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On the Gevrey wellposedness of the Cauchy problem for weakly hyperbolic equations of 4th order

(Dedicated to Professor Kunihiko KAJITANI on his sixtieth birthday)

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Abstract. We shall consider the Cauchy problem for weakly hyperbolic equations of 4th order with coefficients depending on time. Using the suitable energy of 4th order equations, we derive the energy inequality which shows exactly the derivative loss of the solution.

Key words: weakly hyperbolic equations, Gevrey wellposedness, energy inequality.

1. Introduction

F. Colombini, E. De Giorgi, E. Jannelli and S. Spagnolo showed the Gevrey wellposedness for the hyperbolic equations of second order with coefficients depending only on time and Hölder continuous (see [1], [2] and see also [3], [12] for the hyperbolic equations of third order). Later on their results were generalized by T. Nishitani [14] and P. D'Ancona [6] for the second order equations whose coefficients depend also on space variables, and by Y. Ohya and S. Tarama [16] for the hyperbolic equations of higher order. In this paper we restrict to hyperbolic equations of 4th order with less smooth coefficients in time, and show the Gevrey wellposedness, using the suitable energy for the equation of 4th order.

We shall consider the Cauchy problem in $[0,T]\times\mathbf{R}_x^n$

$$\begin{cases} \partial_t^4 u - \sum_{|\omega|=2} a_\omega(t)\partial_x^\omega \partial_t^2 u + \sum_{|\omega|=4} b_\omega(t)\partial_x^\omega u = 0, \\ \partial_t^h u(0,x) = u_h(x) \quad (h=0,1,2,3), \end{cases} \tag{1}$$

where $a_\omega(t)$ and $b_\omega(t)$ ($\omega\in\mathbf{N}^n$) are real coefficients satisfying, for some $\alpha\in(0,1]$,

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$$\Big\{ \sum_{|\omega|=2} a_\omega(t)\xi^\omega \Big\}^2 \in C^\alpha([0,T]),$$

$$\sum_{|\omega|=4} b_\omega(t)\xi^\omega \in C^\alpha([0,T]) \quad \text{for} \quad \xi \in \mathbf{R}^n_\xi. \tag{2}$$

We remark that the condition (2) is more relaxed condition than

$$\sum_{|\omega|=2} a_\omega(t)\xi^\omega \in C^\alpha([0,T]),$$

$$\sum_{|\omega|=4} b_\omega(t)\xi^\omega \in C^\alpha([0,T]) \quad \text{for} \quad \xi \in \mathbf{R}^n_\xi. \tag{3}$$

Obviously if $a_\omega(t)$ and $b_\omega(t)$ satisfy (3), they also satisfy (2). Conversely, supposing $n = 1$, we define

$$a_2(t) = \begin{cases} 2t\sin^2 \dfrac{1}{t} & \text{for} \quad t \in (0,T] \\ 0 & \text{for} \quad t = 0, \end{cases}$$

$$b_4(t) = \begin{cases} t^2\sin^4 \dfrac{1}{t} & \text{for} \quad t \in (0,T] \\ 0 & \text{for} \quad t = 0. \end{cases}$$

The condition (3) with $\alpha = 1$ excludes $a_2(t)$. In fact, if we put $t_j = \frac{1}{j\pi}$, $t'_j = \frac{1}{\frac{\pi}{4}+j\pi}$ $(j \in \mathbf{N})$, then we find that $\frac{4}{\pi}|t_j - t'_j| = \frac{1}{j\pi(\frac{\pi}{4}+j\pi)}$ and $|a_2(t_j) - a_2(t'_j)| = \left|2t_j\sin^2\frac{1}{t_j} - 2t'_j\sin^2\frac{1}{t'_j}\right| = |-t'_j| = \frac{1}{\frac{\pi}{4}+j\pi}$. Hence noting that $\frac{1}{j\pi(\frac{\pi}{4}+j\pi)} \geq \frac{1}{(\frac{\pi}{4}+j\pi)^2} \geq \frac{4}{5}\frac{1}{j\pi(\frac{\pi}{4}+j\pi)}$, we get

$$\frac{2}{\sqrt{\pi}}\,|t_j - t'_j|^{\frac{1}{2}} \geq |a_2(t_j) - a_2(t'_j)| \geq \frac{4}{\sqrt{5\pi}}\,|t_j - t'_j|^{\frac{1}{2}} \quad \text{for} \quad j \in \mathbf{N}.$$

This means that the condition (3) with $\alpha \leq \frac{1}{2}$ holds for $a_2(t)$. In the same way the condition (2) with $\alpha = 1$ holds for $a_2(t)$ and $b_4(t)$.

Remark 1 More generally, if $\sum_{|\omega|=2} a_\omega(t)\xi^\omega \geq 0$, then (2) implies $\sum_{|\omega|=2} a_\omega(t)\xi^\omega \in C^{\frac{\alpha}{2}}([0,T])$; on the other hand, if $\sum_{|\omega|=2} a_\omega(t)\xi^\omega \geq \delta|\xi|^2$ $({}^\exists\delta > 0)$, then (2) implies (3), that is $\sum_{|\omega|=2} a_\omega(t)\xi^\omega \in C^\alpha([0,T])$.

Now we shall assume three types of weakly hyperbolic conditions for the 4th order equation (1)

$$\text{(I)} \qquad \sum_{|\omega|=2} a_\omega(t)\xi^\omega \geq 0, \quad \sum_{|\omega|=4} b_\omega(t)\xi^\omega \geq 0,$$
$$\Big\{\sum_{|\omega|=2} a_\omega(t)\xi^\omega\Big\}^2 - 4\sum_{|\omega|=4} b_\omega(t)\xi^\omega \geq 0,$$

$$\text{(II)} \qquad \sum_{|\omega|=2} a_\omega(t)\xi^\omega \geq 0, \quad \sum_{|\omega|=4} b_\omega(t)\xi^\omega \geq 0,$$
$$\Big\{\sum_{|\omega|=2} a_\omega(t)\xi^\omega\Big\}^2 - 4\sum_{|\omega|=4} b_\omega(t)\xi^\omega \geq \delta_0|\xi|^4,$$

$$\text{(III)} \qquad \sum_{|\omega|=2} a_\omega(t)\xi^\omega \geq 0, \quad \sum_{|\omega|=4} b_\omega(t)\xi^\omega \geq \delta_0|\xi|^4,$$
$$\Big\{\sum_{|\omega|=2} a_\omega(t)\xi^\omega\Big\}^2 - 4\sum_{|\omega|=4} b_\omega(t)\xi^\omega \geq 0,$$

for $t \in [0,T]$, $\xi \in \mathbf{R}^n_\xi$ with some $\delta_0 > 0$. We find that under these conditions there always exist two non-negative characteristic roots and two non-positive ones. Particularly under the condition (I) the multiplicity of the characteristic roots is four. While under the conditions (II) or (III) the multiplicity is at most two.

Remark 2 In the case (II) we have really $\sum_{|\omega|=2} a_\omega(t)\xi^\omega \geq \sqrt{\delta_0}|\xi|^2$, while in the case (III) $\sum_{|\omega|=2} a_\omega(t)\xi^\omega \geq 2\sqrt{\delta_0}|\xi|^2$.

Then we can prove the following theorems.

Theorem 1 (multiplicity 4) *Let $T > 0$, $\rho_0 > 0$. The coefficients satisfy* (2) *and* (I). *Then for any $u_h \in \gamma^s(\mathbf{R}^n_x)$ $(h = 0,1,2,3)$, the Cauchy problem* (1) *has a unique (global) solution $u \in C^4\big([0,T],\gamma^s(\mathbf{R}^n_x)\big)$, provided*

$$1 \leq s < 1 + \frac{\alpha}{4}, \tag{4}$$

and moreover when $u_h \in \gamma^s_0(\mathbf{R}^n_x)$ $(s > 1)$, there exists a positive function $\rho(t)$ satisfying $\rho(0) = \rho_0$, such that for $\xi \in \mathbf{R}^n_\xi$

$$e^{\rho(t)\langle\xi\rangle^{\frac{1}{s}}}\Big\{\langle\xi\rangle^{\frac{12}{4+\alpha}}|\hat{u}|+\langle\xi\rangle^{\frac{8}{4+\alpha}}|\partial_t\hat{u}|+\langle\xi\rangle^{\frac{4-\alpha}{4+\alpha}}|\partial_t^2\hat{u}|+\langle\xi\rangle^{\frac{-2\alpha}{4+\alpha}}|\partial_t^3\hat{u}|+\langle\xi\rangle^{\frac{-4-4\alpha}{4+\alpha}}|\partial_t^4\hat{u}|\Big\}$$
$$\leq {}^{\exists}C\sum_{h=0}^{3} e^{\rho_0\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{3-h}|\hat{u}_h|. \qquad (5)_{\mathrm{I}}$$

Remark 3 The critical assumption (4) coincides with the result of Y. Ohya and S. Tarama [16] under the condition (3) instead of the condition (2). Under the condition (2) (see Remark 1), it follows by [16] the wellposedness in γ^s for $1 \leq s < 1 + \frac{\alpha}{8}$.

Theorem 2 (multiplicity 2) *Let $T > 0$, $\rho_0 > 0$. The coefficients satisfy* (2) *and* (II) (*resp.* (III)). *Then for any $u_h \in \gamma^s(\mathbf{R}^n_x)$ $(h = 0, 1, 2, 3)$, the Cauchy problem* (1) *has a unique* (*global*) *solution $u \in C^4([0,T], \gamma^s(\mathbf{R}^n_x))$, provided*

$$1 \leq s < 1 + \frac{\alpha}{2}, \qquad (6)$$

and moreover when $u_h \in \gamma_0^s(\mathbf{R}^n_x)$ $(s > 1)$, there exists a positive function $\rho(t)$ satisfying $\rho(0) = \rho_0$, such that for $\xi \in \mathbf{R}^n_\xi$

$$e^{\rho(t)\langle\xi\rangle^{\frac{1}{s}}}\Big\{\langle\xi\rangle^{\frac{6+2\alpha}{2+\alpha}}|\hat{u}| + \langle\xi\rangle^2|\partial_t\hat{u}| + \langle\xi\rangle|\partial_t^2\hat{u}| + |\partial_t^3\hat{u}| + \langle\xi\rangle^{\frac{-2-2\alpha}{2+\alpha}}|\partial_t^4\hat{u}|\Big\}$$
$$\leq {}^{\exists}C\sum_{h=0}^{3} e^{\rho_0\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{3-h}|\hat{u}_h|, \qquad (7)_{\mathrm{II}}$$

$$\Big(\textit{resp.}\ \sum_{h=0}^{4} e^{\rho(t)\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{\frac{5+3\alpha}{2+\alpha}-h}|\partial_t^h\hat{u}| \leq {}^{\exists}C\sum_{h=0}^{3} e^{\rho_0\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{3-h}|\hat{u}_h|\Big). \qquad (8)_{\mathrm{III}}$$

Remark 4 The condition (2) is equivalent to the condition (3) under the condition (II) or (III) (see Remark 1 and Remark 2). Therefore the wellposedness result of Theorem 2 follows by [16]. So we shall particularly show how to obtain the estimates $(7)_{\mathrm{II}}$ and $(8)_{\mathrm{III}}$ which will be useful when one considers nonlinear equations of 4th order (see [7], [9] and [17]).

Remark 5 It is interesting to notice that, in Theorem 2, we obtain in the case (II) a better estimate than in the case (III). This is related to the following fact (see [4] where the general case of operators of order m is treated). When the characteristic roots degenerate of finite order and the coefficients a_ω and b_ω are sufficiently regular (say C^5), under the condition (II) the operator L given in (1) is strongly hyperbolic (the Cauchy problem (1) is C^∞-wellposed), while under the condition (III), if really $\left\{\sum_{|\omega|=2} a_\omega \xi^\omega\right\}^2 - 4\sum_{|\omega|=4} b_\omega \xi^\omega = 0$ for some $\xi \in \mathbf{R}^n_\xi$, surely L is not strongly hyperbolic.

The vibrating beam is described by the equation

$$\left(\partial_t^2 - A(t,\partial_x)\right)\left(\partial_t^2 - B(t,\partial_x)\right)u = 0, \tag{9}$$

where $A(t,\xi) = \sum_{|\omega|=2} p_\omega(t)\xi^\omega \geq \delta|\xi|^2$, $B(t,\xi) = \sum_{|\omega|=2} q_\omega(t)\xi^\omega \geq \delta|\xi|^2$ for $t \in [0,T]$, $\xi \in \mathbf{R}^n_\xi$ with some $\delta > 0$ and $A(t,\xi)$, $B(t,\xi) \in C^2([0,T])$ for $\xi \in \mathbf{R}^n_\xi$. If $\sum_{|\omega|=2} a_\omega(t)\partial_x^\omega = A(t,\partial_x) + B(t,\partial_x)$ and $\sum_{|\omega|=4} b_\omega(t)\partial_x^\omega = A(t,\partial_x)B(t,\partial_x)$, the equation (1) with lower term $-2\{\partial_t B(t,\partial_x)\}\partial_t \hat{u} - \{\partial_t^2 B(t,\partial_x)\}u$ can be written by the equation (9). By Fourier transform we change the equation (9) into $\left(\partial_t^2 + A(t,\xi)\right)\left(\partial_t^2 + B(t,\xi)\right)\hat{u} = 0$ which can be regarded as the following equation whose solution is $\left(\partial_t^2 + B(t,\xi)\right)\hat{u}$.

$$\partial_t^2\left\{\left(\partial_t^2 + B(t,\xi)\right)\hat{u}\right\} + A(t,\xi)\left\{\left(\partial_t^2 + B(t,\xi)\right)\hat{u}\right\} = 0.$$

Then we define the energy for the second order equation

$$\begin{aligned} E^{(1)}(t,\xi)^2 =& \left|\partial_t\left(\partial_t^2 + B(t,\xi)\right)\hat{u}\right|^2 + A(t,\xi)\left|\left(\partial_t^2 + B(t,\xi)\right)\hat{u}\right|^2 \\ &\left(= \left|\left(\partial_t^2 + B(t,\xi)\right)\partial_t\hat{u} + \left\{\partial_t B(t,\xi)\right\}\hat{u}\right|^2 + A(t,\xi)\left|\left(\partial_t^2 + B(t,\xi)\right)\hat{u}\right|^2\right). \end{aligned}$$

Here we remark that the term $\{\partial_t B(t,\xi)\}\hat{u}$ appears in the energy due to the lower terms. According to the method of the second order equations, we obtain the energy inequality $E^{(1)}(t,\xi)^2 \leq E^{(1)}(0,\xi)^2$ for $t \in [0,T]$. Hence we find that

$$\left(\partial_t^2 - B(t,\partial_x)\right)u = f(t,x) \in C^2\left([0,T], C^\infty(\mathbf{R}^n_x)\right). \tag{10}$$

Furthermore solving (10), we obtain that $u(t,x) \in C^2\left([0,T], C^\infty(\mathbf{R}^n_x)\right)$.

While the equation (1) with lower terms $-2\{\partial_t A(t,\partial_x)\}\partial_t u - \{\partial_t^2 A(t,\partial_x)\}u$ can be also written by

$$\left(\partial_t^2 - B(t,\partial_x)\right)\left(\partial_t^2 - A(t,\partial_x)\right)u = 0.$$

Similarly defining the energy for the second order equation

$$\begin{aligned} E^{(2)}(t,\xi)^2 =& \left|\partial_t\left(\partial_t^2 + A(t,\xi)\right)\hat{u}\right|^2 + B(t,\xi)\left|\left(\partial_t^2 + A(t,\xi)\right)\hat{u}\right|^2 \\ &\left(= \left|\left(\partial_t^2 + A(t,\xi)\right)\partial_t\hat{u} + \left\{\partial_t A(t,\xi)\right\}\hat{u}\right|^2 + B(t,\xi)\left|\left(\partial_t^2 + A(t,\xi)\right)\hat{u}\right|^2\right), \end{aligned}$$

we obtain $u(t,x) \in C^2\left([0,T], C^\infty(\mathbf{R}^n_x)\right)$.

Remark 6 If $A(t,\xi) \geq 0$, $B(t,\xi) \geq 0$ for $t \in [0,T]$, $\xi \in \mathbf{R}^n_\xi$ and $A(t,\xi)$, $B(t,\xi) \in C^\alpha([0,T])$ for $\xi \in \mathbf{R}^n_\xi$, applying the result of [2], we obtain the solution of the equation (9) or (10), in $\gamma^s(\mathbf{R}^n_x)$, provided $1 \leq s < 1 + \frac{\alpha}{2}$.

Since our equation (1) does not include the lower terms, we shall combine $E^{(1)}(t,\xi)^2$ and $E^{(2)}(t,\xi)^2$ and exclude the terms $\{\partial_t B(t,\xi)\}\hat{u}$ and $\{\partial_t A(t,\xi)\}\hat{u}$ to define the energy for (1) with suitably modified coefficients (see the definition of the partial energy (23)). However it seems very difficult to treat the order $2m$ similarly as the order 4, unless under more restricted assumptions (see [5]). In [5] we consider the equations of higher order (not only $2m$ order).

Notations

$$\langle\xi\rangle_\nu = (|\xi|^2 + \nu^2)^{\frac{1}{2}} \ (\nu \geq 1), \quad \langle\xi\rangle = (|\xi|^2 + 1)^{\frac{1}{2}} \ (= \langle\xi\rangle_1),$$
$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)dx.$$

$C^\alpha([0,T])$ $(0 \leq \alpha \leq 1)$ is the space of Hölder continuous functions with exponent α on $[0,T]$.

$\gamma^s(\mathbf{R}^n_x)$ $(s \geq 1)$ is the space of Gevrey functions $f(x)$ satisfying for any compact set $K \subset \mathbf{R}^n$, $\sup_{x\in K} |\partial_x^\alpha f(x)| \leq C_K \rho_K^{|\alpha|} |\alpha|!^s$ for $\alpha \in \mathbf{N}^n$.

$\gamma_0^s(\mathbf{R}^n_x)$ $(s > 1)$ is the space of Gevrey functions $f(x)$ of the order s having compact support.

2. Proof of the Theorems

When $s = 1$, the problem (1) is well-posed in γ^1 which is the topological vector space of analytic functions on $\mathbf{R}^n$ (see [8] and [10]). Therefore we may suppose $s > 1$ for the proof. Our main task is to investigate the regularity of the solution, namely, to derive the energy inequality $(5)_{\mathrm{I}}$, $(7)_{\mathrm{II}}$ and $(8)_{\mathrm{III}}$.

By Fourier transform the Cauchy problem (1) is changed into

$$\begin{cases} v_{tttt} + \{A(t,\xi) + B(t,\xi)\}v_{tt} + A(t,\xi)B(t,\xi)v = 0, \\ \partial_t^h v(0,\xi) = v_h(\xi) \quad (h = 0,1,2,3), \end{cases} \tag{11}$$

where $v = \hat{u}$, and $v_h = \hat{u}_h$ $(h = 0,1,2,3)$ and $A(t,\xi)$, $B(t,\xi)$ $(A(t,\xi) \geq B(t,\xi))$ are two roots of the algebraic equation $\Lambda^2 - \sum_{|\omega|=2} a_\omega(t)\xi^\omega \Lambda + \sum_{|\omega|=4} b_\omega(t)\xi^\omega = 0$.

Using $A(t,\xi)$ and $B(t,\xi)$, we also change the weakly hyperbolic conditions (I), (II) and (III) into

"$A(t,\xi)$ and $B(t,\xi)$ are real valued" &

$$A(t,\xi) \geq 0, \quad B(t,\xi) \geq 0, \tag{12$_{\mathrm{I}}$}$$

$$A(t,\xi) \geq 0, \quad B(t,\xi) \geq 0, \quad A(t,\xi) - B(t,\xi) \geq \delta_1 |\xi|^2 \\ \big(\text{and so } \ A(t,\xi) \geq \delta_1 |\xi|^2\big), \tag{13$_{\mathrm{II}}$}$$

$$A(t,\xi) \geq 0, \quad B(t,\xi) \geq \delta_1 |\xi|^2 \quad \big(\text{and so } \ A(t,\xi) \geq \delta_1 |\xi|^2\big), \tag{14$_{\mathrm{III}}$}$$

for $t \in [0,T]$, $\xi \in \mathbf{R}^n_\xi$ with some $\delta_1 > 0$.

Now we must separate the proof of Theorems into three parts according to the weakly hyperbolic conditions $(12)_{\mathrm{I}}$, $(13)_{\mathrm{II}}$ and $(14)_{\mathrm{III}}$.

2.1. Case of $(12)_{\mathrm{I}}$ (Theorem 1)

We first treat the case of $(12)_{\mathrm{I}}$ which implies that the multiplicity of the characteristic roots is four. In order to define the energy based on the distinct and smooth characteristic roots, we regularize the coefficient as follows.

$$H_\varepsilon(t,\xi) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \big(A(\tau,\xi) + B(\tau,\xi)\big)^2 \varphi\Big(\frac{t-\tau}{\varepsilon}\Big) d\tau + 5\varepsilon^\alpha |\xi|^4, \tag{15}$$

$$G_\varepsilon(t,\xi) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} A(\tau,\xi) B(\tau,\xi) \varphi\Big(\frac{t-\tau}{\varepsilon}\Big) d\tau + \varepsilon^\alpha |\xi|^4, \tag{16}$$

$(0 < \varepsilon < 1)$, where $\varphi(t) \in C_0^\infty(\mathbf{R}^1_t)$ satisfies $\varphi(t) \geq 0$ and $\int_{-\infty}^{\infty} \varphi(t) dt = 1$. Using $H_\varepsilon(t,\xi)$ and $G_\varepsilon(t,\xi)$, we define that $A_\varepsilon(t,\xi)$ and $B_\varepsilon(t,\xi)$ $(A_\varepsilon(t,\xi) \geq B_\varepsilon(t,\xi))$ are two roots of the algebraic equation $\Lambda^2 - \sqrt{H_\varepsilon(t,\xi)}\Lambda + G_\varepsilon(t,\xi) = 0$.

Then since $A(t,\xi) + B(t,\xi) = \sum_{|\omega|=2} a_\omega(t)\xi^\omega$ and $A(t,\xi)B(t,\xi) = \sum_{|\omega|=4} b_\omega(t)\xi^\omega$, by (2) there exists $M > 0$ such that for $\xi \in \mathbf{R}^\xi_n$

$$\Big|\big(A_\varepsilon(t,\xi) + B_\varepsilon(t,\xi)\big)^2 - \big(A(t,\xi) + B(t,\xi)\big)^2\Big| \leq M\varepsilon^\alpha |\xi|^4, \tag{17}$$

$$\big|A_\varepsilon(t,\xi)B_\varepsilon(t,\xi) - A(t,\xi)B(t,\xi)\big| \leq M\varepsilon^\alpha |\xi|^4, \tag{18}$$

$$\begin{aligned}&\Big|\frac{\partial}{\partial t}\big(A_\varepsilon(t,\xi)+B_\varepsilon(t,\xi)\big)^2\Big|\le M\varepsilon^{\alpha-1}|\xi|^4,\\&\Big|\frac{\partial}{\partial t}\big(A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)\big)\Big|\le M\varepsilon^{\alpha-1}|\xi|^4.\end{aligned}\tag{19}$$

For a proof refer to [1] and [18]. We remark that by (15) and (16) $A_\varepsilon(t,\xi)$, $B_\varepsilon(t,\xi)$ have not been defined yet.

Lemma 1 *Define that $C_\varepsilon(t,\xi)=A_\varepsilon(t,\xi)-B_\varepsilon(t,\xi)$. Then it holds that for $\xi\in\mathbf{R}^n_\xi$*

$$\begin{gathered}\big(A_\varepsilon(t,\xi)+B_\varepsilon(t,\xi)\big)^2\ge 5\varepsilon^\alpha|\xi|^4,\quad A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)\ge\varepsilon^\alpha|\xi|^4\\C_\varepsilon(t,\xi)^2\ge\varepsilon^\alpha|\xi|^4,\end{gathered}\tag{20}$$

and

$$\frac{\big(A_\varepsilon(t,\xi)+B_\varepsilon(t,\xi)\big)^2}{A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)C_\varepsilon(t,\xi)^2}\le 5\varepsilon^{-\alpha}|\xi|^{-4}.\tag{21}$$

Proof. From (15), (16) and $(12)_{\mathrm{I}}$ we can easily see that

$$\big(A_\varepsilon(t,\xi)+B_\varepsilon(t,\xi)\big)^2\ge 5\varepsilon^\alpha|\xi|^4,\quad A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)\ge\varepsilon^\alpha|\xi|^4,$$

and

$$\begin{aligned}C_\varepsilon(t,\xi)^2&=\big(A_\varepsilon(t,\xi)+B_\varepsilon(t,\xi)\big)^2-4A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)\\&=\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\{\big(A(\tau,\xi)+B(\tau,\xi)\big)^2-4A(\tau,\xi)B(\tau,\xi)\}\varphi\Big(\frac{t-\tau}{\varepsilon}\Big)d\tau\\&\qquad\qquad+\big(5\varepsilon^\alpha-4\varepsilon^\alpha\big)|\xi|^4\\&=\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\big(A(\tau,\xi)-B(\tau,\xi)\big)^2\varphi\Big(\frac{t-\tau}{\varepsilon}\Big)d\tau+\varepsilon^\alpha|\xi|^4\\&\ge\varepsilon^\alpha|\xi|^4.\end{aligned}$$

Hence we also get

$$\begin{aligned}\frac{\big((A_\varepsilon(t,\xi)+B_\varepsilon(t,\xi)\big)^2}{A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)C_\varepsilon(t,\xi)^2}&=\frac{C_\varepsilon(t,\xi)^2+4A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)}{A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)C_\varepsilon(t,\xi)^2}\\&=\frac{1}{A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)}+\frac{4}{C_\varepsilon(t,\xi)^2}\le\frac{1}{\varepsilon^\alpha|\xi|^4}+\frac{4}{\varepsilon^\alpha|\xi|^4}\\&=5\varepsilon^{-\alpha}|\xi|^{-4}.\end{aligned}$$

$\square$

With the modified coefficients $A_\varepsilon(t,\xi)$, $B_\varepsilon(t,\xi)$ we shall define the energy

$$\begin{aligned} E_\varepsilon(t,\xi)^2 &= e^{2\rho(t)\langle\xi\rangle_\nu^\kappa}\{|v_{ttt}+B_\varepsilon v_t|^2+|v_{ttt}+A_\varepsilon v_t|^2+A_\varepsilon|v_{tt}+B_\varepsilon v|^2 \\ &\qquad\qquad + B_\varepsilon|v_{tt}+A_\varepsilon v|^2\} \\ &\equiv e^{2\rho(t)\langle\xi\rangle_\nu^\kappa}\{|k_1|^2+|l_1|^2+A_\varepsilon|k_2|^2+B_\varepsilon|l_2|^2\} \\ &\equiv e^{2\rho(t)\langle\xi\rangle_\nu^\kappa}F_\varepsilon(t,\xi)^2, \end{aligned} \tag{22}$$

where $k_1 = v_{ttt}+B_\varepsilon v_t$, $l_1 = v_{ttt}+A_\varepsilon v_t$, $k_2 = v_{tt}+B_\varepsilon v$, $l_2 = v_{tt}+A_\varepsilon v$ and

$$F_\varepsilon(t,\xi)^2 = |k_1|^2+|l_1|^2+A_\varepsilon|k_2|^2+B_\varepsilon|l_2|^2 \tag{23}$$

and $\rho(t)$ is positive and determined later.

Paying attention to $F_\varepsilon(t,\xi)^2$ only, we can find from the following lemma that $F_\varepsilon(t,\xi)^2$ is bounded from below by $|v|^2$, $|v_t|^2$, $|v_{tt}|^2$ and $|v_{ttt}|^2$.

Lemma 2 *It holds that*

$$\begin{aligned} &|v|^2 \le \frac{A_\varepsilon+B_\varepsilon}{A_\varepsilon B_\varepsilon C_\varepsilon^2}F_\varepsilon^2, \quad |v_t|^2 \le \frac{2}{C_\varepsilon^2}F_\varepsilon^2, \\ &|v_{tt}|^2 \le \frac{A_\varepsilon+B_\varepsilon}{C_\varepsilon^2}F_\varepsilon^2, \quad |v_{ttt}|^2 \le \frac{A_\varepsilon^2+B_\varepsilon^2}{C_\varepsilon^2}F_\varepsilon^2. \end{aligned} \tag{24}$$

Moreover there exist $\delta>0$ and $L>0$ such that

$$\begin{aligned} &\delta\max\left\{\varepsilon^{\frac{3\alpha}{2}}|\xi|^6|v|^2,\ \varepsilon^\alpha|\xi|^4|v_t|^2,\ \varepsilon^\alpha|\xi|^2|v_{tt}|^2,\ \varepsilon^\alpha|v_{ttt}|^2\right\} \\ &\qquad \le F_\varepsilon(t,\xi)^2 \le L\big(|\xi|^6|v|^2+|\xi|^4|v_t|^2+|\xi|^2|v_{tt}|^2+|v_{ttt}|^2\big). \end{aligned} \tag{25}$$

Proof. Since $F_\varepsilon(t,\xi)^2 \ge |k_1|^2+|l_1|^2$, by (20) we get

$$\begin{aligned} F_\varepsilon(t,\xi)^2 &\ge \left|v_{ttt}+B_\varepsilon v_t\right|^2+\left|v_{ttt}+A_\varepsilon v_t\right|^2 \\ &= 2|v_{ttt}|^2+2(A_\varepsilon+B_\varepsilon)\Re(v_{ttt},\ v_t)+(A_\varepsilon^2+B_\varepsilon^2)|v_t|^2 \\ &= \begin{cases} \dfrac{(A_\varepsilon-B_\varepsilon)^2}{A_\varepsilon^2+B_\varepsilon^2}|v_{ttt}|^2+\left|\dfrac{A_\varepsilon+B_\varepsilon}{\sqrt{A_\varepsilon^2+B_\varepsilon^2}}\,v_{ttt}+\sqrt{A_\varepsilon^2+B_\varepsilon^2}\,v_t\right|^2 \\ \left|\sqrt{2}\,v_{ttt}+\dfrac{A_\varepsilon+B_\varepsilon}{\sqrt{2}}\,v_t\right|^2+\dfrac{(A_\varepsilon-B_\varepsilon)^2}{2}|v_t|^2 \end{cases} \end{aligned}$$

$$\geq \begin{cases} \dfrac{C_\varepsilon^2}{A_\varepsilon^2 + B_\varepsilon^2}|v_{ttt}|^2 \\ \dfrac{C_\varepsilon^2}{2}|v_t|^2 \end{cases} \geq \begin{cases} \delta\varepsilon^\alpha |v_{ttt}|^2 \\ \delta\varepsilon^\alpha |\xi|^4 |v_t|^2 \end{cases} \quad ({}^\exists\delta > 0).$$

Hence we also get the second and fourth inequalities in (24).

Similarly since $F_\varepsilon(t,\xi)^2 \geq A_\varepsilon|k_2|^2 + B_\varepsilon|l_2|^2$, by (20) we get

$$\begin{aligned} F_\varepsilon(t,\xi)^2 &\geq A_\varepsilon\big|v_{tt} + B_\varepsilon v\big|^2 + B_\varepsilon\big|v_{tt} + A_\varepsilon v\big|^2 \\ &= (A_\varepsilon + B_\varepsilon)|v_{tt}|^2 + 4A_\varepsilon B_\varepsilon \Re(v_{tt}, v) + A_\varepsilon B_\varepsilon (A_\varepsilon + B_\varepsilon)|v|^2 \\ &= \begin{cases} \dfrac{(A_\varepsilon - B_\varepsilon)^2}{A_\varepsilon + B_\varepsilon}|v_{tt}|^2 + \left|\sqrt{\dfrac{4A_\varepsilon B_\varepsilon}{A_\varepsilon + B_\varepsilon}}\, v_{tt} + \sqrt{A_\varepsilon B_\varepsilon(A_\varepsilon + B_\varepsilon)}\, v\right|^2 \\ \left|\sqrt{A_\varepsilon + B_\varepsilon}\, v_{tt} + \dfrac{2A_\varepsilon B_\varepsilon}{\sqrt{A_\varepsilon + B_\varepsilon}}\, v\right|^2 + \dfrac{A_\varepsilon B_\varepsilon (A_\varepsilon - B_\varepsilon)^2}{A_\varepsilon + B_\varepsilon}|v|^2 \end{cases} \\ &\geq \begin{cases} \dfrac{C_\varepsilon^2}{A_\varepsilon + B_\varepsilon}|v_{tt}|^2 \\ \dfrac{A_\varepsilon B_\varepsilon C_\varepsilon^2}{A_\varepsilon + B_\varepsilon}|v|^2 \end{cases} \geq \begin{cases} \delta\varepsilon^\alpha|\xi|^2|v_{tt}|^2 \\ \delta\varepsilon^{\frac{3\alpha}{2}}|\xi|^6|v|^2 \end{cases} \quad ({}^\exists\delta > 0), \end{aligned}$$

here we used by (20) and (21)

$$\frac{A_\varepsilon B_\varepsilon C_\varepsilon^2}{A_\varepsilon + B_\varepsilon} = (A_\varepsilon + B_\varepsilon)\left\{\frac{(A_\varepsilon + B_\varepsilon)^2}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right\}^{-1} \geq \delta\varepsilon^{\frac{3\alpha}{2}}|\xi|^6.$$

Hence we also get the first and third inequalities in (24).

While we can easily get the right side of (25) as follows.

$$\begin{aligned} F_\varepsilon(t,\xi)^2 &= |k_1|^2 + |l_1|^2 + A_\varepsilon|k_2|^2 + B_\varepsilon|l_2|^2 \\ &\leq 2|v_{ttt}|^2 + 2B_\varepsilon^2|v_t|^2 + 2|v_{ttt}|^2 + 2A_\varepsilon^2|v_t|^2 \\ &\qquad + 2A_\varepsilon|v_{tt}|^2 + 2A_\varepsilon B_\varepsilon^2|v|^2 + 2B_\varepsilon|v_{tt}|^2 + 2A_\varepsilon^2 B_\varepsilon|v|^2 \\ &\leq L\big(|\xi|^6|v|^2 + |\xi|^4|v_t|^2 + |\xi|^2|v_{tt}|^2 + |v_{ttt}|^2\big). \end{aligned}$$

□

Secondly differentiating $F_\varepsilon(t,\xi)^2$ in t, by (11) we have

$$\begin{aligned} &\big(F_\varepsilon(t,\xi)^2\big)' \\ &= 2\Re(v_{tttt} + B_\varepsilon v_{tt} + B_\varepsilon' v_t,\ k_1) + 2\Re(v_{tttt} + A_\varepsilon v_{tt} + A_\varepsilon' v_t,\ l_1) \end{aligned}$$

$$
\begin{aligned}
&+ A'_\varepsilon|k_2|^2 + B'_\varepsilon|l_2|^2 + 2A_\varepsilon\Re(v_{ttt} + B_\varepsilon v_t + B'_\varepsilon v,\ k_2) \\
&+ 2B_\varepsilon\Re(v_{ttt} + A_\varepsilon v_t + A'_\varepsilon v,\ l_2) \\
&= 2\Re\big(-(A+B)v_{tt} - ABv + B_\varepsilon v_{tt} + B'_\varepsilon v_t,\ k_1\big) \\
&+ 2\Re\big(-(A+B)v_{tt} - ABv + A_\varepsilon v_{tt} + A'_\varepsilon v_t,\ l_1\big) \\
&+ A'_\varepsilon|k_2|^2 + B'_\varepsilon|l_2|^2 + 2A_\varepsilon\Re(k_1,\ k_2) + 2A_\varepsilon\Re(B'_\varepsilon v,\ k_2) \\
&+ 2B_\varepsilon\Re(l_1,\ l_2) + 2B_\varepsilon\Re(A'_\varepsilon v,\ l_2) \\
&= 2\Re\big(-(A+B)v_{tt} - ABv + B_\varepsilon v_{tt} + A_\varepsilon v_{tt} + A_\varepsilon B_\varepsilon v,\ k_1\big) + 2\Re(B'_\varepsilon v_t,\ k_1) \\
&+ 2\Re\big(-(A+B)v_{tt} - ABv + A_\varepsilon v_{tt} + B_\varepsilon v_{tt} + A_\varepsilon B_\varepsilon v,\ l_1\big) \\
&+ 2\Re(A'_\varepsilon v_t,\ l_1) + A'_\varepsilon|k_2|^2 + B'_\varepsilon|l_2|^2 + 2A_\varepsilon\Re(B'_\varepsilon v,\ k_2) + 2B_\varepsilon\Re(A'_\varepsilon v,\ l_2) \\
&= 2\big\{(A_\varepsilon + B_\varepsilon) - (A+B)\big\}\Re(v_{tt},\ k_1 + l_1) + 2(A_\varepsilon B_\varepsilon - AB)\Re(v,\ k_1 + l_1) \\
&+ 2\big\{B'_\varepsilon\Re(v_t,\ k_1) + A'_\varepsilon\Re(v_t,\ l_1)\big\} + \big\{A'_\varepsilon|k_2|^2 + B'_\varepsilon|l_2|^2\big\} \\
&+ 2\big\{A_\varepsilon B'_\varepsilon\Re(v,\ k_2) + A'_\varepsilon B_\varepsilon\Re(v,\ l_2)\big\} \\
&\equiv I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned}
$$

here we used $2A_\varepsilon\Re(k_1,\ k_2) = 2A_\varepsilon\Re(k_2,\ k_1) = 2\Re(A_\varepsilon v_{tt} + A_\varepsilon B_\varepsilon v,\ k_1)$ and $2B_\varepsilon\Re(l_1,\ l_2) = 2B_\varepsilon\Re(l_2,\ l_1) = 2\Re(B_\varepsilon v_{tt} + A_\varepsilon B_\varepsilon v,\ k_1)$.

We shall pick up each term I_k $(k = 0, 1, \dots, 5)$ in order to estimate $\big(F_\varepsilon(t,\xi)^2\big)'$.

In the following C and C' will denote the constants not depending ε, possibly having different values in different lines.

(i) Estimate for I_1

$$
\begin{aligned}
I_1 &= 2\big\{(A_\varepsilon + B_\varepsilon) - (A+B)\big\}\Re\big(\varepsilon^\eta v_{tt},\ \varepsilon^{-\eta}(k_1 + l_1)\big) \\
&\le \big|(A_\varepsilon + B_\varepsilon) - (A+B)\big|\varepsilon^{2\eta}|v_{tt}|^2 \\
&\qquad + \big|(A_\varepsilon + B_\varepsilon) - (A+B)\big|\varepsilon^{-2\eta}|k_1 + l_1|^2 \\
&\le \big|(A_\varepsilon + B_\varepsilon) - (A+B)\big|\varepsilon^{2\eta}\frac{(A_\varepsilon + B_\varepsilon) + (A+B)}{C_\varepsilon^2}F_\varepsilon^2 \\
&\qquad + 2\big|(A_\varepsilon + B_\varepsilon) - (A+B)\big|\varepsilon^{-2\eta}\frac{(A_\varepsilon + B_\varepsilon) + (A+B)}{\sqrt{5}\varepsilon^{\frac{\alpha}{2}}|\xi|^2}F_\varepsilon^2 \\
&= \big|(A_\varepsilon + B_\varepsilon)^2 - (A+B)^2\big|\frac{\varepsilon^{2\eta}}{C_\varepsilon^2}F_\varepsilon^2 \\
&\qquad + \frac{2}{\sqrt{5}}\big|(A_\varepsilon + B_\varepsilon)^2 - (A+B)^2\big|\varepsilon^{-2\eta-\frac{\alpha}{2}}|\xi|^{-2}F_\varepsilon^2,
\end{aligned}
$$

here we used by (24)

$$|v_{tt}|^2 \le \frac{A_\varepsilon + B_\varepsilon}{C_\varepsilon^2} F_\varepsilon^2 \le \frac{(A_\varepsilon + B_\varepsilon) + (A+B)}{C_\varepsilon^2} F_\varepsilon^2,$$

$$|k_1 + l_1|^2 \le 2|k_1|^2 + 2|l_1|^2 \le 2F_\varepsilon^2 \le 2\frac{(A_\varepsilon + B_\varepsilon) + (A+B)}{\sqrt{5}\varepsilon^{\frac{\alpha}{2}}|\xi|^2} F_\varepsilon^2.$$

Taking $\eta = -\frac{1}{2}$, by (17) and (20) we get

$$I_1 \le M\varepsilon^{2\eta} F_\varepsilon^2 + \frac{2}{\sqrt{5}} M \varepsilon^{\frac{\alpha}{2}-2\eta}|\xi|^2 F_\varepsilon^2 = C\varepsilon^{-1}F_\varepsilon^2 + C'\varepsilon^{\frac{\alpha}{2}+1}|\xi|^2 F_\varepsilon^2. \tag{26}$$

(ii) Estimate for I_2

Taking $\zeta = \frac{\alpha}{4} - \frac{1}{2}$, by (17) and the left side of (25) we get

$$\begin{aligned} I_2 &= 2\big(A_\varepsilon B_\varepsilon - AB\big)\Re\big(\varepsilon^\zeta |\xi| v,\ \varepsilon^{-\zeta}|\xi|^{-1}(k_1 + l_1)\big) \\ &\le \big|A_\varepsilon B_\varepsilon - AB\big|\varepsilon^{2\zeta}|\xi|^2|v|^2 + \big|A_\varepsilon B_\varepsilon - AB\big|\varepsilon^{-2\zeta}|\xi|^{-2}|k_1 + l_1|^2 \\ &\le \big|A_\varepsilon B_\varepsilon - AB\big|\delta^{-1}\varepsilon^{2\zeta - \frac{3\alpha}{2}}|\xi|^{-4}F_\varepsilon^2 + 2\big|A_\varepsilon B_\varepsilon - AB\big|\varepsilon^{-2\zeta}|\xi|^{-2}F_\varepsilon^2 \\ &\le M\delta^{-1}\varepsilon^{2\zeta - \frac{\alpha}{2}}F_\varepsilon^2 + 2M\varepsilon^{\alpha - 2\zeta}|\xi|^2 F_\varepsilon^2 \\ &= C\varepsilon^{-1}F_\varepsilon^2 + C'\varepsilon^{\frac{\alpha}{2}+1}|\xi|^2 F_\varepsilon^2. \end{aligned} \tag{27}$$

(iii) Estimate for I_3

$$\begin{aligned} I_3 &= 2\big\{B_\varepsilon' \Re(v_t,\ v_{ttt} + B_\varepsilon v_t) + A_\varepsilon' \Re(v_t,\ v_{ttt} + A_\varepsilon v)\big\} \\ &= 2(A_\varepsilon + B_\varepsilon)'\Re(v_t,\ v_{ttt}) + \big\{(A_\varepsilon^2)' + (B_\varepsilon^2)'\big\}|v_t|^2 \\ &= 2(A_\varepsilon + B_\varepsilon)'\Re\bigg(\Big(\frac{2}{C_\varepsilon^2}\Big)^{-\frac{1}{4}}\Big(\frac{A_\varepsilon^2 + B_\varepsilon^2}{C_\varepsilon^2}\Big)^{\frac{1}{4}} v_t, \\ &\qquad\qquad \Big(\frac{2}{C_\varepsilon^2}\Big)^{\frac{1}{4}}\Big(\frac{A_\varepsilon^2 + B_\varepsilon^2}{C_\varepsilon^2}\Big)^{-\frac{1}{4}} v_{ttt}\bigg) \\ &\qquad + (A_\varepsilon^2 + 2A_\varepsilon B_\varepsilon + B_\varepsilon^2)'|v_t|^2 - 2(A_\varepsilon B_\varepsilon)'|v_t|^2 \\ &\le \big|(A_\varepsilon + B_\varepsilon)'\big|\Big(\frac{2}{C_\varepsilon^2}\Big)^{\frac{1}{2}}\Big(\frac{A_\varepsilon^2 + B_\varepsilon^2}{C_\varepsilon^2}\Big)^{\frac{1}{2}} \\ &\qquad \times \bigg\{\Big(\frac{2}{C_\varepsilon^2}\Big)^{-1}|v_t|^2 + \Big(\frac{A_\varepsilon^2 + B_\varepsilon^2}{C_\varepsilon^2}\Big)^{-1}|v_{ttt}|^2\bigg\} \\ &\qquad + \big|\{(A_\varepsilon + B_\varepsilon)^2\}'\big||v_t|^2 + 2\big|(A_\varepsilon B_\varepsilon)'\big||v_t|^2. \end{aligned}$$

Noting that

$$\begin{aligned}\left|(A_\varepsilon+B_\varepsilon)'\right|(A_\varepsilon^2+B_\varepsilon^2)^{\frac12} &\le \left|(A_\varepsilon+B_\varepsilon)'\right|(A_\varepsilon+B_\varepsilon)\\ &=\frac12\left|\{(A_\varepsilon+B_\varepsilon)^2\}'\right|,\end{aligned}$$

by (19), (20) and (24) we get

$$\begin{aligned}I_3 &\le \left|(A_\varepsilon+B_\varepsilon)'\right|\frac{2\sqrt2(A_\varepsilon^2+B_\varepsilon^2)^{\frac12}}{C_\varepsilon^2}F_\varepsilon^2\\ &\quad+\left|\{(A_\varepsilon+B_\varepsilon)^2\}'\right|\frac{2}{C_\varepsilon^2}F_\varepsilon^2+2\left|(A_\varepsilon B_\varepsilon)'\right|\frac{2}{C_\varepsilon^2}F_\varepsilon^2\\ &\le \left|\{(A_\varepsilon+B_\varepsilon)^2\}'\right|\frac{\sqrt2+2}{C_\varepsilon^2}F_\varepsilon^2+2\left|(A_\varepsilon B_\varepsilon)'\right|\frac{2}{C_\varepsilon^2}F_\varepsilon^2\\ &\le (\sqrt2+2)M\varepsilon^{-1}F_\varepsilon^2+4M\varepsilon^{-1}F_\varepsilon^2\\ &= C\varepsilon^{-1}F_\varepsilon^2.\end{aligned}\tag{28}$$

(iv) Estimate for I_4

$$\begin{aligned}I_4 &= A_\varepsilon'|v_{tt}+B_\varepsilon v|^2+B_\varepsilon'|v_{tt}+A_\varepsilon v|^2\\ &=(A_\varepsilon+B_\varepsilon)'|v_{tt}|^2+(A_\varepsilon'B_\varepsilon^2+A_\varepsilon^2B_\varepsilon')|v|^2\\ &\quad+2(A_\varepsilon'B_\varepsilon+A_\varepsilon B_\varepsilon')\Re(v,\ v_{tt})\\ &\le \left|(A_\varepsilon+B_\varepsilon)'\right||v_{tt}|^2+\left|\frac{A_\varepsilon'B_\varepsilon}{A_\varepsilon}+\frac{A_\varepsilon B_\varepsilon'}{B_\varepsilon}\right|A_\varepsilon B_\varepsilon|v|^2\\ &\quad+2\left|(A_\varepsilon B_\varepsilon)'\Re\left(\left(\frac{A_\varepsilon+B_\varepsilon}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right)^{-\frac14}\left(\frac{A_\varepsilon+B_\varepsilon}{C_\varepsilon^2}\right)^{\frac14}v,\right.\right.\\ &\qquad\qquad\left.\left.\left(\frac{A_\varepsilon+B_\varepsilon}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right)^{\frac14}\left(\frac{A_\varepsilon+B_\varepsilon}{C_\varepsilon^2}\right)^{-\frac14}v_{tt}\right)\right|\\ &\le \left|\{(A_\varepsilon+B_\varepsilon)^2\}'\right|\frac{1}{2(A_\varepsilon+B_\varepsilon)}|v_{tt}|^2\\ &\quad+\left|\left(\frac{A_\varepsilon'}{A_\varepsilon}+\frac{B_\varepsilon'}{B_\varepsilon}\right)(A_\varepsilon+B_\varepsilon)-(A_\varepsilon+B_\varepsilon)'\right|A_\varepsilon B_\varepsilon|v|^2\\ &\quad+2\left|(A_\varepsilon B_\varepsilon)'\right|\left(\frac{A_\varepsilon+B_\varepsilon}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right)^{\frac12}\left(\frac{A_\varepsilon+B_\varepsilon}{C_\varepsilon^2}\right)^{\frac12}\\ &\qquad\times\left\{\left(\frac{A_\varepsilon+B_\varepsilon}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right)^{-1}|v|^2+\left(\frac{A_\varepsilon+B_\varepsilon}{C_\varepsilon^2}\right)^{-1}|v_{tt}|^2\right\}.\end{aligned}$$

Noting that $\frac{A_\varepsilon'}{A_\varepsilon}+\frac{B_\varepsilon'}{B_\varepsilon}=\frac{(A_\varepsilon B_\varepsilon)'}{A_\varepsilon B_\varepsilon}$, by (19)–(21) and (24) we get

$$\begin{aligned}
I_4 &\le \big|\{(A_\varepsilon+B_\varepsilon)^2\}'\big|\frac{1}{2C_\varepsilon^2}F_\varepsilon^2+\big|(A_\varepsilon B_\varepsilon)'\big|\frac{(A_\varepsilon+B_\varepsilon)^2}{A_\varepsilon B_\varepsilon C_\varepsilon^2}F_\varepsilon^2\\
&\quad+\big|\{(A_\varepsilon+B_\varepsilon)^2\}'\big|\frac{1}{2C_\varepsilon^2}F_\varepsilon^2+4\big|(A_\varepsilon B_\varepsilon)'\big|\frac{1}{C_\varepsilon}\left\{\frac{(A_\varepsilon+B_\varepsilon)^2}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right\}^{\frac12}F_\varepsilon^2\\
&\le \frac{M}{2}\varepsilon^{-1}F_\varepsilon^2+5M\varepsilon^{-1}F_\varepsilon^2+\frac{M}{2}\varepsilon^{-1}F_\varepsilon^2+2\sqrt{5}M\varepsilon^{-1}F_\varepsilon^2\\
&= C\varepsilon^{-1}F_\varepsilon^2. \qquad (29)
\end{aligned}$$

(v) Estimate for I_5

By (19)–(21) and (24) we get

$$\begin{aligned}
I_5 &= 2\big\{A_\varepsilon B_\varepsilon'\Re(v,\ v_{tt}+B_\varepsilon v)+A_\varepsilon' B_\varepsilon\Re(v,\ v_{tt}+A_\varepsilon v)\big\}\\
&= 2(A_\varepsilon B_\varepsilon)'\Re(v,\ v_{tt})+2A_\varepsilon B_\varepsilon(A_\varepsilon+B_\varepsilon)'|v|^2\\
&= 2(A_\varepsilon B_\varepsilon)'\Re\Bigg(\left(\frac{A_\varepsilon+B_\varepsilon}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right)^{-\frac14}\left(\frac{A_\varepsilon+B_\varepsilon}{C_\varepsilon^2}\right)^{\frac14}v,\\
&\qquad\qquad \left(\frac{A_\varepsilon+B_\varepsilon}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right)^{\frac14}\left(\frac{A_\varepsilon+B_\varepsilon}{C_\varepsilon^2}\right)^{-\frac14}v_{tt}\Bigg)\\
&\qquad + 2A_\varepsilon B_\varepsilon\big|(A_\varepsilon+B_\varepsilon)'\big||v|^2\\
&\le \big|(A_\varepsilon B_\varepsilon)'\big|\left(\frac{Ae+Be}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right)^{\frac12}\left(\frac{A_\varepsilon+B_\varepsilon}{C_\varepsilon^2}\right)^{\frac12}\\
&\qquad\times\left\{\left(\frac{A_\varepsilon+B_\varepsilon}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right)^{-1}|v|^2+\left(\frac{A_\varepsilon+B_\varepsilon}{C_\varepsilon^2}\right)^{-1}|v_{tt}|^2\right\}\\
&\qquad + 2\big|(A_\varepsilon+B_\varepsilon)'\big|\left(\frac{A_\varepsilon+B_\varepsilon}{C_\varepsilon^2}\right)\left(\frac{A_\varepsilon+B_\varepsilon}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right)^{-1}|v|^2.\\
&\le \big|(A_\varepsilon B_\varepsilon)'\big|\left\{\frac{(A_\varepsilon+B_\varepsilon)^2}{A_\varepsilon B_\varepsilon C_\varepsilon^2}\right\}^{\frac12}\frac{2}{C_\varepsilon}F_\varepsilon^2+\frac{\big|\{(A_\varepsilon+B_\varepsilon)^2\}'\big|}{C_\varepsilon^2}F_\varepsilon^2\\
&\le 2\sqrt{5}M\varepsilon^{-1}F_\varepsilon^2+M\varepsilon^{-1}F_\varepsilon^2=C\varepsilon^{-1}F_\varepsilon^2. \qquad (30)
\end{aligned}$$

Thus, summing up the right sides of (26)–(30), consequently we have

$$\big(F_\varepsilon(t,\xi)^2\big)'\le C\varepsilon^{-1}F_\varepsilon^2+C'\varepsilon^{\frac{\alpha}{2}+1}|\xi|^2F_\varepsilon^2\ \big(\le C\varepsilon^{-1}F_\varepsilon^2+C'\varepsilon^{\frac{\alpha}{2}+1}\langle\xi\rangle_\nu^2F_\varepsilon^2\big).$$

Moreover choosing $\varepsilon=\langle\xi\rangle_\nu^{-\frac{1}{1+\frac{\alpha}{4}}}$, we can get $\big(F_\varepsilon(t,\xi)^2\big)'\le C\langle\xi\rangle_\nu^{\frac{1}{1+\frac{\alpha}{4}}}F_\varepsilon(t,\xi)^2$.

Hence we can also get

$$\begin{aligned}\big(E_\varepsilon(t,\xi)^2\big)' &= 2\rho'(t)e^{2\rho(t)\langle\xi\rangle_\nu^\kappa}\langle\xi\rangle_\nu^\kappa F_\varepsilon(t,\xi)^2 + e^{2\rho(t)\langle\xi\rangle_\nu^\kappa}\big(F_\varepsilon(t,\xi)^2\big)' \\ &\le 2\rho'(t)e^{2\rho(t)\langle\xi\rangle_\nu^\kappa}\langle\xi\rangle_\nu^\kappa F_\varepsilon(t,\xi)^2 + Ce^{2\rho(t)\langle\xi\rangle_\nu^\kappa}\langle\xi\rangle_\nu^{\frac{1}{1+\frac{\alpha}{4}}} F_\varepsilon(t,\xi)^2.\end{aligned}$$

From the assumption (4) we find that $\frac{1}{1+\frac{\alpha}{4}} < \frac{1}{s} \le 1$. Therefore putting $\kappa = \frac{1}{s}$ and noting that $\langle\xi\rangle_\nu^{\frac{1}{1+\frac{\alpha}{4}}} = \langle\xi\rangle_\nu^{\frac{1}{1+\frac{\alpha}{4}}-\frac{1}{s}}\langle\xi\rangle_\nu^{\frac{1}{s}} \le \nu^{\frac{1}{1+\frac{\alpha}{4}}-\frac{1}{s}}\langle\xi\rangle_\nu^{\frac{1}{s}}$, we obtain

$$\big(E_\varepsilon(t,\xi)^2\big)' \le \Big(2\rho'(t) + C\nu^{\frac{1}{1+\frac{\alpha}{4}}-\frac{1}{s}}\Big)e^{2\rho(t)\langle\xi\rangle_\nu^\kappa}\langle\xi\rangle_\nu^{\frac{1}{s}} F_\varepsilon(t,\xi)^2. \tag{31}$$

At last, taking large enough $\nu = \nu(T) > 0$ to a given positive T such that $\rho_0 - \frac{T}{2}C\nu^{\frac{1}{1+\frac{\alpha}{4}}-\frac{1}{s}} > 0$, we define the positive function $\rho(t) = \rho_0 - \frac{t}{2}C\nu^{\frac{1}{1+\frac{\alpha}{4}}-\frac{1}{s}}$. Then by (31) we have $\big(E_\varepsilon(t,\xi)^2\big)' \le 0$ which gives the energy inequality

$$E_\varepsilon(t,\xi)^2 \le E_\varepsilon(0,\xi)^2 \quad \text{for } \ t \in [0,T]. \tag{32}$$

By (25) and (32) we also have the following energy inequality based on v, v_t, v_{tt} and v_{ttt}.

$$\begin{aligned}&\delta e^{2\rho(t)\langle\xi\rangle_\nu^{\frac{1}{s}}}\max\{\varepsilon^{\frac{3\alpha}{2}}|\xi|^6|v|^2,\ \varepsilon^\alpha|\xi|^4|v_t|^2,\ \varepsilon^\alpha|\xi|^2|v_{tt}|^2,\ \varepsilon^\alpha|v_{ttt}|^2\} \\ &\qquad\le L\big(e^{2\rho_0\langle\xi\rangle_\nu^{\frac{1}{s}}}|\xi|^6|v_0|^2 + |\xi|^4|v_1|^2 + |\xi|^2|v_2|^2 + |v_3|^2\big).\end{aligned} \tag{33}$$

Furthermore there exist $C_\nu > 0$ and $\delta_\nu > 0$ such that

$$e^{\langle\xi\rangle^{\frac{1}{s}}} \le e^{\langle\xi\rangle_\nu^{\frac{1}{s}}} = e^{\{(|\xi|^2+1)+(\nu^2-1)\}^{\frac{1}{2s}}} \le e^{(|\xi|^2+1)^{\frac{1}{2s}}+(\nu^2-1)^{\frac{1}{2s}}} = C_\nu e^{\langle\xi\rangle^{\frac{1}{s}}}$$

and

$$\begin{aligned}\varepsilon = \langle\xi\rangle_\nu^{-\frac{1}{1+\frac{\alpha}{4}}} &= \{\langle\xi\rangle_\nu^{-2}\}^{\frac{1}{2+\frac{\alpha}{2}}} = \left\{\frac{|\xi|^2+1}{|\xi|^2+\nu^2}\right\}^{\frac{1}{2+\frac{\alpha}{2}}}\{\langle\xi\rangle^{-2}\}^{\frac{1}{2+\frac{\alpha}{2}}} \\ &\ge \delta_\nu\langle\xi\rangle^{-\frac{1}{1+\frac{\alpha}{4}}}.\end{aligned}$$

Hence we can change (33) into

$$\sum_{h=0}^{3} e^{2\rho(t)\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{2q_h}|\partial_t^h v|^2 \le C\sum_{h=0}^{3} e^{2\rho_0\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{2(3-h)}|v_h|^2, \tag{34}$$

where $q_0 = \frac{12}{4+\alpha}$, $q_1 = \frac{8}{4+\alpha}$, $q_2 = \frac{4-\alpha}{4+\alpha}$, $q_3 = \frac{-2\alpha}{4+\alpha}$, $q_4 = \frac{-4-4\alpha}{4+\alpha}$. Finally we remark that the information for $\partial_t^4 \hat{u}$ in $(5)_{\mathrm{I}}$ follows immediately from the equation (11) and obtain $(5)_{\mathrm{I}}$. If we integrate both sides of (34) over $\mathbf{R}_\xi^n$, we also get

$$\sum_{h=0}^{4} \left\| e^{\rho(t)\langle\xi\rangle^{\frac{1}{s}}} \langle\xi\rangle^{q_h} \partial_t^h v \right\| \le C \sum_{h=0}^{3} \left\| e^{\rho_0\langle\xi\rangle^{\frac{1}{s}}} \langle\xi\rangle^{3-h} v_h \right\|, \tag{35}$$

where $\|\cdot\|$ denotes L^2-norm.

To prove the existence and uniqueness, the energy inequality plays an important role. Following the argument in L^2 of [14], we shall give a rapid sketch of the proof. For the simple notation we suppose that the spatial dimension $n = 1$, and put $U = e^{\rho(t)\langle D\rangle^{\frac{1}{s}}} \cdot {}^t(u, \partial_t u, \partial_t^2 u, \partial_t^3 u)$ and $U_0 = e^{\rho_0\langle D\rangle^{\frac{1}{s}}} \cdot {}^t(u_0, u_1, u_2, u_3)$. Then the Cauchy problem (1) can be written in the following form in $[0,T] \times \mathbf{R}_x^1$:

$$\begin{cases} \partial_t U = P(t,D)U, \\ U(0,x) = U_0(x), \end{cases}$$

$$\text{where } P(t,D_x) = \begin{pmatrix} \rho'(t)\langle D\rangle^{\frac{1}{s}} & 1 & 0 & 0 \\ 0 & \rho'(t)\langle D\rangle^{\frac{1}{s}} & 1 & 0 \\ 0 & 0 & \rho'(t)\langle D\rangle^{\frac{1}{s}} & 1 \\ -b_4(t)D^4 & 0 & -a_2(t)D^2 & \rho'(t)\langle D\rangle^{\frac{1}{s}} \end{pmatrix}. \tag{36}$$

Now we approximate the differential operator $P(t,D)$ by a sequence of operators $P_k(t,D) = P(t,\zeta_k(D))$, where $\zeta_k(\xi) = k\sin(\frac{\xi}{k})$ $(k \ge 1)$ satisfies $|\zeta_k(\xi)| \le |\xi|$ and $\zeta_k(\xi) \to \xi$ $(k \to \infty)$ uniformly on any compact set. We shall find the solution $U(t)$ of (36) as the weak limit of the solutions $U_k(t)$ $\left(= e^{\rho(t)\langle\zeta_k(D)\rangle^{\frac{1}{s}}} \cdot {}^t(u, \partial_t u, \partial_t^2 u, \partial_t^3 u)\right)$ of the Cauchy problem

$$\begin{cases} \partial_t U_k = P_k(t,D)U_k, \\ U_k(0,x) = e^{\rho_0\langle\zeta_k(D)\rangle^{\frac{1}{s}}} \cdot {}^t(u_0, u_1, u_2, u_3). \end{cases} \qquad (k \ge 1)$$

Since $P_k(t,D)$ is a bounded operator for any fixed $k \ge 1$, $U_k(t)$ is uniquely determined by successive approximations as the solution of the integral equation

$$U_k(t) = U_0 + \int_0^t P_k(\tau)U_k(\tau)d\tau. \tag{37}$$

We can replace ξ by $\zeta_k(\xi)$ in (35) with $C > 0$ independent of k. Therefore there exist $C > 0$ and $C' > 0$ independent of k such that

$$\|\Lambda_1(\zeta_k(\xi))\hat{U}_k(t)\| \le C\|\Lambda_0(\zeta_k(\xi))\hat{U}_k(0)\| \le C\|\Lambda_0(\xi)\hat{U}_0\| \tag{38}$$

$$\begin{aligned}&\|\Lambda_2(\zeta_k(\xi))\hat{U}_k(t) - \Lambda_2(\zeta_k(\xi))\hat{U}_k(t')\| \\ &\le \left\|\int_{t'}^{t} \partial_\tau\{\Lambda_2(\zeta_k(\xi))\hat{U}_k(\tau)\}d\tau\right\| \le C'|t-t'|\|\Lambda_0(\zeta_k(\xi))\hat{U}_k(0)\| \\ &\le C'|t-t'|\|\Lambda_0(\xi)\hat{U}_0\|,\end{aligned} \tag{39}$$

where $\Lambda_0(\xi) = \mathrm{diag}\{\langle\xi\rangle^3, \langle\xi\rangle^2, \langle\xi\rangle, 1\}, \Lambda_1(\xi) = \mathrm{diag}\{\langle\xi\rangle^{q_0}, \langle\xi\rangle^{q_1}, \langle\xi\rangle^{q_2}, \langle\xi\rangle^{q_3}\}$, $\Lambda_2(\xi) = \mathrm{diag}\{\langle\xi\rangle^{q_1}, \langle\xi\rangle^{q_2}, \langle\xi\rangle^{q_3}, \langle\xi\rangle^{q_4}\}$. Then, using the Ascoli-Arzela theorem, by (38) and (39) we find that the sequence $\{\Lambda_1(\zeta_k(\xi))\hat{U}_k(t)\}_{k=1}^{\infty}$ is bounded in L_2 and has the weak limit $\Lambda_1(\xi)\hat{U}(t)$ which also satisfies

$$\|\Lambda_2(\xi)\hat{U}(t) - \Lambda_2(\xi)\hat{U}(t')\| \le C|t-t'|\|\Lambda_0(\xi)\hat{U}_0\|^2.$$

Considering $\int_0^T \int_{\mathbf{R}} U_k(t,x)\psi(t,x)dt\,dx$ for $\psi(t,x) = \psi_1(t)\psi_2(x) \in C_0^\infty\big((0,T)\times\mathbf{R}\big)$, we see that the limit of (37) is $U(t) = U_0 + \int_0^t P(\tau)U(\tau)d\tau$ and $U(t) \in C^1\big([0,T], L^2(\mathbf{R})\big)$. This means that $u \in C^4\big([0,T], \gamma^s(\mathbf{R})\big)$.

2.2. Case of $(13)_{\mathrm{II}}$ (Theorem 2)

We next treat the case of $(13)_{\mathrm{II}}$ which implies that the multiplicity of the characteristic roots is at most two. In this case the definitions of the regularized coefficients A_ε and B_ε are quite same as the previous case.

But paying attention to $(13)_{\mathrm{II}}$, we find that (20) in Lemma 1 becomes

$$\big(A_\varepsilon(t,\xi) + B_\varepsilon(t,\xi)\big)^2 \ge \delta_1^2|\xi|^4, \quad A_\varepsilon(t,\xi)B_\varepsilon(t,\xi) \ge \varepsilon^\alpha|\xi|^4$$
$$C_\varepsilon(t,\xi)^2 \ge \delta_1^2|\xi|^4,$$

and (25) in Lemma 2 becomes

$$\begin{aligned}&\delta\max\{\varepsilon^\alpha|\xi|^6|v|^2,\ |\xi|^4|v_t|^2,\ |\xi|^2|v_{tt}|^2,\ |v_{ttt}|^2\} \\ &\qquad \le F_\varepsilon(t,\xi)^2 \le L\big(|\xi|^6|v|^2 + |\xi|^4|v_t|^2 + |\xi|^2|v_{tt}|^2 + |v_{ttt}|^2\big).\end{aligned} \tag{40}$$

Hence taking $\eta = -\frac{\alpha}{2} - \frac{1}{2}$ and $\zeta = -\frac{1}{2}$, we get the followings instead of (26) and (27).

$$
\begin{aligned}
I_1 &\le M\varepsilon^{\alpha+2\eta}F_\varepsilon^2 + \frac{2}{\sqrt{5}}M\varepsilon^{\alpha-2\eta}|\xi|^2F_\varepsilon^2 = M\varepsilon^{-1}F_\varepsilon^2 + \frac{2}{\sqrt{5}}M\varepsilon^{2\alpha+1}|\xi|^2F_\varepsilon^2 \\
&\le C\varepsilon^{-1}F_\varepsilon^2 + C'\varepsilon^{\alpha+1}|\xi|^2F_\varepsilon^2, \\
I_2 &\le M\delta^{-1}\varepsilon^{2\zeta}F_\varepsilon^2 + 2M\varepsilon^{\alpha-2\zeta}|\xi|^2F_\varepsilon^2 = C\varepsilon^{-1}F_\varepsilon^2 + C'\varepsilon^{\alpha+1}|\xi|^2F_\varepsilon^2. \quad (41)
\end{aligned}
$$

Consequently we have $\big(F_\varepsilon(t,\xi)^2\big)' \le C\varepsilon^{-1}F_\varepsilon(t,\xi)^2 + C'\varepsilon^{\alpha+1}\langle\xi\rangle_\nu^2 F_\varepsilon(t,\xi)^2$. Similarly choosing $\varepsilon = \langle\xi\rangle_\nu^{-\frac{1}{1+\frac{\alpha}{2}}}$ and using the assumption (6), we also have the energy inequality $E_\varepsilon(t,\xi)^2 \le E_\varepsilon(0,\xi)^2$ for $t \in [0,T]$. Finally by (40) we have the following energy inequality based on $\partial_t^h v$ $(h = 0,1,2,3,4)$.

$$
\sum_{h=0}^{4} e^{2\rho(t)\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{2q_h}|\partial_t^h v|^2 \le C\sum_{h=0}^{3} e^{2\rho_0\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{2(3-h)}|v_h|^2,
$$

where $q_0 = \frac{6+2\alpha}{2+\alpha}$, $q_1 = 2$, $q_2 = 1$, $q_3 = 0$, $q_4 = \frac{-2-2\alpha}{2+\alpha}$. This is equivalent to $(7)_{\mathrm{II}}$.

2.3. Case of $(14)_{\mathrm{III}}$ (Theorem 2)

We next treat the case of $(14)_{\mathrm{III}}$ which also implies that the multiplicity of the characteristic roots is at most two. But in this case the definitions of the regularized coefficients A_ε and B_ε are replaced by $A_\varepsilon(t,\xi) = \max\{\Lambda_{1\varepsilon}(t,\xi),\ \Lambda_{2\varepsilon}(t,\xi)\}$ and $B_\varepsilon(t,\xi) = \min\{\Lambda_{1\varepsilon}(t,\xi),\ \Lambda_{2\varepsilon}(t,\xi)\}$, where $\Lambda_{1\varepsilon}(t,\xi)$, $\Lambda_{2\varepsilon}(t,\xi)$ are two roots of the algebraic equation $\Lambda^2 - \tilde{H}_\varepsilon(t,\xi)\Lambda + G_\varepsilon(t,\xi) = 0$ with

$$
\begin{aligned}
&\tilde{H}_\varepsilon(t,\xi)\ \big(= A_\varepsilon(t,\xi) + B_\varepsilon(t,\xi)\big) \\
&\quad = \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\big(A(\tau,\xi) + B(\tau,\xi)\big)\varphi\Big(\frac{t-\tau}{\varepsilon}\Big)d\tau \qquad (42) \\
&G_\varepsilon(t,\xi)\ \big(= A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)\big) \\
&\quad = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} A(\tau,\xi)B(\tau,\xi)\varphi\Big(\frac{t-\tau}{\varepsilon}\Big)d\tau - K\varepsilon^\alpha|\xi|^4, \\
&\qquad\qquad (0 < \varepsilon < 1,\ K > 0). \qquad (43)
\end{aligned}
$$

We can use the condition (3) instead of the condition (2) (see Remark 4). Then by (3) there exists $M > 0$ such that for $\xi \in \mathbf{R}_\xi^n$

$$
\Big|\big(A_\varepsilon(t,\xi) + B_\varepsilon(t,\xi)\big) - \big(A(t,\xi) + B(t,\xi)\big)\Big| \le M\varepsilon^\alpha|\xi|^2, \qquad (44)
$$

$$\left|A_\varepsilon(t,\xi)B_\varepsilon(t,\xi) - A(t,\xi)B(t,\xi)\right| \le M\varepsilon^\alpha|\xi|^4, \tag{45}$$

$$\begin{aligned}&\left|\frac{\partial}{\partial t}\big(A_\varepsilon(t,\xi) + B_\varepsilon(t,\xi)\big)\right| \le M\varepsilon^{\alpha-1}|\xi|^2,\\ &\left|\frac{\partial}{\partial t}\big(A_\varepsilon(t,\xi)B_\varepsilon(t,\xi)\big)\right| \le M\varepsilon^{\alpha-1}|\xi|^4,\end{aligned} \tag{46}$$

and we get the following lemma.

Lemma 3 *Let* $\varepsilon = \langle\xi\rangle_\nu^{-\frac{1}{1+\frac{\alpha}{2}}}$. *Then there exist* $\nu > 0$ *and* $K > 0$ *such that for* $\xi \in \mathbf{R}^n_\xi$

$$\begin{gathered}A_\varepsilon(t,\xi) + B_\varepsilon(t,\xi) \ge 2\delta_1|\xi|^2, \quad A_\varepsilon(t,\xi)B_\varepsilon(t,\xi) \ge \frac{\delta_1^2}{2}|\xi|^4\\ C_\varepsilon(t,\xi)^2 \ge \varepsilon^\alpha|\xi|^4,\end{gathered} \tag{47}$$

where $\delta_1 > 0$ *is given by* $(14)_{\mathrm{III}}$.

Proof. Noting (44) and that
$\left|\big(A(t)+B(t)\big)^2 - \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\big(A(\tau)+B(\tau)\big)^2\varphi\big(\frac{t-\tau}{\varepsilon}\big)d\tau\right| \le C\varepsilon^\alpha|\xi|^4$, we find that

$$\begin{aligned}C_\varepsilon(t,\xi)^2 &= (A_\varepsilon + B_\varepsilon)^2 - 4A_\varepsilon B_\varepsilon\\ &= (A_\varepsilon + B_\varepsilon)^2 - (A_\varepsilon + B_\varepsilon)(A+B) + (A_\varepsilon + B_\varepsilon)(A+B)\\ &\qquad - (A+B)^2 + (A+B)^2 - \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\big(A(\tau)+B(\tau)\big)^2\varphi\Big(\frac{t-\tau}{\varepsilon}\Big)d\tau\\ &\qquad + \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\big(A(\tau)+B(\tau)\big)^2\varphi\Big(\frac{t-\tau}{\varepsilon}\Big)d\tau - 4A_\varepsilon B_\varepsilon\\ &= (A_\varepsilon + B_\varepsilon)\big\{(A_\varepsilon + B_\varepsilon) - (A+B)\big\}\\ &\qquad + (A+B)\big\{(A_\varepsilon + B_\varepsilon) - (A+B)\big\}\\ &\qquad + \left\{(A+B)^2 - \frac{1}{\varepsilon}\int_{-\infty}^{\infty}\big(A(\tau)+B(\tau)\big)^2\varphi\Big(\frac{t-\tau}{\varepsilon}\Big)d\tau\right\}\\ &\qquad + \left\{\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\big(A(\tau)+B(\tau)\big)^2\varphi\Big(\frac{t-\tau}{\varepsilon}\Big)d\tau\right.\\ &\qquad\qquad \left. - \frac{1}{\varepsilon}\int_{-\infty}^{\infty}4A(\tau)B(\tau)\varphi\Big(\frac{t-\tau}{\varepsilon}\Big)d\tau\right\} + 4K\varepsilon^\alpha|\xi|^4\end{aligned}$$

$$\begin{aligned}
&\geq -C\varepsilon^{\alpha}|\xi|^4 - C\varepsilon^{\alpha}|\xi|^4 - C\varepsilon^{\alpha}|\xi|^4 \\
&\quad + \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \big(A(\tau,\xi) - B(\tau,\xi)\big)^2 \varphi\Big(\frac{t-\tau}{\varepsilon}\Big) d\tau + 4K\varepsilon^{\alpha}|\xi|^4 \\
&\geq (4L - 3C)\varepsilon^{\alpha}|\xi|^4.
\end{aligned}$$

Therefore if we take $K > 0$ large enough, we can obtain $C_\varepsilon(t,\xi)^2 \geq \varepsilon^{\alpha}|\xi|^4$.

Paying attention to $(14)_{\mathrm{III}}$, from (42) and (43) we also find that

$$A_\varepsilon(t,\xi) + B_\varepsilon(t,\xi) \geq 2\delta_1|\xi|^2$$

and taking $\nu > 0$ such that $\delta_1^2 - K\nu^{\frac{-2\alpha}{2+\alpha}} \geq \frac{\delta_1^2}{2}$,

$$\begin{aligned}
A_\varepsilon(t,\xi)B_\varepsilon(t,\xi) &\geq \{\delta_1^2 - K\varepsilon^{\alpha}\}|\xi|^4 = \Big\{\delta_1^2 - K\langle\xi\rangle_\nu^{\frac{-2\alpha}{2+\alpha}}\Big\}|\xi|^4 \\
&\geq \Big\{\delta_1^2 - K\nu^{\frac{-2\alpha}{2+\alpha}}\Big\}|\xi|^4 \geq \frac{\delta_1^2}{2}|\xi|^4.
\end{aligned}$$

□

Moreover we remark that (25) in Lemma 2 becomes

$$\begin{aligned}
&\delta \max\{\varepsilon^{\alpha}|\xi|^6|v|^2,\ \varepsilon^{\alpha}|\xi|^4|v_t|^2,\ \varepsilon^{\alpha}|\xi|^2|v_{tt}|^2,\ \varepsilon^{\alpha}|v_{ttt}|^2\} \\
&\quad \leq F_\varepsilon(t,\xi)^2 \leq L\big(|\xi|^6|v|^2 + |\xi|^4|v_t|^2 + |\xi|^2|v_{tt}|^2 + |v_{ttt}|^2\big).
\end{aligned} \tag{48}$$

Hence taking $\eta = -\frac{1}{2}$, by (44) we get the following instead of (26).

$$\begin{aligned}
I_1 &= 2\{(A_\varepsilon + B_\varepsilon) - (A+B)\}\Re\big(\varepsilon^{\eta}v_{tt},\ \varepsilon^{-\eta}(k_1 + l_1)\big) \\
&\leq \big|(A_\varepsilon + B_\varepsilon) - (A+B)\big|\varepsilon^{2\eta}|v_{tt}|^2 \\
&\qquad + \big|(A_\varepsilon + B_\varepsilon) - (A+B)\big|\varepsilon^{-2\eta}|k_1 + l_1|^2 \\
&\leq C\varepsilon^{2\eta}F_\varepsilon^2 + C'\varepsilon^{\alpha-2\eta}|\xi|^2F_\varepsilon^2 = C\varepsilon^{-1}F_\varepsilon^2 + C'\varepsilon^{\alpha+1}|\xi|^2F_\varepsilon^2.
\end{aligned}$$

Concerned with I_2, we also get (41) instead of (27).

Consequently we have $\big(F_\varepsilon(t,\xi)^2\big)' \leq C\varepsilon^{-1}F_\varepsilon(t,\xi)^2 + C'\varepsilon^{\alpha+1}\langle\xi\rangle_\nu^2F_\varepsilon(t,\xi)^2$. Similarly choosing $\varepsilon = \langle\xi\rangle_\nu^{-\frac{1}{1+\frac{\alpha}{2}}}$ and using the assumption (6), we also have the energy inequality $E_\varepsilon(t,\xi)^2 \leq E_\varepsilon(0,\xi)^2$ for $t \in [0,T]$. Finally by (48) we have the following energy inequality based on $\partial_t^h v$ $(h = 0,1,2,3,4)$.

$$\sum_{h=0}^{4} e^{2\rho(t)\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{\frac{10+6\alpha}{2+\alpha}-2h}|\partial_t^h v|^2 \leq C\sum_{h=0}^{3} e^{2\rho_0\langle\xi\rangle^{\frac{1}{s}}}\langle\xi\rangle^{2(3-h)}|v_h|^2.$$

This is equivalent to $(8)_{\mathrm{III}}$.

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