $C^1$ vector fields on $M$, endowed with the $C^1$ topology and by $X_t$ $(t \in \mathbb{R})$ the $C^1$ flow on $M$ generated by $X \in \chi^1(M)$. $\Omega(X)$, $per(X)$, $Sing(X)$ are the sets of nonwandering, periodic and singular points of $X$ respectively. Recall that a set $\Lambda \subset M$ is called a hyperbolic set of $X$ if compact, invariant and there exists a continuous splitting $TM/\Lambda = E^s \oplus E^X \oplus E^u$, invariant under the derivative of flow $X_t$, $DX_t$, where $E^s$ and $E^u$ are exponentially contracted and expanded respectively by $DX_t$ and $E^X$ is tangent to $X$. We say that $X \in \chi^1(M)$ satisfies Axiom A if $\Omega(X)$ is a hyperbolic set of $X$ and $\Omega(X) = \overline{Sing(X) \cup per(X)}$ (We denote by $\overline{A}$ the closure of $A$ in $M$). Let $\mathcal{G}^1(M)$ be the interior of the set of vector fields in $\chi^1(M)$ whose critical elements (singularities and periodic orbits) are hyperbolic.

In [3], Hayashi showed that diffeomorphisms in $\mathcal{F}^1(M)$ satisfy Axiom A where $\mathcal{F}^1(M)$ is the diffeomorphism version of $\mathcal{G}^1(M)$ and this naturally give rise to the following question: Do vector fields in $\mathcal{G}^1(M)$ satisfy Axiom A? Unfortunately this does not hold generally and the geometric Lorenz attractor in [2] is well-known as one of the counter examples. Vector field generating this attractor is an element of $\mathcal{G}^1(M)$ but has singularities accu-

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mulated by the periodic orbits, hence its nonwandering set cannot be the hyperbolic set. So we hope for the other characterization of $\mathcal{G}^1(M)$, or at least dense subset of $\mathcal{G}^1(M)$, replacing Axiom A but until now no such a characterization exists for any dimension $\geq 3$.

Recently Morales-Pacifico-Pujals introduced the notion of singular hyperbolic set by generalizing both the geometric Lorenz attactor and the concept of hyperbolic set, and showed that $C^1$ robust transitive sets of 3-flows are singular hyperbolic sets ([8], [9]). Based on their papers, we shall characterize a dense subset of $\mathcal{G}^1(M)$ on 3-manifold in this paper. Throughout the rest of this paper, we assume $\dim M = 3$.

Before stating our theorem, we need the following definiton.

**Definition 1.1** ([8], Definition 1) A compact invariant set $\Lambda$ is a singular hyperbolic set of $X \in \chi^1(M)$ if it has singularities, all of them hyperbolic and there is a continuous splitting $TM/\Lambda = E^s \oplus E^{cu}$ invariant under $DX_t$ such that $E^{cu}$ contains the direction of the flow $X_t$, $E^s$ is one-dimensional and there exist two numbers $\lambda > 0$ and $C > 0$ satisfying

- $\|DX_t/E^s_x\| \cdot \|DX_{-t}/E^{cu}_{X_t(x)}\| \leq Ce^{-\lambda t}$
- $\|DX_t/E^s_x\| \leq Ce^{-\lambda t}$
- $\left|\det(DX_t/E^{cu}_x)\right| \geq Ce^{\lambda t}$

for all $t \geq 0$ and $x \in \Lambda$. Here $\det(A)$ means the determinant of $A$.

Singular hyperbolic set is as sort of "partially" hyperbolic set which has volume expanding central direction. Here partially hyperbolicity implies that $TM/\Lambda$ can be decomposed into $E^s \oplus E^{cu}$, $E^s$ being a uniformly contracting one-dimensional bundle that dominates $E^{cu}$. Definiton 1.1 requires the additional condition that $E^{cu}$ is volume expanding. It is easy to see that if singular hyperbolic set $\Lambda$ has a compact invariant subset which is isolated from singularities in $\Lambda$, then this subset is hyperbolic. Hence we see that singular hyperbolicity is a generalized concept of hyperbolicity for 3-flows which can handle the invariant sets with singularities.

Now we state our result.

**Theorem** *There exists a dense subset $\mathcal{U} \subset \mathcal{G}^1(M)$ such that for any $X \in \mathcal{U}$, replacing $X$ by $-X$ if necessary, $X$ satisfies Axiom A or exhibits a singular hyperbolic set.*

By above Theorem, any vector field in $\mathcal{G}^1(M)$ can be approximated by the vector fields which satisfy Axiom A or have the structure like the geometric Lorenz attractor.

## 2. Some Preliminaries

In this section we will state several results needed for the proof of Theorem. Let $S$ be any vector field in $\mathcal{G}^1(M)$ and $\sigma$ be any singularity of $S$. Since $\sigma$ is hyperbolic, there exist neighborhoods $\mathcal{N}(S) \subset \mathcal{G}^1(M)$, $U_\sigma \subset M$ of $S$ and $\sigma$ respectively and a continuous function $\rho : \mathcal{N}(S) \to U_\sigma$ which to each vector field $X \in \mathcal{N}(S)$ associates the unique singularity of $X$ in $U_\sigma$. We call $\rho(X)$ the continuaton of $\sigma$ for $X$. By $\rho$ and compactness of $M$, $S$ has finite number of singularities, denoted by $\sigma_1(S), \ldots, \sigma_l(S)$ respectively, and next lemma is immediate.

**Lemma 2.1** *For any $S \in \mathcal{G}^1(M)$, there exists a neighborhood $\mathcal{U}_0 = \mathcal{U}_0(S)$ of $S$ in $\mathcal{G}^1(M)$ such that singularities of each $X \in \mathcal{U}_0$ are only $\sigma_1(X), \ldots,$ $\sigma_l(X)$, which are the continuations of $\sigma_1(S), \ldots, \sigma_l(S)$ for $X$ respectively.*

It is well-known that each element of $\mathcal{G}^1(M)$ has finitely many attracting and repelling periodic orbits ([10]). Denote the number of these periodic orbits of $S \in \mathcal{G}^1(M)$ by $a(S)$ and $r(S)$ respectively.

**Lemma 2.2** *Any $S \in \mathcal{G}^1(M)$ can be approximated by neighborhood $\mathcal{U}_1 = \mathcal{U}_1(S)$ in $\mathcal{G}^1(M)$ such that $a(X) = a(Y)$, $r(X) = r(Y)$ for all $X, Y \in \mathcal{U}_1$.*

*Proof.* By contradiction, suppose that there exists a vector field $S_0$ and its neighborhood $\mathcal{B}_0 \subset \mathcal{G}^1(M)$ such that each $X \in \mathcal{B}_0$ is approximated by $Y$ satisfying $a(X) + r(X) < a(Y) + r(Y)$. Let $\mathcal{B}_n = \{X \in \mathcal{B}_0 : a(X) + r(X) \geq n\}$ $(n \geq 1)$ and then $\mathcal{B}_n$ is open and dense in $\mathcal{B}_0$. Since $\mathcal{B}_0$ is a Baire space, we can take $X_0 \in \bigcap_n \mathcal{B}_n (\subset \mathcal{B}_0)$. But clearly $a(X_0) + r(X_0) = \infty$, i.e., $X_0$ has infinite number of attracting or repelling periodic orbits. This is a contradiction to [10]. □

We take $\mathcal{U}_1 \subset \mathcal{U}_0$ in the following. Next we will characterize singularities of vector fields in $\mathcal{U}_1$ accumulated by the periodic orbits. To do this, we need the following definition.

**Definition 2.3** ([8], Definition 3) Let $X \in \chi^1(M)$. We say that $\sigma \in$ *Sing*$(X)$ is Lorenz-like if the eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ of the derivative $D_\sigma X$ are real and satisfy

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

In the following we denote by $per_1(X)$ and $\overline{per_1}(X)$ the sets of saddle periodic points of $X$ and its closure respectively. We also denote by $\operatorname{ind}\sigma$ the index of $\sigma$, i.e., the dimension of stable subspace of $D_\sigma X$.

**Proposition 2.4** *Let $X \in \mathcal{U}_1$ and $\sigma_X \in Sing(X) \cap \overline{per_1}(X)$ be given. Assume that $\operatorname{ind}\sigma_X = 2$. Then $\sigma_X$ is Lorenz-like and satisfies*

$$W^{ss}(\sigma_X) \cap \overline{per_1}(X) = \{\sigma_X\},$$

*where $W^{ss}(\sigma_X)$ is the stable manifold associated to the strong contracting eigenvalue $\lambda_2$.*

*Proof.* Proposition 2.4 is obtained by using the methods in the proof of [9, Lemmas 4.1 and 4.2] and the following lemma.

**Lemma 2.5** ([4], the $C^1$ Connecting Lemma) *Let $X \in \chi^1(M)$ and $p$, $q$ be two points which are not periodic. Assume that for all neighborhood $U$ and $V$ of $p$ and $q$ respectively, there is $x \in U$ such that $X_t(x) \in V$ for some $t \geq 0$. Then $\forall \epsilon > 0$, $\exists L > 0$ such that for any $\delta > 0$, there is $Y$ $C^1\epsilon$-close to $X$ satisfying*

- $Y = X$ *on* $M - B_\delta\left(X_{[0,L]}(p) \cup X_{[-L,0]}(q)\right)$
- *$q$ is on the forward $Y$-orbit of $p$.*

*Here $X_{[a,b]}(x)$ denotes the segment of orbit $\{X_t(x) \mid a \leq t \leq b\}$ and $B_\delta(A)$ is $\delta$-neighborhood of $A$ in $M$.*

By Lemma 2.5, we can perturb $X$ to $Y$ so that $Y \in \mathcal{U}_1$ and $Y$ exhibits a homoclinic loop associated to $\sigma_Y$ (continuation of $\sigma_X$ for $Y$). We can further perturb $Y$ to obtain $C^\infty$ vector field $Z \in \mathcal{U}_1$ such that $Z$ still has a homoclinic loop associated to $\sigma_Z$.

Assume that there exists $X \in \mathcal{U}_1$ with $\sigma_X \in Sing(X) \cap \overline{per_1}(X)$ having a complex eigenvalue. Then $\sigma_Z$ also has a complex eigenvalue. The argument of [14, p.247] shows that we can perturb $Z$ to $Z_1$, arbitrarily $C^1$ close to $Z$, to generate a new attracting periodic orbit. This contradicts $Z \in \mathcal{U}_1$, proving that the eigenvalues of $\sigma_X$ are real.

We can arrange the eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$ of $D_\sigma X$ such that

$$\lambda_2 \leq \lambda_3 \leq \lambda_1.$$

Then $X \in \mathcal{U}_1$ implies that $\lambda_3 < 0$ and $\lambda_1 > 0$. So if $\sigma_X$ is not Lorenz-like, we have that $|\lambda_2|, |\lambda_3| > \lambda_1$. Since the eigenvalues of $\sigma_Z$ satisfy this inequality, [14, Theorem 3.2.12] enable us to perturb $Z$ to generate a new attracting periodic orbit. This contradicts $Z \in \mathcal{U}_1$, proving $\sigma_X$ is Lorenz-like. For the rest of Propotition, we can perturb $X$ as in the proof of [9, Lemma 4.2] to generate an attracting periodic orbit again, which contradicts $X \in \mathcal{U}_1$. $\square$

## 3. Proof of Theorem

In this section we complete the proof of Theorem. For this let us state two key results. In [6], it was shown that on 2-manifolds compact invariant sets of diffeomorphisms in $\mathcal{F}^1(M)$ which have dense periodic orbits are hyperbolic. By applying the same method, we can immediately obtain the corresponding result for 3-flows as in the following (see also [12]).

**Proposition 3.1** *Let $X \in \mathcal{G}^1(M)$ and $\Lambda$ is a compact invariant set of $X$ which has dense periodic orbits. Then if $\Lambda \cap Sing(X) = \phi$, $\Lambda$ is a hyperbolic set of $X$.*

If $\Lambda \cap Sing(X) \neq \phi$, $\Lambda$ cannot be a hyperbolic set of $X$. However the following result holds in parallel with above Proposition.

**Proposition 3.2** *Let $X \in \mathcal{G}^1(M)$ and $\Lambda$ is a compact invariant set of $X$ which has dense periodic orbits. Suppose that $X$ has a neighborhood $\mathcal{U}_X$ such that $a(Y_1) = a(Y_2)$ and $r(Y_1) = r(Y_2)$ for all $Y_1, Y_2 \in \mathcal{U}_X$. Then if $\Lambda \cap Sing(X) \neq \phi$ and every singularity in $\Lambda$ is Lorenz-like, $\Lambda$ is a singular hyperbolic set of $X$.*

Proposition 3.2 is obtained from the methods in the proof of [9, Theorem C]. Let $\mathcal{P}_\Lambda$ be the set of periodic orbits contained in $\Lambda$. Then we have the hyperbolic splitting $TM/\mathcal{P}_\Lambda = E^s \oplus E^X \oplus E^u$, where $E^s$ is the stable bundle, $E^u$ is the unstable bundle and $E^X$ is tangent to the flow direction. We set $E^{cu} = E^X \oplus E^u$ and define over $\mathcal{P}_\Lambda$ the splitting

$$TM/\mathcal{P}_\Lambda = E^s \oplus E^{cu}. \tag{1}$$

Suppose that we can extend this splitting continuously to the closure $\overline{\mathcal{P}_\Lambda} = \Lambda$, denoted by $TM/\Lambda = \tilde{E}^s \oplus \tilde{E}^{cu}$, and $\tilde{E}^s$ dominates $\tilde{E}^{cu}$. Then we can show that $\tilde{E}^s$ and $\tilde{E}^{cu}$ is actually uniformly contracting and volume expanding bundle respectively. In fact if $\tilde{E}^s$ (resp. $\tilde{E}^{cu}$) is not contracting (resp. volume expanding), we can perturb $X$ to generate a new repelling (resp. attracting) periodic orbit by [9, §5.3 and 5.4], the argument similar to [6, pp.521–524]. But this contradicts $X \in \mathcal{U}_1$. Thus we see that $\Lambda$ is a singular hyperbolic set of $X$.

The proof of the continuous extension of (1) to $\Lambda$ and the domination property of $(\tilde{E}^s, \tilde{E}^{cu})$ is rather technical, but the basic idea follow from [6] substantially. In fact it is primarily proved that $(E^s, E^{cu})$ has the domination property, then by this property and [7, Proposition 1.3], (1) can be extended continuously to $\Lambda$. To prove the domination property of $(E^s, E^{cu})$, roughly speaking, it is enough to show that the angle between $E^s$ and $E^{cu}$ is uniformly bounded away from 0 over $\mathcal{P}_\Lambda$, which corresponds to the result of [6, Lemma II.9].

The hypothesis of robust transitivity is necessary to prove these facts for two reasons: to utilize the property of the periodic orbits of [9, Theorem 3.11] and prohibit the generation of new attracting or repelling periodic orbit by small $C^1$ perturbation. But our situation that $X \in \mathcal{U}_1$ satisfies these properties, hence we can directly use the proof of [9, Theorem C] to show Proposition 3.2.

To complete the proof of Theorem, we shall show that there exists a vector field in $\mathcal{U}_1$ of Lemma 2.2 which satisfies Axiom A or has a compact invariant subset satisfying the condition of Proposition 3.2. At first we state a well-known generic property without proof.

**Lemma 3.3** *There exists a residual subset $\mathcal{R}$ of $\chi^1(M)$ such that, for any $X \in \mathcal{R}$, if $K$ is a compact subset in $M$ satisfying $\overline{per_1}(X) \cap K = \phi$, then $\overline{per_1}(Y) \cap K = \phi$ for all $Y$ sufficiently $C^1$ close to $X$.*

Since we now take $\mathcal{U}_1 \subset \mathcal{U}_0$, each $X \in \mathcal{U}_1$ has $l$ singularities, $\sigma_1(X), \ldots, \sigma_l(X)$. By using Lemma 3.3 and arranging the subscript of these singularities appropriately, we obtain the following lemma.

**Lemma 3.4** *There exist a neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ and number $k$ with $0 \leq k \leq l$ such that*

1) *$X \in \mathcal{U}_2 \Rightarrow \sigma_1(X), \ldots, \sigma_k(X)$ are not accumulated by the periodic orbits of $X$.*
2) *$X \in \mathcal{U}_2 \cap \mathcal{R} \Rightarrow \sigma_{k+1}(X), \ldots, \sigma_l(X)$ are accumulated by the periodic orbits of $X$.*

*Proof.* First assume that there exist $X_1 \in \mathcal{U}_1 \cap \mathcal{R}$ and $\sigma_1(X_1) \in Sing(X_1)$ such that $\overline{per_1}(X_1) \cap \{\sigma_1(X_1)\} = \phi$, otherwise the lemma is proved by setting $\mathcal{U}_2 = \mathcal{U}_1$ and $k = 0$. Then by Lemma 3.3, there exists a neighborhood $\mathcal{V}_1$ of $X_1$ in $\mathcal{U}_1$ such that $\overline{per_1}(Y) \cap \{\sigma_1(Y)\} = \phi$ for any $Y \in \mathcal{V}_1$. Next assume that there exist $X_2 \in \mathcal{V}_1 \cap \mathcal{R}$ and $\sigma_2(X_2) \in Sing(X_2)$ such that $\overline{per_1}(X_2) \cap \{\sigma_2(X_2)\} = \phi$, otherwise the lemma is proved by setting $\mathcal{U}_2 = \mathcal{V}_1$ and $k = 1$. Again by Lemma 3.3, there exists a neighborhood $\mathcal{V}_2$ of $X_2$ in $\mathcal{V}_1$ such that $\overline{per_1}(Y) \cap \{\sigma_2(Y)\} = \phi$ for any $Y \in \mathcal{V}_2$. Since the number of singularities of vector fields in $\mathcal{U}_1$ is $l$, we can prove the lemma by continuing this process at most $l$ times. □

**Remark** After continuing this process, suppose that we have reached the situation such that $Sing(X_0) \cap \overline{per_1}(X_0) = \phi$ for some $X_0 \in \mathcal{U}_1 \cap \mathcal{R}$. Then by Proposition 3.1, we have that $\overline{per_1}(X_0)$ (and therefore $\overline{Sing(X_0) \cup per(X_0)}$) is a hyperbolic set of $X_0$. Moreover by the general density theorem ([11]), we may assume that $\Omega(X_0) = \overline{Sing(X_0) \cup per(X_0)}$. This implies $X_0$ satisfies Axiom A, proving our Theorem. So we will assume $k \lneqq l$ in the following.

**Lemma 3.5** *Let $X \in \mathcal{U}_2 \cap \mathcal{R}$. Then for any $\sigma_X \in Sing(X) \cap \overline{per_1}(X)$ and any $\delta > 0$, there exists a neighborhood $\mathcal{W}$ of $X$ in $\mathcal{U}_2$ such that each $Y \in \mathcal{W}$ satisfies*

$$W^u(\sigma_Y) \subset B_\delta(\overline{per_1}(X)) \quad or \quad W^s(\sigma_Y) \subset B_\delta(\overline{per_1}(X))$$

*according to* $\operatorname{ind}\sigma_X = 2$ *or* $1$ *respectively. Here $\sigma_Y$ is the continuation of $\sigma_X$ for $Y$.*

*Proof.* This Lemma corresponds to [9, Corollary 3.5], for which the property of robust transitivity is essential. The proof here is, however, the conseqence from Lemma 3.3.

Suppose that on the contrary, there exist $\sigma_X \in Sing(X) \cap \overline{per_1}(X)$, $\delta_0 > 0$ and $Y_0$ arbitrarily $C^1$ close to $X$ such that the lemma is false. Replacing $X$ by $-X$ if necessary, we may assume $\operatorname{ind}\sigma_X=2$, i.e., $\sigma_X$ is Lorenz-like. So we assume that

$$W^u(\sigma_{Y_0}) \not\subset B_{\delta_0}(\overline{per_1}(X)).$$

Then there exists a neighborhood $\mathcal{V}(Y_0)$ of $Y_0$ in $\mathcal{U}_2$ such that for each $Z \in \mathcal{V}(Y_0)$, $\sigma_Z$ is Lorenz-like and

$$W^u(\sigma_Z) \not\subset B_{\delta_0}(\overline{per_1}(X)). \tag{2}$$

Shrinking $\mathcal{V}(Y_0)$ if necessary, we may also assume that

$$\overline{per_1}(Z) \subset B_{\delta_0}\big(\overline{per_1}(X)\big) \tag{3}$$

from Lemma 3.3. Now we carry out the following sequence of pertubations of $Y_0$.

1. By Lemma 3.4 and (2), we can perturb $Y_0$ to $Y_1$, arbitrarily $C^1$ close to $Y_0$, so that $\sigma_{Y_1} \in Sing(Y_1) \cap \overline{per_1}(Y_1)$ and $W^u(\sigma_{Y_1}) \not\subset B_{\delta_0}\big(\overline{per_1}(X)\big)$.
2. Then by using Lemma 2.5, we can perturb $Y_1$ to $Y_2$, arbitrarily $C^1$ close to $Y_1$, so that $Y_2$ exhibits a homoclinic loop $\Gamma$ associated to $\sigma_{Y_2}$ and still $W^u(\sigma_{Y_2}) \not\subset B_{\delta_0}\big(\overline{per_1}(X)\big)$.
3. Since $W^u(\sigma_{Y_2}) \not\subset B_{\delta_0}\big(\overline{per_1}(X)\big)$, there are orbits which are through points arbitrarily near $\Gamma$ and leave $B_{\delta_0}\big(\overline{per_1}(X)\big)$ along one branch of $W^u(\sigma_{Y_2})$. So we can perturb $Y_2$ to $Y_3$, arbitrarily $C^1$ close to $Y_2$, so that both branches of $W^u(\sigma_{Y_3})$ leave $B_{\delta_0}\big(\overline{per_1}(X)\big)$.
4. Again by Lemma 3.4 we can perturb $Y_3$ to $Y_4$, arbitrarily $C^1$ close to $Y_3$, so that $\sigma_{Y_4} \in Sing(Y_4) \cap \overline{per_1}(Y_4)$ and both branches of $W^u(\sigma_{Y_4})$ still leave $B_{\delta_0}\big(\overline{per_1}(X)\big)$. Clearly we have $\overline{per_1}(Y_4) - B_{\delta_0}\big(\overline{per_1}(X)\big) \neq \phi$.

As we can perturb $Y_0$ to $Y_4$ arbitrarily $C^1$ close to $Y_0$, this is a contradiction to (3). Thus we have completed the proof of Lemma 3.5. □

**Lemma 3.6** *Let $X \in \mathcal{U}_2 \cap \mathcal{R}$. Then we have the following*:

*Let $\sigma_1$, $\sigma_2$ be any two singularities of $X$. If there are points $p_n \in per_1(X)$ and numbers $t_n > 0$ such that $\lim_{n\to\infty} p_n = \sigma_1$ and $\lim_{n\to\infty} X_{t_n}(p_n) = \sigma_2$, then* $\operatorname{ind}\sigma_1 = \operatorname{ind}\sigma_2$.

*Proof.* Suppose that $X$ has two singularities $\sigma_1(X)$, $\sigma_2(X)$ such that the lemma is false. Let $\operatorname{ind}\sigma_1(X) = 1$ and $\operatorname{ind}\sigma_2(X) = 2$ respectively. By Proposition 2.4, we have $W^{uu}\big(\sigma_1(X)\big) \not\subset V$ for a small neighborhood $V$ of $\overline{per_1}(X)$. Then using the argument in the proof of [9, Lemma 4.3], we can perturb $X$ to $Y$, arbitrarily $C^1$ close to $X$ so that $W^u\big(\sigma_2(Y)\big) \not\subset V$. This contradicts Lemma 3.5. □

Now we conclude the proof of Theorem. Let $S$ be any vector field in $\mathcal{G}^1(M)$. Then $S$ can be $C^1$ approximated by $X \in \mathcal{U}_2 \cap \mathcal{R}$. By Lemma 3.4 and Remark after it, $X$ has singularities accumulated by the periodic orbits, so we can take a sequence of saddle periodic orbits of $X$, $\{\gamma_n\}_{n\geq 1}$, such that

$\overline{\bigcup_{n=1}^{\infty}\gamma_n} \cap Sing(X) \neq \phi$. Set $K = \overline{\bigcup_{n=1}^{\infty}\gamma_n}$. Since all the singularities in $K$ have the same index by Lemma 3.6, replacing $X$ by $-X$ if necessary, we may assume that those index is two. So given $\sigma \in K \cap Sing(X)$, $\sigma$ is Lorenz-like by Proposition 2.4. Then Propositon 3.2 implies that $K$ is a singular hyperbolic set of $X$. We have completed the proof of Theorem.

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