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# Extensions of some 2-groups

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**Abstract.** Let $H$ be a 2-group with faithful irreducible characters all which are algebraically conjugate to each other, and $\phi$ be any faithful irreducible character of $H$. We are interested in 2-group $G$ with the normal subgroup $H$ such that induced character $\phi^G$ is irreducible. For example, for 2-groups $H$ that are the cyclic groups, the dihedral groups $D_n$ and the generalized quaternion groups $Q_n$, all of such 2-groups $G$ was determined ([3]–[5]). In paticular, we showed that such a 2-group $G$ for $H = D_n$ or $Q_n$ is uniquely determined. Let $G_t(D_n)$ and $G_t(Q_n)$ be those 2-groups, respectively. The purpose of this paper is to determine all 2-groups $G$ for $H = G_t(D_n)$ and $G_t(Q_n)$ and faithful irreducible characters $\phi$ of $H$. In this paper we determine the character tables of $G_t(D_n)$ and $G_t(Q_n)$ in order to show that these groups have faithful irreducible characters all which are algebraically conjugate to each other. As result it is shown that these 2-groups have identical character tables.

*Key words*: 2-group, group extension, identical character.

## 1. Introduction

Let $D_n$, $Q_n$ and $SD_n$ be the dihedral group, the generalized quaternion group and the semidihedral group, respectively, of order $2^{n+1}$:

$$D_n = \left\langle a,\ b \;\middle|\; a^{2^n} = 1,\ b^2 = 1,\ bab^{-1} = a^{-1} \right\rangle \quad (n \geq 2),$$
$$Q_n = \left\langle a,\ b \;\middle|\; a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ bab^{-1} = a^{-1} \right\rangle \quad (n \geq 2),$$
$$SD_n = \left\langle a,\ b \;\middle|\; a^{2^n} = 1,\ b^2 = 1,\ bab^{-1} = a^{-1+2^{n-1}} \right\rangle \quad (n \geq 3).$$

And we define 2-groups $G_t(D_n)$ and $G_t(Q_n)$ of order $2^{n+t+1}$ $(0 \leq t \leq n-2)$ as follows:

$$G_t(D_n) = \left\langle a,\, b,\, x \;\middle|\; \begin{matrix} a^{2^n} = 1,\ b^2 = 1,\ x^{2^t} = 1, \\ bab^{-1} = a^{-1},\ xax^{-1} = a^{1+2^{n-t}},\ xbx^{-1} = b \end{matrix} \right\rangle,$$
$$G_t(Q_n) = \left\langle a,\, b,\, x \;\middle|\; \begin{matrix} a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ x^{2^t} = 1, \\ bab^{-1} = a^{-1},\ xax^{-1} = a^{1+2^{n-t}}, xbx^{-1} = b \end{matrix} \right\rangle.$$

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We note that $G_0(D_n) = D_n$ and $G_0(Q_n) = Q_n$. Let $\mathrm{Irr}(G)$ be the set of irreducible characters of a finite group $G$ and $\mathrm{FIrr}(G)$ ($\subset \mathrm{Irr}(G)$) be the set of faithful irreducible characters of $G$. We considered the following problem (see [3]).

**Problem** Let $H$ be a 2-group with faithful irreducible characters all which are algebraically conjugate to each other. Take a $\phi \in \mathrm{FIrr}(H)$.
(I) Characterize a 2-group $G$ such that $H \lhd G$ and $\phi^G \in \mathrm{Irr}(G)$.
(II) Determine all the 2-groups $G$ such that $H \lhd G$ and $\phi^G \in \mathrm{Irr}(G)$.

And for example, we showed the following in [4].

**Theorem 1** ([4, Theorem 1]) *Let $H = D_n$ $(n \geq 2)$, $Q_n$ $(n \geq 2)$ or $SD_n$ $(n \geq 3)$. Let $G$ be a 2-group with $G \rhd H$ and $|G : H| = 2^t$ $(t \geq 1)$. Take a $\phi \in \mathrm{FIrr}(H)$. If $\phi^G \in \mathrm{Irr}(G)$, then $t \leq n-2$ and one of the following holds:*
*(1) $G \cong G_t(D_n)$ when $H = D_n$,*
*(2) $G \cong G_t(Q_n)$ when $H = Q_n$,*
*(3) $G \cong G_t(D_n)$ or $G_t(Q_n)$ when $H = SD_n$.*

*In paticular, when $H = D_n$ $(n \geq 3)$ or $Q_n$ $(n \geq 3)$, $G$ is uniquely determined, respectively, for each integer $t$ $(1 \leq t \leq n-2)$.*

Set $H_n = D_n$ or $Q_n$. Theorem 1 implies that there exists no 2-groups $G$ for $H_2$ and two kinds of series of 2-groups for $n \geq 3$:

$$H_n = G_0(H_n) \subset G_1(H_n) \subset G_2(H_n) \subset \cdots \subset G_{n-2}(H_n)$$

with $|G_{i+1}(H_n) : G_i(H_n)| = 2$ $(0 \leq i \leq n-3)$.

The purpose of this paper is to consider Problem (II) for $H = G_t(H_n)$ and show the following. 2-groups $G_{t+s}(H_n)$ in the following theorem are defined in Section 4.

**Theorem A** *Let $H_n = D_n$ $(n \geq 3)$ or $Q_n$ $(n \geq 3)$ and $t$ be an integer such that $1 \leq t \leq n-2$. Let $G$ be a 2-group with $G \rhd G_t(H_n)$ and $|G : G_t(H_n)| = 2^s$ $(s \geq 1)$. Take a $\phi \in \mathrm{FIrr}(G_t(H_n))$. If $\phi^G \in \mathrm{Irr}(G)$, then $s \leq n-t-2$ and $G \cong G_{t+s}(H_n)$ or $\tilde{G}_{t+s}(H_n)$.*

In Section 2 we completely determine irreducible representations and characters of $G_t(H_n)$ $(1 \leq t \leq n-2)$ in order to show that these groups have faithful irreducible characters all which are algebraically conjugate to each other.

By the way, the following definition is well-known.

**Definition** Let $G_1$ and $G_2$ be finite groups. We shall say that $G_1$ and $G_2$ have identical character tables if the following three conditions are satisfied:
(1) There exists a bijection $\alpha$ from $G_1$ to $G_2$.
(2) There exists a bijection $\beta$ from $\mathrm{Irr}(G_1)$ to $\mathrm{Irr}(G_2)$.
(3) It shall be possible to choose a pair of bijections $(\alpha, \beta)$ such that $\chi^\beta(g^\alpha) = \chi(g)$ for all $g \in G_1$ and all $\chi \in \mathrm{Irr}(G_1)$.

Some pairs of nonisomorphic groups with identical character tables are well-known. The most famous pair is the dihedral group and the generalized quaternion group of order $4m$ ($m \geq 2$). Two nonisomorphic extraspecial $p$-groups of the same order have also identical character tables. And for example Fisher in [2] and Mattarei in [8]–[10] exhibited some $p$-groups with identical character tables, respectively. From the argument in Section 2 we have

**Theorem B** *The 2-groups $G_t(D_n)$ and $G_t(Q_n)$ $(0 \leq t \leq n-2)$ have identical character tables.*

In fact it is easy to see that $G_t(D_n) = D_n \rtimes \langle x \rangle$ and $G_t(Q_n) = Q_n \rtimes \langle x \rangle$ have identical character tables by comparing the actions of $x$ on $D_n$ and $Q_n$, because $D_n$ and $Q_n$ have identical character tables. As result, we exhibit series of groups with identical character tables. The character tables of $D_n$ and $Q_n$ are well-known. So we have also an interest to character tables of $G_t(D_n)$ and $G_t(Q_n)$. In Section 3 we explicitly determine character tables of these groups.

**Notation** For positive numbers $n$ and $k$, $2^n \mid k$ and $2^n \nmid k$ imply that $2^n$ devides $k$ and $2^n$ doesn't devide $k$, respectively. We write $2^n \parallel k$ when $2^n \mid k$ and $2^{n+1} \nmid k$. And a primitive $n$-th root of 1 is denoted by $\zeta_n$.

## 2. Irreducible representations and characters of $G_t(D_n)$ and $G_t(Q_n)$

In this section we determine all irreducible representations and chatacters of $G_t(D_n)$ and $G_t(Q_n)$ $(1 \leq t \leq n-2)$. We will use the following lemmas.

**Lemma 2** ([1, Corollary(45.5)]) *Let $H \triangleleft G$, and let $T$ be an irreducible representation of $H$. Then the induced representation $T^G$ is irreducible if and only if, for all $x \notin H$, the representations $T$ and $T^{(x)}\colon h \mapsto T(xhx^{-1})$ of $H$ are disjoint.*

**Lemma 3** *For any integers $n$, $t$, $l$ and $k$ ($1 \leq t \leq n-2$, $l \geq 0$, $k \geq 1$, $2 \nmid k$), there exists an odd number $\kappa$ such that*

$$(1+2^{n-t})^{2^l k} \equiv 1+2^{n-t+l}\kappa \pmod{2^n}.$$

*Proof.* Clear. □

We set $G = G_t(D_n)$ or $G_t(Q_n)$. Let $G'$ be the commutator subgroup of $G$. It is easily seen that $G' \supset \langle bab^{-1}a^{-1}\rangle = \langle a^2\rangle$ and $G/\langle a^2\rangle$ is abelian. So we have $G' = \langle a^2\rangle$. Then $G/G' \cong \langle \overline{a}\rangle \times \langle \overline{b}\rangle \times \langle \overline{x}\rangle$ and the relations $\overline{a}^2 = \overline{b}^2 = \overline{x}^{2^t} = \overline{1}$. So we have $2^{t+2}$ one-dimensional representations $\chi_{\mu,\gamma,\nu}$ of $G$:

$$\chi_{\mu,\gamma,\nu}\colon\ a \mapsto (-1)^{\mu}, \quad b \mapsto (-1)^{\gamma}, \quad x \mapsto \zeta_{2^t}^{\nu},$$

where $\mu = 1, 2$, $\gamma = 1, 2$ and $1 \leq \nu \leq 2^t$.

Next it follows from Yamada [11, Theorem 1] that the rest of irreducible representations of $G$ are induced from one-dimensional representation of $H_s = \langle a, x^{2^s}\rangle$ ($0 \leq s \leq t$). We note that $H_s$ is a normal subgroup of $G$. From now we write $H_s$ by $H$ simply, and let $H'$ be the commutator subgroup of $H$. We consider into two cases for integers $s$ ($0 \leq s \leq t$).

(Case 2-I) $s = 0$, i.e., $H = \langle a, x\rangle$.

It is easily seen that $H' = \langle xax^{-1}a^{-1}\rangle = \langle a^{2^{n-t}}\rangle$, $H/H' \cong \langle \overline{a}\rangle \times \langle \overline{x}\rangle$ and the relations $\overline{a}^{2^{n-t}} = \overline{x}^{2^t} = \overline{1}$. So we have $2^n$ one-dimensional representations $\phi_{0,\mu,\nu}$ of $H$:

$$\phi_{0,\mu,\nu}\colon\ a \mapsto \zeta_{2^{n-t}}^{\mu}, \quad x \mapsto \zeta_{2^t}^{\nu},$$

where $1 \leq \mu \leq 2^{n-t}$ and $1 \leq \nu \leq 2^t$. Set $\phi_{\mu,\nu} = \phi_{0,\mu,\nu}$ for simplicity. We have the decomposition into disjoint right cosets: $G = H \cup Hb$. Using this, we have induced representations $\Phi_{\mu,\nu}^G$ of $G$ affording the character $\phi_{\mu,\nu}^G$:

$$a \mapsto \begin{pmatrix} \zeta_{2^{n-t}}^{\mu} & 0 \\ 0 & \zeta_{2^{n-t}}^{-\mu} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x \mapsto \begin{pmatrix} \zeta_{2^t}^{\nu} & 0 \\ 0 & \zeta_{2^t}^{\nu} \end{pmatrix}.$$

By Lemma 2, $\Phi_{\mu,\nu}^G$ is irreducible, if and only if $\mu \not\equiv -\mu \pmod{2^{n-t}}$ i.e.,

$2^{n-t-1} \nmid \mu$.

Now we have $\phi^G_{\mu,\nu}(g) = 0$ for any $g \in G - H$ since $H \lhd G$. And $\phi^G_{\mu,\nu}(a^k x^l) = \phi_{\mu,\nu}(a^k x^l) + \phi^b_{\mu,\nu}(a^k x^l) = \zeta^{\nu l}_{2^t}(\zeta^{\mu k}_{2^{n-t}} + \zeta^{-\mu k}_{2^{n-t}})$. It is easy to see that $\phi^G_{\mu,\nu} \neq \phi^G_{\mu',\nu'}$, if and only if $\zeta^{\mu}_{2^{n-t}} + \zeta^{-\mu}_{2^{n-t}} \neq \zeta^{\mu'}_{2^{n-t}} + \zeta^{-\mu'}_{2^{n-t}}$ or $\zeta^{\nu}_{2^t} \neq \zeta^{\nu'}_{2^t}$. This is clearly equivalent to the condition $\mu \not\equiv \pm\mu' \pmod{2^{n-t}}$ or $\nu \not\equiv \nu'$ (mod $2^t$). As result we have $(2^{n-t-1} - 1) \times 2^t = 2^{n-1} - 2^t$ irreducible characters of $G$:

$$\phi^G_{\mu,\nu}(a^k b^m x^l) = \begin{cases} 0, & m = 1, \\ \zeta^{\nu l}_{2^t}(\zeta^{\mu k}_{2^{n-t}} + \zeta^{-\mu k}_{2^{n-t}}), & m = 0, \end{cases}$$

where $1 \leq \mu < 2^{n-t-1}$ and $1 \leq \nu \leq 2^t$.

(Case 2-II) $1 \leq s \leq t$.

We note that if $s = t$, then $H = \langle a \rangle$. It is easily seen that $H' = \langle x^{2^s} a x^{-2^s} a^{-1} \rangle = \langle a^{2^{n-t+s}} \rangle$, $H/H' \cong \langle \overline{a} \rangle \times \langle \overline{x^{2^s}} \rangle$ and the relations $\overline{a}^{2^{n-t+s}} = (\overline{x^{2^s}})^{2^{t-s}} = \overline{1}$. So we have $2^n$ one-dimensional representations $\phi_{s,\mu,\nu}$ of $H$:

$$\phi_{s,\mu,\nu}\colon\ a \mapsto \zeta^{\mu}_{2^{n-t+s}}, \quad x^{2^s} \mapsto \zeta^{\nu}_{2^{t-s}},$$

where $1 \leq \mu \leq 2^{n-t+s}$ and $1 \leq \nu \leq 2^{t-s}$. Set $\phi_{\mu,\nu} = \phi_{s,\mu,\nu}$ for simplicity. We have the decomposition into disjoint right cosets:

$$G = \left( \bigcup_{i=0}^{2^s-1} Hx^i \right) \cup \left( \bigcup_{i=0}^{2^s-1} Hbx^i \right).$$

Using this, we have induced representations $\Phi^G_{\mu,\nu}$ of $G$ affording $\phi^G_{\mu,\nu}$. Indeed we define $2^s \times 2^s$ metrices as follows:

$$A = \begin{pmatrix} \zeta^{\mu}_{2^{n-t+s}} & 0 & \cdots & 0 \\ 0 & \zeta^{\mu(1+2^{n-t})^1}_{2^{n-t+s}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta^{\mu(1+2^{n-t})^{2^s-1}}_{2^{n-t+s}} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \phi_{\mu,\nu}(b^2) & 0 & \cdots & 0 \\ 0 & \phi_{\mu,\nu}(b^2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \phi_{\mu,\nu}(b^2) \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ \zeta_{2^{t-s}}^{\nu} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where

$$\phi_{\mu,\nu}(b^2) = \begin{cases} -1 & s = t \text{ and } G = G_t(Q_n), \\ 1 & \text{othewise.} \end{cases}$$

And we denote the $2^s \times 2^s$ zero matrix by $O$.

Then we have an induced representation $\Phi_{\mu,\nu}^G$ of $G$ affording the character $\phi_{\mu,\nu}^G$:

$$a \mapsto \begin{pmatrix} A & O \\ O & A^{-1} \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} O & B_1 \\ B_2 & O \end{pmatrix}, \qquad x \mapsto \begin{pmatrix} X & O \\ O & X \end{pmatrix}.$$

By Lemma 2, $\Phi_{\mu,\nu}^G$ is irreducible, if and only if for each integer $i$ $(1 \leq i \leq 2^s - 1)$, $\mu \not\equiv \mu(1+2^{n-t})^i \pmod{2^{n-t+s}}$, $\mu \not\equiv -\mu(1+2^{n-t})^i \pmod{2^{n-t+s}}$ and $\mu \not\equiv -\mu \pmod{2^{n-t+s}}$. This is equivalent to the condition $2 \nmid \mu$. Indeed, because $\mu \not\equiv \pm\mu(1+2^{n-t})^{2^{s-1}} \pmod{2^{n-t+s}}$, it follows from $s \geq 1$ that $2 \nmid \mu$. Clearly if $2 \nmid \mu$, the above condition for $\phi_{\mu,\nu}^G \in \mathrm{Irr}(G)$ holds.

Now we have $\phi_{\mu,\nu}^G(g) = 0$ for any $g \in G - H$ since $H \triangleleft G$. And for each integer $l$ $(0 \leq l < s)$, $k$ $(1 \leq k < 2^{n-l}$, $2 \nmid k)$, $\alpha$ $(0 \leq \alpha < 2^l)$ and $\beta$ $(0 \leq \beta < 2^{s-l})$, it follows from Lemma 3 that

$$\begin{aligned} \zeta_{2^{n-t+s}}^{\pm 2^l \mu k (1+2^{n-t})^{2^{s-l}\alpha+\beta}} &= \zeta_{2^{n-t+s}}^{\pm 2^l \mu k (1+2^{n-t+s-l})^{\alpha}(1+2^{n-t})^{\beta}} \\ &= \zeta_{2^{n-t+s}}^{\pm 2^l \mu k (1+2^{n-t})^{\beta}} \end{aligned}$$

Similarly, for each integer $l$ $(0 \leq l < s)$, $k$ $(1 \leq k < 2^{n-l}$, $2 \nmid k)$ and $\beta$ $(0 \leq \beta < 2^{s-l-1})$, it follows from Lemma 3 and $2 \nmid \mu$ that

$$\zeta_{2^{n-t+s}}^{\pm 2^l \mu k (1+2^{n-t})^{2^{s-l-1}+\beta}} = \zeta_{2^{n-t+s}}^{\pm 2^l \mu k (1+2^{n-t+s-l-1})(1+2^{n-t})^{\beta}}$$

$$
\begin{aligned}
&= \zeta_{2^{n-t+s}}^{\pm 2^l \mu k (1+2^{n-t})^\beta} \zeta_{2^{n-t+s}}^{\pm 2^{n-t+s-1} \mu k (1+2^{n-t})^\beta} \\
&= -\zeta_{2^{n-t+s}}^{\pm 2^l \mu k (1+2^{n-t})^\beta}.
\end{aligned}
$$

So we have $\phi^G_{\mu,\nu}(a^{2^l k}) = 0$, where $0 \leq l < s$, $1 \leq k < 2^{n-l}$ and $2 \nmid k$. And since $\zeta_{2^{n-t}} = \zeta_{2^{n-t+s}}^{2^s}$, we have clearly $\phi^G_{\mu,\nu}(a^{2^l k}) = 2^s\big(\zeta_{2^{n-t}}^{2^{l-s}\mu k} + \zeta_{2^{n-t}}^{-2^{l-s}\mu k}\big)$, where $s \leq l < n$, $1 \leq k < 2^{n-l}$, $2 \nmid k$. Here we have

$$
\begin{aligned}
\phi^G_{\mu,\nu}(a^k x^{2^s j}) &= \sum_{i=0}^{2^s-1} \phi^{x^i}_{\mu,\nu}(a^k x^{2^s j}) + \sum_{i=0}^{2^s-1} \phi^{bx^i}_{\mu,\nu}(a^k x^{2^s j}) \\
&= \phi_{\mu,\nu}(x^{2^s j}) \left( \sum_{i=0}^{2^s-1} \phi^{x^i}_{\mu,\nu}(a^k) + \sum_{i=0}^{2^s-1} \phi^{bx^i}_{\mu,\nu}(a^k) \right) \\
&= \zeta_{2^{t-s}}^{\nu j} \phi^G_{\mu,\nu}(a^k)
\end{aligned}
$$

It is easy to see that $\phi^G_{\mu,\nu} \neq \phi^G_{\mu',\nu'}$, if and only if $\zeta_{2^{n-t}}^{\mu} + \zeta_{2^{n-t}}^{-\mu} \neq \zeta_{2^{n-t}}^{\mu'} + \zeta_{2^{n-t}}^{-\mu'}$ or $\zeta_{2^{t-s}}^{\nu} \neq \zeta_{2^{t-s}}^{\nu'}$. This is clearly equivalent to the condition $\mu \not\equiv \pm\mu'$ (mod $2^{n-t}$) or $\nu \not\equiv \nu'$ (mod $2^{t-s}$). As result we have $2^{n-t-2} \times 2^{t-s} = 2^{n-s-2}$ irreducible characters of $G$:

$$
\begin{aligned}
&\phi^G_{\mu,\nu}(a^k b^m x^l) \\
&\quad = \begin{cases} 0, & 2^s \nmid k \text{ or } m = 1 \text{ or } 2^s \nmid l, \\ 2^s \zeta_{2^t}^{\nu l} (\zeta_{2^{n-t+s}}^{\mu k} + \zeta_{2^{n-t+s}}^{-\mu k}), & 2^s \mid k \text{ and } m = 0 \text{ and } 2^s \mid l, \end{cases}
\end{aligned}
$$

where $1 \leq \mu < 2^{n-t-1}$, $2 \nmid \mu$ and $1 \leq \nu \leq 2^{t-s}$.

The total number of irreducible characters of $G$ which we have now is

$$
2^{t+2} + 2^{n-1} - 2^t + \sum_{s=1}^{t} 2^{n-s-2} = 3 \cdot 2^t + 2^{n-t-2}(3 \cdot 2^t - 1).
$$

We know easily these irreducible characters is all ones of $G = G_t(D_n)$ or $G_t(Q_n)$ from orthogonality relation (for example, see [1, (31.14)]). In fact we have

$$
2^{t+2} \times 1 + (2^{n-1} - 2^t) \times 2^2 + \sum_{s=1}^{t} (2^{n-s-2} \times (2^{s+1})^2) = 2^{n+t+1} = |G|.
$$

Consequently we have all irreducible characters of $G_t(D_n)$ or $G_t(Q_n)$ as follows, the number of which is $3 \cdot 2^t + 2^{n-t-2}(3 \cdot 2^t - 1)$:

(1) $2^{t+2}$ one-dimensional characters $\chi_{\mu,\gamma,\nu}$ ($\mu=1$, 2, $\gamma=1$, 2 and $1\leq\nu\leq 2^t$):

$$\chi_{\mu,\gamma,\nu}(a^k b^m x^l) = (-1)^{\mu k}(-1)^{\gamma m}\zeta_{2^t}^{\nu l}.$$

(2) $2^{n-1}-2^t$ irreducible characters $\phi_{0,\mu,\nu}^G$ ($1 \leq \mu < 2^{n-t-1}$ and $1 \leq \nu \leq 2^t$):

$$\phi_{0,\mu,\nu}^G(a^k b^m x^l) = \begin{cases} 0, & m=1, \\ \zeta_{2^t}^{\nu l}(\zeta_{2^{n-t}}^{\mu k} + \zeta_{2^{n-t}}^{-\mu k}), & m=0. \end{cases}$$

(3) for each integer $s$ ($1 \leq s \leq t$), $2^{n-s-2}$ irreducible characters $\phi_{s,\mu,\nu}^G$ ($1 \leq \mu < 2^{n-t-1}$, $2 \nmid \mu$ and $1 \leq \nu \leq 2^{t-s}$):

$$\begin{aligned} &\phi_{s,\mu,\nu}^G(a^k b^m x^l) \\ &\quad = \begin{cases} 0, & 2^s \nmid k \text{ or } m=1 \text{ or } 2^s \nmid l, \\ 2^s\zeta_{2^t}^{\nu l}(\zeta_{2^{n-t+s}}^{\mu k} + \zeta_{2^{n-t+s}}^{-\mu k}), & 2^s \mid k \text{ and } m=0 \text{ and } 2^s \mid l. \end{cases} \end{aligned}$$

## 3. Conjugacy classes of $G_t(D_n)$ and $G_t(Q_n)$

Now it is sufficient to determine the set of conjugacy classes in order to give character tables of $G_t(D_n)$ and $G_t(Q_n)$. Let $G = G_t(D_n)$ or $G_n(Q_n)$. Since $\langle a \rangle \lhd G$, we have the set of conjugacy classes concluded in $\langle a \rangle$ in $G$ by Lemma 3:

$$\begin{aligned} &\{1\}, \\ &\{a^{2^{n-1}}\}, \\ &\{a^{i(1+2^{n-t})^\mu}, a^{-i(1+2^{n-t})^\mu} \mid 1 \leq \mu \leq 2^t\} \qquad (1 \leq i \leq 2^n,\ 2 \nmid i), \\ &\{a^{i(1+2^{n-t})^\mu},\ a^{-i(1+2^{n-t})^\mu} \mid 1 \leq \mu \leq 2^{t-s}\} \\ &\qquad\qquad\qquad\qquad\qquad (1 \leq i \leq 2^n,\ 2^s \parallel i,\ 1 \leq s \leq t-1), \\ &\{a^i, a^{-i}\} \qquad\qquad\qquad\quad\ \ (1 \leq i \leq 2^n,\ 2^t \mid i,\ 2^{n-1} \nmid i). \end{aligned}$$

The total number of these conjugacy classes is

$$\begin{aligned} &1+1+(2^n/2)/2^{t+1} + \sum_{s=1}^{t-1}(2^n/2^{s+1})/2^{t-s+1} + (2^n/2^t-2)/2 \\ &= 2 + 2^{n-t-2} + (t-1)\times 2^{n-t-2} + 2^{n-t-1} - 1 \\ &= 1 + (t+2)\cdot 2^{n-t-2}. \end{aligned}$$

Next we consider conjugacy classes concluding elements $a^i b x^j$ ($1 \leq i$

$\leq 2^n$, $1 \leq j \leq 2^t$.) Since $a^k b x^j a^{-k} = a^{2k\{1+\{(1+2^{n-t})^j-1\}/2\}} b x^j$, $a^k a b x^j a^{-k} = a^{1+2k\{1+\{(1+2^{n-t})^j-1\}/2\}} b x^j$ and $xb = bx$, we have the following conjugacy classes:

$$\begin{aligned}
&\{a^{2\mu} b x^j \mid 1 \leq \mu \leq 2^{n-1}\} && (1 \leq j \leq 2^t),\\
&\{a^{1+2\mu} b x^j \mid 1 \leq \mu \leq 2^{n-1}\} && (1 \leq j \leq 2^t).
\end{aligned}$$

The total number of these conjugacy classes is $2 \times 2^t = 2^{t+1}$.

Next we consider conjugacy classes concluding the elements $a^i x^j$ $(1 \leq i \leq 2^n,\ 1 \leq j \leq 2^t)$. Since

$$\begin{aligned}
(a^\mu x^\nu)(a^i x^j)(a^\mu x^\nu)^{-1} &= (x^\nu a^i x^{-\nu})(a^\mu x^j a^{-\mu})\\
&= a^{i(1+2^{n-t})^\nu}(a^\mu x^j a^{-\mu}),\\
\text{and}\quad (a^\mu b x^\nu)(a^i x^j)(a^\mu b x^\nu)^{-1} &= (x^\nu a^{-i} x^{-\nu})(a^\mu x^j a^{-\mu})\\
&= a^{-i(1+2^{n-t})^\nu}(a^\mu x^j a^{-\mu}),
\end{aligned}$$

we consider the set of conjugacy classes concluding elements $a^\mu x^j a^{-\mu}$ $(1 \leq \mu \leq 2^n,\ 1 \leq j \leq 2^t)$ in three cases.

(Case 3-I) $1 \leq j \leq 2^t$, $2 \nmid j$.

Since there exists an odd number $\kappa_1$ such that $a^\mu x^j a^{-\mu} = a^{\mu\{1-(1+2^{n-t})^j\}} x^j = a^{-2^{n-t}\mu\kappa_1} x^j$ by Lamma 3, we have

$$\left\{a^\mu x^j a^{-\mu} \;\middle|\; 1 \leq \mu \leq 2^n\right\} = \left\{a^{2^{n-t}\mu} x^j \;\middle|\; 1 \leq \mu \leq 2^t\right\}.$$

And so we have for integers $i$ $(1 \leq i \leq 2^n)$

$$\begin{aligned}
&\big\{a^{i(1+2^{n-t})^\nu}(a^\mu x^j a^{-\mu}),\ a^{-i(1+2^{n-t})^\nu}(a^\mu x^j a^{-\mu}) \mid\\
&\qquad\qquad\qquad\qquad 1 \leq \mu \leq 2^n,\ 1 \leq \nu \leq 2^t\big\}\\
=&\big\{a^{i(1+2^{n-t})^\nu + 2^{n-t}\mu} x^j,\ a^{-i(1+2^{n-t})^\nu + 2^{n-t}\mu} x^j \mid\\
&\qquad\qquad\qquad\qquad 1 \leq \mu \leq 2^t,\ 1 \leq \nu \leq 2^t\big\}\\
=&\begin{cases}
\left\{a^{2^{n-t}\mu} x^j \;\middle|\; 1 \leq \mu \leq 2^t\right\} \quad (i = 2^n),\\
\left\{a^{i+2^{n-t}\mu} x^j, a^{-i+2^{n-t}\mu} x^j \;\middle|\; 1 \leq \mu \leq 2^t\right\} \quad (2 \nmid i),\\
\left\{a^{i+2^{n-t}\mu} x^j, a^{-i+2^{n-t}\mu} x^j \;\middle|\; 1 \leq \mu \leq 2^t\right\}\\
\qquad (2^u \parallel i,\ 1 \leq u \leq n-t-2),\\
\left\{a^{i+2^{n-t}\mu} x^j \;\middle|\; 1 \leq \mu \leq 2^t\right\} = \left\{a^i x^j \;\middle|\; 1 \leq i \leq 2^n, 2^{n-t-1} \parallel i\right\}\\
\qquad (2^{n-t-1} \parallel i).
\end{cases}
\end{aligned}$$

We note that $i+2^{n-t}\kappa_1 \equiv -i+2^{n-t}\kappa_2 \pmod{2^n}$ for some integers $\kappa_1$ and $\kappa_2$, if and only if $2^{n-t-1} \mid i$. Consequently the total number of these conjugacy classes is

$$\begin{aligned}
&1 + (2^{n-1}/2^{t+1}) + \sum_{u=1}^{n-t-2} (2^{n-u-1}/2^{t+1}) + 1 \\
&\quad = 2 + 2^{n-t-2} + \sum_{u=1}^{n-t-2} 2^{n-t-2-u} \\
&\quad = 2 + 2^{n-t-2} + 2^{n-t-2} - 1 \\
&\quad = 1 + 2^{n-t-1}.
\end{aligned}$$

Since the number of elements of $\{j \mid 1 \le j \le 2^t,\ 2 \nmid j\}$ is $2^{t-1}$, the total number of conjugacy classes in this case is $(1+2^{n-t-1}) \times 2^{t-1} = 2^{t-1}+2^{n-2}$.

(Case 3-II) $1 \le j \le 2^t$, $2^s \parallel j$ for each integer $s$ $(1 \le s \le t-2)$.

Since there exists an odd number $\kappa_2$ such that $a^{\mu}x^j a^{-\mu} = a^{\mu\{1-(1+2^{n-t})^j\}}x^j = a^{-2^{n-t+s}\mu\kappa_2}x^j$ by Lemma 3, we have

$$\{a^{\mu}x^j a^{-\mu} \mid 1 \le \mu \le 2^n\} = \{a^{2^{n-t+s}\mu}x^j \mid 1 \le \mu \le 2^{t-s}\}.$$

And so we have for integers $i$ $(1 \le i \le 2^n)$

$$\begin{aligned}
&\{a^{i(1+2^{n-t})^{\nu}}(a^{\mu}x^j a^{-\mu}),\ a^{-i(1+2^{n-t})^{\nu}}(a^{\mu}x^j a^{-\mu}) \mid \\
&\qquad\qquad\qquad\qquad 1 \le \mu \le 2^n,\ 1 \le \nu \le 2^t\} \\
&= \{a^{i(1+2^{n-t})^{\nu}+2^{n-t+s}\mu}x^j,\ a^{-i(1+2^{n-t})^{\nu}+2^{n-t+s}\mu}x^j \mid \\
&\qquad\qquad\qquad\qquad 1 \le \mu \le 2^{t-s},\ 1 \le \nu \le 2^t\} \\
&= \begin{cases}
\{a^{2^{n-t+s}\mu}x^j \mid 1 \le \mu \le 2^{t-s}\} \quad (i = 2^n), \\
\{a^{i+2^{n-t}\mu}x^j, a^{-i+2^{n-t}\mu}x^j \mid 1 \le \mu \le 2^t\} \quad (2 \nmid i), \\
\{a^{i+2^{n-t+u}\mu}x^j, a^{-i+2^{n-t+u}\mu}x^j \mid 1 \le \mu \le 2^{t-u}\} \\
\qquad (2^u \parallel i,\ 1 \le u \le s), \\
\{a^{i+2^{n-t+s}\mu}x^j, a^{-i+2^{n-t+s}\mu}x^j \mid 1 \le \mu \le 2^{t-s}\} \\
\qquad (2^u \parallel i,\ s+1 \le u \le n-t+s-2), \\
\{a^{i+2^{n-t+s}\mu}x^j \mid 1 \le \mu \le 2^{t-s}\} \\
\quad = \{a^i x^j \mid 1 \le i \le 2^n, 2^{n-t+s-1} \parallel i\} \quad (2^{n-t+s-1} \parallel i).
\end{cases}
\end{aligned}$$

We note that $i + 2^{n-t+s}\kappa_1 \equiv -i + 2^{n-t+s}\kappa_2 \pmod{2^n}$ for some integers $\kappa_1$

and $\kappa_2$, if and only if $2^{n-t+s-1} \mid i$. Consequently the number of these conjugacy classes for each integer $s$ $(1 \le s \le t-2)$ is

$$\begin{aligned}
&1+(2^{n-1}/2^{t+1})+\sum_{u=1}^{s}(2^{n-u-1}/2^{t-u+1})+\sum_{u=s+1}^{n-t+s-2}(2^{n-u-1}/2^{t-s+1})+1\\
&=2+2^{n-t-2}+\sum_{u=1}^{s}2^{n-t-2}+\sum_{u=s+1}^{n-t+s-2}2^{n-t+s-2-u}\\
&=1+2^{n-t-1}+2^{n-t-2}s.
\end{aligned}$$

Since the number of elements of $\{j \mid 1 \le j \le 2^t,\ 2^s \parallel j\}$ is $2^{t-s-1}$, the total number of conjugacy classes in this case is

$$\begin{aligned}
&\sum_{s=1}^{t-2}\big((1+2^{n-t-1}+2^{n-t-2}s)\times 2^{t-s-1}\big)\\
&=\sum_{s=1}^{t-2}(2^{t-s-1}+2^{n-s-2}+s^{n-s-3}s)\\
&=(2^{t-1}-2)+(2^{n-2}-2^{n-t})+2^{n-3}\sum_{s=1}^{t-2}2^{-s}s\\
&=2^{t-1}-2+2^{n-2}-2^{n-t}+2^{n-3}(2-2^{3-t}-(t-2)2^{2-t})\\
&=2^{n-1}+2^{t-1}-(t+2)2^{n-t-1}-2.
\end{aligned}$$

(Case 3-III) $1 \le j \le 2^t$, $2^{t-1} \parallel j$, i.e., $j = 2^{t-1}$.

Since we have $a^{\mu}x^{2^{t-1}}a^{-\mu} = a^{\mu\{1-(1+2^{n-t})^{2^{t-1}}\}}x^{2^{t-1}} = a^{2^{n-1}\mu}x^{2^{t-1}}$ by Lemma 3, we have

$$\begin{aligned}
\big\{a^{\mu}x^{2^{t-1}}a^{-\mu} \bigm| 1 \le \mu \le 2^n\big\} &= \big\{a^{2^{n-1}\mu}x^{2^{t-1}} \bigm| \mu = 0,1\big\}\\
&= \big\{x^{2^{t-1}},\ a^{2^{n-1}}x^{2^{t-1}}\big\}.
\end{aligned}$$

And so we have for integers $i$ $(1 \le i \le 2^n)$

$$\begin{aligned}
&\big\{a^{i(1+2^{n-t})^{\nu}}(a^{\mu}x^{2^{t-1}}a^{-\mu}),\ a^{-i(1+2^{n-t})^{\nu}}(a^{\mu}x^{2^{t-1}}a^{-\mu}) \bigm|\\
&\qquad\qquad\qquad\qquad 1 \le \mu \le 2^n,\ 1 \le \nu \le 2^t\big\}\\
&= \big\{a^{i(1+2^{n-t})^{\nu}+2^{n-1}\mu}x^{2^{t-1}},\ a^{-i(1+2^{n-t})^{\nu}+2^{n-1}\mu}x^{2^{t-1}} \bigm|\\
&\qquad\qquad\qquad\qquad \mu = 0,\ 1,\ 1 \le \nu \le 2^t\big\}
\end{aligned}$$

$$= \begin{cases} \{a^{2^{n-1}\mu}x^{2^{t-1}} \mid \mu = 0,1\} = \{x^{2^{t-1}},\ a^{2^{n-1}}x^{2^{t-1}}\} \quad (i = 2^n), \\ \{a^{i+2^{n-t}\mu}x^{2^{t-1}},\ a^{-i+2^{n-t}\mu}x^{2^{t-1}} \mid 1 \le \mu \le 2^t\} \quad (2 \nmid i), \\ \{a^{i+2^{n-t+u}\mu}x^{2^{t-1}},\ a^{-i+2^{n-t+u}\mu}x^{2^{t-1}} \mid 1 \le \mu \le 2^{t-u}\} \\ \qquad (2^u \parallel i,\ 1 \le u \le t-1), \\ \{a^{i+2^{n-1}\mu}x^{2^{t-1}},\ a^{-i+2^{n-1}\mu}x^{2^{t-1}} \mid \mu = 0,1\} \\ \qquad (2^u \parallel i,\ t \le u \le n-3), \\ \{a^{i+2^{n-1}\mu}x^{2^{t-1}} \mid \mu = 0,1\} = \{a^{2^{n-2}}x^{2^{t-1}}, a^{3\cdot 2^{n-2}}x^{2^{t-1}}\} \\ \qquad (2^{n-2} \parallel i). \end{cases}$$

We note that $i + 2^{n-1}\kappa_1 \equiv -i + 2^{n-1}\kappa_2 \pmod{2^n}$ for some integers $\kappa_1$ and $\kappa_2$, if and only if $2^{n-2} \mid i$. And we remark $t-1 < n-2$. Consequently the total number of conjugacy classes in this case is

$$\begin{aligned} &1 + 2^{n-1}/2^{t+1} + \sum_{u=1}^{t-1} 2^{n-u-1}/2^{t-u+1} + \sum_{u=t}^{n-3} 2^{n-u-1}/2^2 + 1 \\ &= 1 + 2^{n-t-2} + \sum_{u=1}^{t-1} 2^{n-t-2} + \sum_{u=t}^{n-3} 2^{n-3-u} + 1 \\ &= 1 + 2^{n-t-2}(t+1). \end{aligned}$$

The total number of conjugacy classes of $G$ in this section is

$$\begin{aligned} &(1 + 2^{n-t-2}(t+2)) + 2^{t+1} + (2^{t-1} + 2^{n-2}) \\ &\quad + (2^{n-1} + 2^{t-1} - (t+2)2^{n-t-1} - 2) + 1 + 2^{n-t-2}(t+1) \\ &= 3\cdot 2^{n-2} + 3\cdot 2^t + 2^{n-t-1} - 2^{n-t} + 2^{n-t-2} \\ &= 3\cdot 2^{n-2} + 3\cdot 2^t - 2^{n-t-2} \\ &= 3\cdot 2^t + 2^{n-t-2}(3\cdot 2^t - 1), \end{aligned}$$

which is equal to the number of irreducible characters of $G$ (see Secion 3).

So we have now the set of conjugacy classes of $G$.

Consequently we have the conjugacy classes of $G$ as follows, the number of which is $3\cdot 2^t + 2^{n-t-2}(3\cdot 2^t - 1)$:

(1) $\{1\}$, $\{a^{2^{n-1}}\}$,

$\{a^{i(1+2^{n-t})^\mu},\ a^{-i(1+2^{n-t})^\mu} \mid 1 \le \mu \le 2^t\}$ $(2 \nmid i)$,

$\{a^{i(1+2^{n-t})^\mu},\ a^{-i(1+2^{n-t})^\mu} \mid 1 \le \mu \le 2^{t-s}\}$ $(2^s \parallel i,\ 1 \le s \le t-1)$,

$\{a^i,\ a^{-i}\}$ $(2^t \mid i,\ 2^{n-1} \nmid i)$,

(2) $1 \le j \le 2^t$

$\{a^{2\mu}bx^j \mid 1 \le \mu \le 2^{n-1}\}$, $\{a^{1+2\mu}bx^j \mid 1 \le \mu \le 2^{n-1}\}$,

(3) $1 \le j \le 2^t$, $2 \nmid j$

$\{a^{2^{n-t}\mu}x^j \mid 1 \le \mu \le 2^t\}$,

$\{a^{i+2^{n-t}\mu}x^j,\ a^{-i+2^{n-t}\mu}x^j \mid 1 \le \mu \le 2^t\}$ $(2 \nmid i)$,

$\{a^{i+2^{n-t}\mu}x^j,\ a^{-i+2^{n-t}\mu}x^j \mid 1 \le \mu \le 2^t\}$ $(2^u \parallel i,\ 1 \le u \le n-t-2)$,

$\{a^i x^j \mid 1 \le i \le 2^n,\ 2^{n-t-1} \parallel i\}$,

(4) $1 \le j \le 2^t$, $2^s \parallel j$ for each integer $s$ $(1 \le s \le t-2)$

$\{a^{2^{n-t+s}\mu}x^j \mid 1 \le \mu \le 2^{t-s}\}$,

$\{a^{i+2^{n-t}\mu}x^j,\ a^{-i+2^{n-t}\mu}x^j \mid 1 \le \mu \le 2^t\}$ $(2 \nmid i)$,

$\{a^{i+2^{n-t}\mu}x^j,\ a^{-i+2^{n-t}\mu}x^j \mid 1 \le \mu \le 2^{t-u}\}$ $(2^u \parallel i,\ 1 \le u \le s)$,

$\{a^{i+2^{n-t+s}\mu}x^j,\ a^{-i+2^{n-t+s}\mu}x^j \mid 1 \le \mu \le 2^{t-s}\}$

$(2^u \parallel i,\ s+1 \le u \le n-t+s-2)$,

$\{a^i x^j \mid 1 \le i \le 2^n,\ 2^{n-t+s-1} \parallel i\}$,

(5) $\{x^{2^{t-1}},\ a^{2^{n-1}}x^{2^{t-1}}\}$,

$\{a^{i+2^{n-t}\mu}x^{2^{t-1}},\ a^{-i+2^{n-t}\mu}x^{2^{t-1}} \mid 1 \le \mu \le 2^t\}$ $(2 \nmid i)$,

$\{a^{i+2^{n-t+u}\mu}x^{2^{t-1}},\ a^{-i+2^{n-t+u}\mu}x^{2^{t-1}} \mid 1 \le \mu \le 2^{t-u}\}$

$(2^u \parallel i,\ 1 \le u \le t-1)$,

$\{a^i x^{2^{t-1}}, a^{-i}x^{2^{t-1}},\ a^{i+2^{n-1}}x^{2^{t-1}},\ a^{-i+2^{n-1}}x^{2^{t-1}}\}$

$(2^u \parallel i,\ t \le u \le n-3)$,

$\{a^{2^{n-2}}x^{2^{t-1}},\ a^{3\cdot 2^{n-2}}x^{2^{t-1}}\}$,

where $1 \le i \le 2^n$.

## 4. Extensions of $G_t(D_n)$ and $G_t(Q_n)$

Let $H_n = D_n$ or $Q_n$ $(n \ge 3)$. From the argument in Section 2 it follows that $G_t(H_n)$ $(1 \le t \le n-2)$ has faithful irreducible characters all which are algebraically conjugate to each other. In fact the induced character $\phi_{t,\mu,1}^{G_t(H_n)}$ from $\phi_{t,\mu,1}$ $(1 \le \mu < 2^{n-t-1},\ 2 \nmid \mu)$ of $H_t = \langle a \rangle$ is faithful. So we consider Problem (II) in Section 1 for $H = G_t(H_n)$ and $\phi_{t,\mu,1}^{G_t(H_n)} \in \mathrm{FIrr}(H)$. We difine some 2-groups:

$$\tilde{G}_2(D_n) = \left\langle a,\, b,\, x,\, y \;\middle|\; \begin{array}{l} a^{2^n} = 1,\, b^2 = 1,\, x^2 = 1,\, y^2 = 1 \\ bab^{-1} = a^{-1},\, xax^{-1} = a^{1+2^{n-1}},\, xbx^{-1} = b \\ yay^{-1} = ax,\, yby^{-1} = bx,\, yxy^{-1} = x \end{array} \right\rangle,$$

$$\tilde{G}_2(Q_n) = \left\langle a,\, b,\, x,\, y \;\middle|\; \begin{array}{l} a^{2^n} = 1,\, b^2 = a^{2^{n-1}},\, x^2 = 1,\, y^2 = 1 \\ bab^{-1} = a^{-1},\, xax^{-1} = a^{1+2^{n-1}},\, xbx^{-1} = b \\ yay^{-1} = ax,\, yby^{-1} = a^{2^{n-1}}bx,\, yxy^{-1} = x \end{array} \right\rangle,$$

Moreover we define some 2-groups for integers $t$ $(3 \le t \le n-2)$:

$$\tilde{G}_t(D_n) = \left\langle a,\, b,\, x,\, y \;\middle|\; \begin{array}{l} a^{2^n} = 1,\, b^2 = 1,\, x^{2^{t-1}} = 1,\, y^2 = x^{e_t} \\ bab^{-1} = a^{-1},\, xax^{-1} = a^{1+2^{n-t+1}},\, xbx^{-1} = b \\ yay^{-1} = a^{1+2^{n-t}}x^{2^{t-2}},\, yby^{-1} = bx^{2^{t-2}},\, yxy^{-1} = x \end{array} \right\rangle,$$

where $e_t$ is the odd number satisfying $(1+2^{n-t+1})^{e_t} \equiv (1+2^{n-t})^2 \pmod{2^n}$,

$$\tilde{G}_t(Q_n) = \left\langle a,\, b,\, x,\, y \;\middle|\; \begin{array}{l} a^{2^n} = 1,\, b^2 = a^{2^{n-1}},\, x^{2^{t-1}} = 1,\, y^2 = x^{e_t} \\ bab^{-1} = a^{-1},\, xax^{-1} = a^{1+2^{n-t+1}},\, xbx^{-1} = b \\ yay^{-1} = a^{1+2^{n-t}}x^{2^{t-2}},\, yby^{-1} = bx^{2^{t-2}},\, yxy^{-1} = x \end{array} \right\rangle,$$

where $e_t$ is the odd number satisfying $(1+2^{n-t+1})^{e_t} \equiv (1+2^{n-t})^2 \pmod{2^n}$.

In [10] Sekiguchi showed the following theorem.

**Theorem 4** *Let $H = D_n$ $(n \ge 3)$ or $Q_n$ $(n \ge 3)$. Let $G$ be a 2-group with $G \supset H$ and $|G : H| = 2^t$ $(t \ge 1)$. Take a $\phi \in \mathrm{FIrr}(H)$. If $\phi^G \in \mathrm{Irr}(G)$, then $t \le n-2$ and one of the following holds:*
*(1) $G \cong G_1(H)$ when $t = 1$,*
*(2) $G \cong G_2(H)$ or $\tilde{G}_2(H)$ when $t = 2$,*
*(3) $G \cong G_t(H)$ or $\tilde{G}_t(H)$ when $3 \le t \le n-2$.*

*Proof of* Theorem A. From the results in Section 2 we have $\phi = \phi_{t,\mu,1}^{G_t(H_n)}$ for some integer $\mu$ $(1 \le \mu < 2^{n-t-1}$ and $2 \nmid \mu)$. It is clear that $\phi_{t,\mu,1}^{H_n} \in \mathrm{FIrr}(H_n)$ and $\phi_{t,\mu,1}^{G_t(H_n)} = (\phi_{t,\mu,1}^{H_n})^{G_t(H_n)}$. So it follows from Theorem 4 that $G \cong G_{t+s}(H_n)$ or $\tilde{G}_{t+s}(H_n)$ for some integers $s$ $(1 \le s \le n-t-2)$. It is easily known that $G_{t+s}(H_n) \triangleright G_t(H_n)$ and $\tilde{G}_{t+s}(H_n) \triangleright G_t(H_n)$. Theorem A is proved. $\square$

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