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Univalent functions with missing Taylor coefficients

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Abstract. For $n \geq 2$, let $\mathcal{U}(\lambda)$ denote the class of all analytic functions f in the unit disc Δ of the form

$$f(z) = z + a_{n+1}z^{n+1} + \cdots$$

satisfying the condition

$$f'(z) \; \frac{z}{f(z)} \;^{2} - 1 \; < \lambda, \quad z \in \Delta.$$

In this paper, among other results, we find condition on λ so that each function in $\mathcal{U}(\lambda)$ is starlike, strongly starlike or convex of some order. In addition, we discuss the mapping properties of the integral operator

$$[I(f)](z) = \frac{c}{z^{c-1}} \int_0^z \frac{\zeta^c}{f(\zeta)}\, d\zeta, \quad c > 0.$$

Key words: Univalent, starlike and convex functions, and integral transform.

1. Introduction

Let $\mathcal{H}$ denote the class of all functions f analytic in the unit disc $\Delta = \{z \colon |z| < 1\}$. For $n \geq 1$, a positive integer, let

$$\mathcal{A}_n = \left\{ f \in \mathcal{H} \colon f(z) = z + \sum_{k=1}^{\infty} a_{n+k} z^{n+k} \right\}$$

with $\mathcal{A}_1 = \mathcal{A}$, where $\mathcal{A}$ is referred to as the normalized analytic functions in the unit disc. A function $f \in \mathcal{A}$ is called starlike in Δ if $f(\Delta)$ is starlike with respect to the origin. The class of all starlike functions is denoted by $\mathcal{S}^* \equiv \mathcal{S}^*(0)$. For $\alpha < 1$, we define

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} \colon \mathrm{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \Delta \right\}$$

and is called the class of all starlike functions of order α. Clearly, $\mathcal{S}^*(\alpha) \subsetneq \mathcal{S}^*$ for $0 < \alpha < 1$. For $0 < \alpha \leq 1$, a function $f \in \mathcal{A}$ is called strongly starlike

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of order α if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi\alpha}{2}, \quad z \in \Delta.$$

We denote by $\mathcal{S}_\alpha$, the set of all strongly starlike functions of order α and it is a well-known fact that each function $f \in \mathcal{S}_\alpha$ is not only starlike but bounded for each $\alpha \in (0, 1)$. Clearly, $\mathcal{S}_1 \equiv \mathcal{S}^*$. These classes of functions are investigated by several authors [D, G]. A function $f \in \mathcal{A}$ is said to be convex of order α, denoted by $\mathcal{K}(\alpha)$ $(\alpha < 1)$, if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$. Thus, the correspondence between the families $\mathcal{K}(\alpha)$ and $\mathcal{S}^*(\alpha)$ is given by

$$f(z) = \int_0^z \frac{g(t)}{t}\, dt, \quad g \in \mathcal{S}^*(\alpha),$$

which is actually referred to as Alexander transform. Now, for $0 \le \alpha < 1$ and $0 < \gamma \le 1$, we introduce the following subclasses for our investigation:

$$\begin{aligned}
\mathcal{S}^*_\alpha &= \left\{f \in \mathcal{S}^*(\alpha)\colon \left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\alpha, \quad z \in \Delta\right\},\\
\mathcal{K}_\alpha &= \left\{f \in \mathcal{K}(\alpha)\colon \left|\frac{zf''(z)}{f'(z)}\right| < 1-\alpha, \quad z \in \Delta\right\},\\
\mathcal{R}_\gamma &= \left\{f \in \mathcal{A}\colon |\arg f'(z)| < \frac{\pi\gamma}{2}, \quad z \in \Delta\right\},
\end{aligned}$$

and for $\lambda > 0$,

$$\mathcal{U}(\lambda) = \left\{f \in \mathcal{A}\colon \left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < \lambda, \quad z \in \Delta\right\},$$
$$f(z) \neq 0 \quad \text{for} \quad z \in \Delta \setminus \{0\}.$$

The following result is well-known [ON, OP, OPSV]:

$$\mathcal{U}(\lambda) \subsetneq \mathcal{S}$$

whenever $0 < \lambda \le 1$ and we note that functions in $\mathcal{U}(\lambda)$ for $\lambda > 1$ need not be univalent in Δ. The typical elements of $\mathcal{U}(1)$ is the Koebe function $z/(1-z)^2$ and the function $z/(1-z^2)$. Further, it is also known that if $f \in \mathcal{U}(\lambda)$ with $f''(0) = 0$ then $f \in \mathcal{S}^*$ whenever $0 < \lambda \le 1/\sqrt{2}$. A number of problems has been discussed in [PV] for functions in $\mathcal{U}(\lambda)$ of the form $f(z) = z + a_3z^3 + \cdots$. In view of these inclusions, it is natural to look for the counterparts of these results for those functions f in $\mathcal{U}(\lambda)$ with $f^{(k)}(0) = 0$

for $k = 2, 3, \ldots, n$ ($n \geq 2$) rather than assuming just the condition $f''(0) = 0$. The aim of this paper is to fill this gap.

Let us start discussing certain basic properties of the functions $f \in \mathcal{U}(\lambda)$ of the form

$$f(z) = z + a_{n+1}z^{n+1} + \cdots, \quad n \geq 2.$$

As $f \in \mathcal{U}(\lambda)$, we can write

$$\begin{aligned} -z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} = \left(\frac{z}{f(z)}\right)^2 f'(z) &= 1 + (n-1)a_{n+1}z^n + \cdots \\ &= 1 + \lambda w(z) \end{aligned} \tag{1.1}$$

with $w \in \mathcal{B}_n$, where

$$\mathcal{B}_n = \left\{ w \in \mathcal{H}\colon \begin{array}{l} |w(z)| < 1 \quad \text{and} \\ w^{(k)}(0) = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, n-1 \end{array} \right\}.$$

By the Schwarz lemma, we have $|w(z)| \leq |z|^n$. By (1.1), it is a simple exercise to see that

$$\frac{z}{f(z)} = 1 - \lambda \int_0^1 \frac{w(tz)}{t^2}\, dt. \tag{1.2}$$

As $|w(z)| \leq |z|^n$, by (1.2), it follows that

$$\left| \frac{z}{f(z)} - 1 \right| \leq \frac{\lambda |z|^n}{n-1}, \quad z \in \Delta, \tag{1.3}$$

and

$$1 - \frac{\lambda|z|^n}{n-1} \leq \mathrm{Re}\left(\frac{z}{f(z)}\right) \leq 1 + \frac{\lambda|z|^n}{n-1}, \quad z \in \Delta. \tag{1.4}$$

- By (1.2), we note that the equality holds in each of the last two inequalities (1.3) and (1.4) for the function

$$f(z) = \frac{z}{1 \pm \lambda z^n/(n-1)}.$$

- By (1.3), it follows that

$$\frac{z}{f(z)} \in \{w\colon |w - 1| < 1\} \quad \text{for} \quad |z| < \sqrt[n]{\frac{n-1}{\lambda}}$$

and thus,

$$\mathrm{Re}\left(\frac{z}{f(z)}\right) > 0 \quad \text{for} \quad |z| < \sqrt[n]{\frac{n-1}{\lambda}}.$$

In particular, for $0 < \lambda \leq n-1$, we have

$$\mathrm{Re}\left(\frac{z}{f(z)}\right) > 0 \quad \text{for} \quad z \in \Delta.$$

On the other hand, (1.3) is equivalent to

$$\left|\frac{f(z)}{z} - \frac{(n-1)^2}{(n-1)^2 - \lambda^2|z|^{2n}}\right| \leq \frac{\lambda|z|^n/(n-1)}{1 - \big(\lambda|z|^n/(n-1)\big)^2}$$

which implies that

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) \geq \frac{1}{1+\lambda|z|^n/(n-1)} > \frac{n-1}{n-1+\lambda}, \quad z \in \Delta.$$

In this paper, we address the following

1.5. Problems (1) Find conditions on λ so that functions in $\mathcal{U}(\lambda)$ belongs to $\mathcal{S}^*(\alpha)$, $\mathcal{S}^*_\alpha$, $\mathcal{S}_\alpha$, $\mathcal{K}(\alpha)$ or $\mathcal{R}_\alpha$.
(2) Find conditions on λ so that the integral operator defined by

$$[I(f)](z) = \frac{c}{z^{c-1}} \int_0^z \frac{\zeta^c}{f(\zeta)}\, d\zeta, \quad c > 0,$$

carries the class $\mathcal{U}(\lambda)$ into $\mathcal{S}^*_\alpha$, $\mathcal{K}(\alpha)$ or $\mathcal{K}_\alpha$.

In these two problems, we consider only those functions $f \in \mathcal{U}(\lambda)$ such that $f^{(k)}(0) = 0$ for $k = 2, \ldots, n$ $(n \geq 2)$.

2. Inclusion results for $U(\lambda)$

2.1. Theorem *If $f(z) = z + a_{n+1}z^{n+1} + \cdots$ belongs to $\mathcal{U}(\lambda)$ for some $n \geq 2$, then $f \in \mathcal{S}^*(\alpha)$ whenever $0 < \lambda \leq \lambda(\alpha, n)$, where*

$$\lambda(\alpha, n) = \begin{cases} \dfrac{(n-1)\sqrt{(1-2\alpha)[(n-1)^2+1-2\alpha]}}{(n-1)^2+1-2\alpha} & \text{if } \ 0 \leq \alpha \leq \dfrac{1}{n+1} \\[2ex] \dfrac{(n-1)(1-\alpha)}{n+\alpha-1} & \text{if } \ \dfrac{1}{n+1} < \alpha < 1. \end{cases}$$

Proof. Suppose that $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$. Then, by the representations (1.1) and (1.2), we see that

$$\frac{zf'(z)}{f(z)} = \frac{1+\lambda w(z)}{1-\lambda\int_0^1 \frac{w(tz)}{t^2}\,dt}$$

and therefore,

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)}-\alpha\right) = \frac{1+\lambda w(z)/(1-\alpha)+\big(\alpha\lambda/(1-\alpha)\big)\int_0^1 \frac{w(tz)}{t^2}\,dt}{1-\lambda\int_0^1 \frac{w(tz)}{t^2}\,dt}.$$

We need to show that $f \in \mathcal{S}^*(\alpha)$. To do this, according to a well-known result [R] and the last equation, it suffices to show that

$$\frac{1+\lambda w(z)/(1-\alpha)+\big(\alpha\lambda/(1-\alpha)\big)\int_0^1 \frac{w(tz)}{t^2}\,dt}{1-\lambda\int_0^1 \frac{w(tz)}{t^2}\,dt} \neq -iT, \quad T\in\mathbb{R},$$

which is equivalent to

$$\lambda\left[\frac{w(z)+\big(\alpha-i(1-\alpha)T\big)\int_0^1 \frac{w(tz)}{t^2}\,dt}{(1-\alpha)(1+iT)}\right] \neq -1, \quad T\in\mathbb{R}.$$

If we let

$$M = \sup_{z\in\Delta,\, w\in\mathcal{B}_n,\, T\in\mathbb{R}}\left|\frac{w(z)+\big(\alpha-i(1-\alpha)T\big)\int_0^1 \frac{w(tz)}{t^2}\,dt}{(1-\alpha)(1+iT)}\right|$$

then, in view of the rotation invariance property of the space $\mathcal{B}_n$, we obtain that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{if} \quad \lambda M \leq 1.$$

This observation shows that it suffices to find M. First we notice that

$$M \leq \sup_{T\in\mathbb{R}}\left\{\frac{1+\sqrt{\alpha^2+(1-\alpha)^2T^2}/(n-1)}{(1-\alpha)\sqrt{1+T^2}}\right\}.$$

Define ϕ: $[0,\,\infty)\to\mathbb{R}$ by

$$\phi(x) = \frac{n-1+\sqrt{\alpha^2+(1-\alpha)^2x}}{(n-1)(1-\alpha)\,\sqrt{1+x}}. \tag{2.2}$$

Differentiating ϕ with respect to x, we get

$$\phi'(x)=\frac{1}{(n-1)(1-\alpha)(1+x)}$$
$$\times\left[\sqrt{1+x}\left(\frac{(1-\alpha)^2}{2\sqrt{\alpha^2+(1-\alpha)^2x}}\right)-\left(\frac{n-1+\sqrt{\alpha^2+(1-\alpha^2)x}}{2\sqrt{1+x}}\right)\right],$$
$$x\in[0,\infty).$$

Case (i). Let $0<\alpha\leq 1/(n+1)$. Then we see that ϕ has its only critical point in the positive real line at

$$x_0=\frac{1}{(1-\alpha)^2}\left[\left(\frac{1-2\alpha}{n-1}\right)^2-\alpha^2\right].$$

Further, we easily observe that $\phi'(x)>0$ for $0\leq x<x_0$ and $\phi'(x)<0$ for $x>x_0$. Therefore, $\phi(x)$ attains its maximum at x_0 and hence,

$$\phi(x)\leq\phi(x_0)$$
$$=\frac{(n-1)^2+1-2\alpha}{(n-1)\sqrt{(1-2\alpha)[(n-1)^2+1-2\alpha]}}\quad\text{for}\quad x\geq 0. \tag{2.3}$$

Case (ii). Let $\alpha>1/(n+1)$ We can easily observe that.

$$\phi'(x)\leq 0\Longleftrightarrow 1-2\alpha\leq(n-1)\sqrt{\alpha^2+(1-\alpha)^2x},\quad\text{for}\quad x\geq 0.$$

This observation shows that $\phi'(x)\leq 0$ for all $x\geq 0$ whenever $1-2\alpha\leq\alpha(n-1)$. Therefore, if $\alpha\geq 1/(n+1)$, ϕ is decreasing on $[0,\infty)$ and hence,

$$\phi(x)\leq\phi(0)=\frac{n-1+\alpha}{(n-1)(1-\alpha)}\quad\text{for all}\quad x\geq 0. \tag{2.4}$$

The required conclusion follows from (2.3) and (2.4). □

The case $n=2$ of Theorem 2.1 has been obtained by Ponnusamy and Vasundhra in [PV].

2.5. Corollary *Let $n\geq 2$ be fixed and $f(z)=z+a_{n+1}z^{n+1}+\cdots\in\mathcal{U}(\lambda)$. Then, we have*
(i) *$f\in\mathcal{S}^*$ whenever $0<\lambda\leq(n-1)/\sqrt{(n-1)^2+1}$.*
(ii) *$f\in\mathcal{S}^*(1/2)$ whenever $0<\lambda\leq(n-1)/(2n-1)$.*

Notice that in the first case, the upper bound on λ is increasing to 1 as $n\to\infty$ and in the second case, the corresponding bound on λ is increasing to $1/2$ as $n \to \infty$.

2.6. Theorem *Let $\gamma \in (0, 1]$ and $n \geq 2$ be fixed. Let $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$,*

$$\lambda^*(\gamma, n) = \frac{(n-1)\sin(\pi\gamma/2)}{\sqrt{n^2 - 4(n-1)\sin^2(\pi\gamma/4)}}$$

and $\lambda^{\mathcal{R}}(\gamma, n)$ be the largest positive $\lambda > 0$ satisfying the equation

$$\sqrt{1-\lambda^2}\sin\frac{\pi\gamma}{2} = 2\left(\frac{\lambda}{n-1}\right)\sqrt{1-\left(\frac{\lambda}{n-1}\right)^2} + \lambda\cos\frac{\pi\gamma}{2}.$$

Then

(i) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{S}_\gamma$ *for* $0 < \lambda \leq \lambda^*(\gamma, n)$.

(ii) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{R}_\gamma$ *for* $0 < \lambda \leq \lambda^{\mathcal{R}}(\gamma, n)$.

Proof. Suppose that $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$ for some $\lambda \in (0, 1]$ and $n \geq 2$. Then, by the definition of $\mathcal{U}(\lambda)$, we have

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < \lambda, \quad z \in \Delta, \tag{2.7}$$

and, by (1.3),

$$\left|\frac{z}{f(z)} - 1\right| < \frac{\lambda|z|^n}{n-1} < \frac{\lambda}{n-1}, \quad z \in \Delta. \tag{2.8}$$

Therefore, it follows that

$$\left|\arg\left(\frac{z}{f(z)}\right)^2 f'(z)\right| < \arcsin(\lambda)$$

and

$$\left|\arg\left(\frac{z}{f(z)}\right)\right| < \arcsin\left(\frac{\lambda}{n-1}\right).$$

Now

$$\begin{aligned}\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| &\leq \left|\arg\left(\frac{z}{f(z)}\right)^2 f'(z)\right| + \left|\arg\left(\frac{z}{f(z)}\right)\right| \\ &< \arcsin(\lambda) + \arcsin\left(\frac{\lambda}{n-1}\right),\end{aligned}$$

which shows that

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi\gamma}{2} \quad \text{for} \quad \lambda \in \big(0,\, \lambda^*(n,\,\gamma)\big],$$

whenever $\lambda^*(n,\,\gamma)$ is the solution of the equation

$$\arcsin(\lambda) + \arcsin\left(\frac{\lambda}{n-1}\right) = \frac{\pi\gamma}{2}. \tag{2.9}$$

Simplifying (2.9) gives

$$\lambda^*(n,\,\gamma) = \frac{\sin(\pi\gamma/2)}{\sqrt{\big(n/(n-1)\big)^2 - \big(4/(n-1)\big)\sin^2(\pi\gamma/4)}}.$$

For the proof of the second part, we estimate

$$\begin{aligned}|\arg f'(z)| &\leq \left|\arg\left(\frac{z}{f(z)}\right)^2 f'(z)\right| + \left|\arg\left(\frac{f(z)}{z}\right)^2\right| \\ &< \arcsin\lambda + 2\arcsin\left(\frac{\lambda}{n-1}\right)\end{aligned}$$

so that $f \in \mathcal{R}_\gamma$ if and only if the right hand side of the previous equation equals $\pi\gamma/2$. This observation implies that

$$\arcsin(\lambda) + \arcsin\left(2\left(\frac{\lambda}{n-1}\right)\sqrt{1-\left(\frac{\lambda}{n-1}\right)^2}\right) = \frac{\pi\gamma}{2}$$

and the required result follows from simplifying this equation. □

3. Integral Transforms

In [PSV], Ponnusamy, Singh and Vasundhra introduced and studied the following integral transform in detail:

$$[I(f)](z) = F(z) = \frac{c}{z^{c-1}}\int_0^z \frac{\zeta^c}{f(\zeta)}\,d\zeta, \quad c > 0. \tag{3.1}$$

In [PSV], the authors found condition on $\lambda_0 = \lambda_0(|f''(0)|/2,\, \alpha,\, c)$ so that, for $0 < \lambda \leq \lambda_0$, $f \in \mathcal{U}(\lambda)$ belong to $\mathcal{S}^*_\alpha$, $\mathcal{K}_\alpha$ and $\mathcal{K}(\alpha)$, respectively. In this section, we address the same problem but now for functions $f \in \mathcal{U}(\lambda)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

where $n \geq 2$ is fixed.

3.2. Theorem *Let $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$ for some $\lambda > 0$ and $n \geq 2$. For $c > 0$ and $\alpha < 1$, let F be defined by (3.1). Then, $F \in \mathcal{S}^*_\alpha$ whenever c and λ are related by*

$$0 < \lambda \leq \frac{(1-\alpha)(n+c)(n-1)}{c(n+1-\alpha)}. \tag{3.3}$$

Proof. Assume that $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$. Then, it is a simple exercise to show that $F = I(f)$ defined by (3.1) satisfies the second order differential equation

$$2\left(1 - \frac{1}{c}\right)\frac{F(z)}{z} + \left(\frac{2}{c} - 1\right)F'(z) - \frac{1}{c}zF''(z)$$

$$= \left(\frac{z}{f(z)}\right)^2 f'(z) = 1 + \lambda w(z) \tag{3.4}$$

where $w \in \mathcal{B}_n$. It follows that

$$\frac{F(z)}{z} = 1 - \frac{c\lambda}{c+1}\int_0^1 \frac{w(tz)(1-t^{c+1})}{t^2}\, dt \tag{3.5}$$

and

$$F'(z) = 1 - \frac{c\lambda}{c+1}\int_0^1 \frac{w(tz)}{t^2}(1 - t^{c+1} + 1 + ct^{c+1})\, dt. \tag{3.6}$$

Using (3.5) and (3.6), we compute

$$\frac{zF'(z)}{F(z)} - 1 = -\frac{\big(c\lambda/(c+1)\big)\int_0^1 \frac{w(tz)}{t^2}(1+ct^{c+1})\, dt}{1 - \big(c\lambda/(c+1)\big)\int_0^1 \frac{w(tz)}{t^2}(1-t^{c+1})\, dt} \tag{3.7}$$

and therefore, as $|w(z)| \leq |z|^n$,

$$\begin{aligned}\left|\frac{zF'(z)}{F(z)} - 1\right| &< \frac{\big(c\lambda/(c+1)\big)\int_0^1 t^{n-2}(1+ct^{c+1})\, dt}{1 - \big(c\lambda/(c+1)\big)\int_0^1 t^{n-2}(1-t^{c+1})\, dt}\\ &= \frac{\big(c\lambda/(c+1)\big)\big[1/(n-1) + c/(c+n)\big]}{1 - \big(c\lambda/(c+1)\big)\big[1/(n-1) - 1/(c+n)\big]}\end{aligned}$$

$$\leq 1-\alpha, \quad \text{by (3.3)}.$$

This completes the proof. □

3.8. Corollary *Suppose that* $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$ *for some* λ *such that* $0 < \lambda \leq (n+c)(n-1)/[c(n+1)]$ *and* $n \geq 2$. *Then* F *defined by* (3.1) *satisfies the condition*

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| < 1.$$

For $c = 1$, Corollary 3.8 shows that if $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$, then

$$\int_0^z \frac{\zeta}{f(\zeta)}\, d\zeta \in \mathcal{S}_0^*$$

whenever $0 < \lambda \leq n - 1$.

3.9. Theorem *Let* $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$ *for* $n \geq 2$, $\alpha < 1$ *and* $c > 0$. *Then* F *defined by* (3.1) *is in* $\mathcal{K}(\alpha)$ *for* $0 < \lambda \leq \lambda_0 = \lambda_0(c, n, \alpha)$, *where*

$$\lambda_0 = \begin{cases} \dfrac{(c+n)(n-1)(1-\alpha)}{c(n+1)(n+1-\alpha)} & \text{for } \ 0<c\leq 2 \\[2ex] \dfrac{(1-\alpha)(n-1)(c+n)}{c\left[2t_0^{n-1}+(1-\alpha)(n+1)+(n-1)(n+c)\right]} & \\ & \text{for } \ 2<c\leq 2(2-\alpha) \\[2ex] \dfrac{(1-\alpha)(n-1)(c+n)}{c\left[2t_0^{n-1}+2(3-2\alpha)t_1^{n-1}+2c(n-1)+(n+1)\alpha+n^2-4n-1\right]} & \\ & \text{for } \ c>2(2-\alpha) \end{cases}$$

where

$$t_0 = \left(\frac{2}{c(c-1)}\right)^{1/(c+1)} \quad \textit{and} \quad t_1 = \left(\frac{2(1+2(1-\alpha))}{[c-2(1-\alpha)](c-1)}\right)^{1/(c+1)}.$$

Proof. From the differential equation (3.4) and the representation (3.6), we can easily see that

$$1 + \frac{zF''(z)}{F'(z)} = 1 - \frac{\frac{c\lambda}{c+1}\int_0^1 \frac{w(tz)}{t^2}(2 + c(1-c)t^{c+1})\, dt + c\lambda w(z)}{1 - \frac{c\lambda}{c+1}\int_0^1 \frac{w(tz)}{t^2}(2 - (1-c)t^{c+1})\, dt}. \quad (3.10)$$

Recall that F is convex of order α if and only if

$$1+\frac{zF''(z)}{F'(z)} \neq \alpha - i(1-\alpha)T, \quad T \in \mathbb{R}. \tag{3.11}$$

From (3.10), we observe that (3.11) is equivalent to

$$\frac{c\lambda}{c+1}\left[\frac{1}{(1-\alpha)(1+iT)}\int_0^1 \frac{w(tz)}{t^2}\left(2+c(1-c)t^{c+1}\right)dt \right.$$
$$\left. + \int_0^1 \frac{w(tz)}{t^2}\left(2-(1-c)t^{c+1}\right)dt\right] + \frac{c\lambda w(z)}{(1-\alpha)(1+iT)} \neq 1$$

which can be rewritten as

$$\frac{c\lambda}{2(c+1)(1-\alpha)}$$
$$\times\left[\left(\frac{1-iT}{1+iT}\right)\int_0^1 \frac{w(tz)}{t^2}\left[2+c(1-c)t^{c+1}\right]dt \right.$$
$$\left. + \int_0^1 \frac{w(tz)}{t^2}\left[2+4(1-\alpha)+\big(c-2(1-\alpha)\big)(1-c)t^{c+1}\right]dt\right]$$
$$+\frac{c\lambda w(z)}{(1-\alpha)(1+iT)} \neq 1.$$

By the triangle inequality and $|w(z)| \leq |z|^n$, the left hand side is bounded by

$$\frac{c\lambda}{(1-\alpha)(c+1)}\left[I_1+I_2+2(c+1)\right]$$

where

$$I_1 = \int_0^1 t^{n-2}\left|2+c(1-c)t^{c+1}\right| dt,$$
$$I_2 = \int_0^1 t^{n-2}\left|2\big(1+2(1-\alpha)\big)+\big(c-2(1-\alpha)\big)(1-c)t^{c+1}\right| dt.$$

These observations show that (3.11) holds whenever

$$\frac{c\lambda}{(1-\alpha)(c+1)}\left[I_1+I_2+2(c+1)\right] \leq 1. \tag{3.12}$$

We compute these integrals and show that the hypotheses imply that (3.12) holds. To do this, we first notice that, for $0 \le c \le 2$,

$$2 + c(1-c)t^{c+1} \ge t^{c+1}(2-c)(c+1)$$

and

$$\begin{aligned} &2(1+2(1-\alpha)) + (c-2(1-\alpha))(1-c)t^{c+1} \\ \ge\, &t^{c+1}(c+1)(2(2-\alpha)-c). \end{aligned}$$

Case (i). If $0 < c \le 2$, then both the integrands are nonnegative and therefore, evaluating the integrals in (3.12), we get the required result.
Case (ii). If $2(2-\alpha) \ge c > 2$, then we see that $2 + c(1-c)t^{c+1}$ takes negative and positive value in the interval [0, 1] and it has only one positive real root at

$$t_0 = \left[\frac{2}{c(c-1)}\right]^{1/(c+1)}.$$

Now $I_1 + I_2$ becomes

$$\begin{aligned} &\int_0^{t_0} t^{n-2}\big[2 + c(1-c)t^{c+1}\big]\,dt \\ &- \int_{t_0}^1 t^{n-2}\big[2 + c(1-c)t^{c+1}\big]\,dt \\ &+ \int_0^1 t^{n-2}\big[2(1+2(1-\alpha)) + (c-2(1-\alpha))(1-c)t^{c+1}\big]\,dt. \end{aligned}$$

Notice that the integrand in the second integral is nonnegative here. By evaluating these integrals we get the required result from the inequality (3.12).
Case (iii). Finally, for $c > 2(2-\alpha)$, we see that the function

$$2\big(1+2(1-\alpha)\big) + \big(c-2(1-\alpha)\big)(1-c)t^{c+1}$$

changes its sign once in the interval [0, 1] and has positive real root at

$$t_1 = \left[\frac{2\big(1+2(1-\alpha)\big)}{\big(c-2(1-\alpha)\big)(c-1)}\right]^{1/(c+1)}.$$

Therefore, the sum $I_1 + I_2$ equals

$$I_1' - I_1'' + I_2' - I_2''$$

where

$$I_1' = \int_0^{t_0} t^{n-2}\left[2 + c(1-c)t^{c+1}\right] dt$$

$$I_1'' = \int_{t_0}^{1} t^{n-2}\left[2 + c(1-c)t^{c+1}\right] dt$$

$$I_2' = \int_0^{t_1} t^{n-2}\left[2\big(1+2(1-\alpha)\big) + \big(c-2(1-\alpha)\big)(1-c)t^{c+1}\right] dt$$

$$I_2'' = \int_{t_1}^{1} t^{n-2}\left[2\big(1+2(1-\alpha)\big) + \big(c-2(1-\alpha)\big)(1-c)t^{c+1}\right] dt.$$

Again, evaluating these integrals we get the required result from the inequality (3.12). □

3.13. Theorem *Let $n \geq 2$ and $\lambda > 0$ be fixed. Let $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{U}(\lambda)$, $\alpha < 1$ and $c > 0$. Then F defined by* (3.1) *satisfies the condition*

$$\left|\frac{zF''(z)}{F'(z)}\right| < 1 - \alpha$$

for $0 < \lambda \leq \lambda_1 = \lambda_1(\alpha, c, n)$, where λ_1 is given by

$$\lambda_1 = \begin{cases} \dfrac{(n-1)(1-\alpha)(c+n)}{c(n+1)(n+1-\alpha)} & \text{for} \quad 0 < c \leq 2 \\[2ex] \dfrac{(1-\alpha)(c+n)(n-1)}{c\left[4t_0^{n-1} + (2c+n-1)(n-1) - (n+1)\alpha\right]} & \text{for} \quad c > 2. \end{cases}$$

Here t_0 is defined as in Theorem 3.9.

Proof. Recall (3.10)

$$\frac{zF''(z)}{F'(z)} = -\frac{\frac{c\lambda}{c+1}\int_0^1 \frac{w(tz)}{t^2}(2 + c(1-c)t^{c+1})\,dt + c\lambda w(z)}{1 - \frac{c\lambda}{c+1}\int_0^1 \frac{w(tz)}{t^2}(2 - (1-c)t^{c+1})\,dt}.$$

Then, since $|w(z)| \leq |z|^2$ for $z \in \Delta$, we have

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \frac{\frac{c\lambda}{c+1}\int_0^1 t^{n-2}|2 + c(1-c)t^{c+1}|\,dt + c\lambda}{1 - \frac{c\lambda}{c+1}\int_0^1 t^{n-2}(2 - (1-c)t^{c+1})\,dt}.$$

Writing

$$k(t) = 2 + c(1-c)t^{c+1} = 2(1-t^{c+1}) + t^{c+1}(2-c)(1+c)$$

we observe that $k(t) \geq 0$ for all $t \in [0, 1]$ whenever $0 < c \leq 2$. Note that $k(t)$ is a decreasing function of t for $c > 2$ and the only positive root t_0 in the interval $(0, 1)$ is given by

$$t_0 = \left(\frac{2}{c(c-1)}\right)^{1/(c+1)}.$$

Therefore, for $0 < c \leq 2$, it is easy to see that

$$I = \int_0^1 t^{n-2}|2 + c(1-c)t^{c+1}|\, dt = \frac{[(2-c)n + c](c+1)}{(n-1)(n+c)}$$

and for $c > 2$, we write

$$I = \int_0^{t_0} t^{n-2}(2 + c(1-c)t^{c+1})\, dt - \int_{t_0}^1 t^{n-2}(2 + c(1-c)t^{c+1})\, dt,$$

so that, by a simple computation, we get

$$I = \frac{c+1}{(n-1)(c+n)}\left[4t_0^{n-1} - (2-c)n - c\right].$$

Therefore, we have

$$\left|\frac{zF''(z)}{F'(z)}\right| < 1 - \alpha$$

where

$$1 - \alpha = \begin{cases} \dfrac{n(n+1)c\lambda}{(n-1)(n+c) - c\lambda(n+1)} & \text{if} \quad 0 < c \leq 2 \\ c\lambda\left[\dfrac{(n-1)(n+c) + 4t_0^{n-1} - (2-c)n - c}{(n-1)(n+c) - c\lambda(n+1)}\right] & \text{if} \quad c > 2 \end{cases}$$

which gives $\lambda_1(\alpha, n, c)$. □

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