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Extrinsic upper bounds for the first eigenvalue of elliptic operators

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Abstract. We consider operators defined on a Riemannian manifold M^m by $L_T(u) = -\mathbf{div}(T\nabla u)$ where T is a positive definite symmetric (1, 1)-tensor such that $\mathbf{div}(T) = 0$. We give an upper bound for the first nonzero eigenvalue $\lambda_{1,T}$ of L_T in terms of the second fundamental form of an immersion ϕ of M^m into a Riemannian manifold of sectional curvature bounded above by δ. We apply these results to a particular family of operators defined on hypersurfaces of the space forms and we prove a stability result.

Key words: r-th mean curvature, Reilly's inequality.

1. Introduction

Let (M^m, g) be a compact, connected m-dimensional Riemannian manifold. In this paper, we are interested in extrinsic upper bounds for the first nonzero eigenvalue of elliptic operators defined on (M^m, g) (i.e. in terms of the second fundamental form of an isometric immersion of (M^m, g) into an n-dimensional Riemannian manifold (N^n, h)). The elliptic second order differential operators L_T, which we are interested in, are of the form

$$L_T u = -\mathbf{div}_M(T\nabla^M u), \quad u \in C^\infty(M),$$

where T is a $(1, 1)$-tensor on M (which will be divergence-free and symmetric), and $\mathbf{div}_M$ and ∇^M denote respectively the divergence and the gradient with respect to the metric g. In the sequel, we will denote by $\lambda_{1,T}$, the first nonzero eigenvalue of such operator L_T.

When T is the identity, $L_T = L_{\mathrm{Id}}$ is nothing but the Laplace operator of (M^m, g). In this case, it is well known that if (M^m, g) is isometrically immersed in the simply connected space form $N^n(c)$ ($c = 0, 1, -1$ respectively for the Euclidean space $\mathbb{R}^n$, the sphere $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}^n$), then we have the following estimate of $\lambda_1 = \lambda_{1,\mathrm{Id}}$ in terms of the square of the length of the mean curvature

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$$\lambda_1 V(M) \leq m \int_M \left(|H|^2 + c\right) \, dv_g, \tag{1}$$

where dv_g and $V(M)$ denote respectively the Riemannian volume element and the volume of (M^m, g) and where H denotes the mean curvature vector of the immersion of (M^m, g) into $N^n(c)$. Furthermore, equality holds in (1) if and only if (M^m, g) is immersed as a minimal submanifold of some geodesic hypersphere of $N^n(c)$. For $c = 0$, this inequality was proved by Reilly ([17]) and can easily be extended to the spherical case $c = 1$ by considering the canonical embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ and by applying the inequality (1) for $c = 0$ to the obtained immersion of (M^m, g) in $\mathbb{R}^{n+1}$. For immersions of (M^m, g) in the hyperbolic space $\mathbb{H}^n$, Heintze ([14]) first proved an L_∞ equivalent of (1) and conjectured (1) which was finally obtained by El Soufi and Ilias ([9]). Note that, the estimates shown in ([14]) and ([9]) are given for immersions of (M^m, g) in a space which is not necessarily of constant sectional curvature.

Later, these estimates were extended to more general operators called L_r $(0 \leq r \leq n)$ defined on hypersurfaces (M^m, g) of $N^{m+1}(c)$. Let us first define these operators. Let ϕ be an isometric immersion of (M^m, g) into $N^{m+1}(c)$ and denote by A its shape (or Weingarten) operator. For any integer $r \in \{0, \ldots, n\}$, the $(1, 1)$-tensors T_r of Newton are defined inductively by: $T_0 = \mathrm{Id}$ and $T_r = S_r \, \mathrm{Id} - AT_{r-1}$, where S_r is the r-th elementary symmetric function of the eigenvalues of A (i.e. the principal curvatures). Note that T_r is a divergence free tensor because the ambient space is of constant curvature (see for instance [19]). The r-th mean curvature of ϕ is given by $H_r = \left(1/\binom{m}{r}\right) S_r$. Now, the operator L_r is defined by $L_r = L_{T_r}$ which is the linearized operator of the first variation of S_{r+1} ([18]). It is important for us to know when L_r is elliptic. Walter proved in [21] that if $H_{r+1} > 0$ and if the immersion ϕ is convex (i.e. the second fundamental form is semi-definite), then T_r is positively definite (i.e. L_r is elliptic). This result was strengthened by Barbosa and Colares ([6]). They proved without any convexity assumption that if $H_{r+1} > 0$ and if, in the case $c = 1$, $\phi(M)$ is contained in a hemisphere, then L_r is elliptic. For simplicity the first nonzero eigenvalue of L_r will be denoted $\lambda_{1,r}$ (instead of λ_{1,T_r}). The first extension of the Reilly inequality (1) to such operators L_r was obtained by Alencar, do Carmo and Rosenberg ([4] and [5]). They proved that, if (M^m, g) is an m-dimensional compact immersed hypersurface of the Euclidean space $\mathbb{R}^{m+1}$ and if $H_{r+1} > 0$, then

$$\lambda_{1,r} \int_M H_r \, dv_g \leq (m-r)\binom{m}{r} \int_M H_{r+1}^2 \, dv_g,$$

and equality holds if and only if (M^m, g) is a geodesic sphere of $\mathbb{R}^{m+1}$. In our paper [12] (Theorem 1.1, see also [11]), we obtained a similar optimal upper bound for $\lambda_{1,r}$ of hypersurfaces of any space form $N^{m+1}(c)$. We proved for all $0 \leq r \leq m-2$, that if $H_{r+1} > 0$ and if ϕ is convex (i.e. the second fundamental form is semi-definite) then

$$\lambda_{1,r} V(M) \leq (m-r)\binom{m}{r} \int_M \frac{H_{r+1}^2 + cH_r^2}{H_r} \, dv_g, \tag{2}$$

and equality holds if and only if ϕ immerses M as a geodesic sphere of $N^{m+1}(c)$.

Our approach to obtain such estimates was a generalization of the conformal technique used by El Soufi and Ilias and in this approach the convexity assumption was essential to obtain the estimate (2). Nevertheless, it is natural to ask if such estimates still valid without the convexity assumption. In this paper, to answer this purpose, we use a different approach inspired by the method of Heintze ([14]). In fact, an L_∞ estimate similar to (2) will be a consequence of an estimate (Theorem 1) obtained in a more general setting: for the operators L_T defined above and for ambient spaces not necessarly of constant sectional curvature.

Before stating the results, we need to define the following normal vector field H_T. If ϕ is an isometric immersion of (M^m, g) in (N^n, h) and B is its second fundamental form, then we define H_T at a point $x \in M$, by

$$H_T(x) = \sum_{i=1}^{m} B(Te_i, e_i),$$

where $(e_i)_{1 \leq i \leq m}$ is an orthonormal basis of the tangent space to M at x.

The main result of our paper is

Theorem 1 *Let (M^m, g) be a compact, connected, m-dimensional Riemannian manifold ($m \geq 2$) and let ϕ be an isometric immersion of (M^m, g) in an n-dimensional complete Riemannian manifold (N^n, h) of sectional curvature bounded above by δ. If $\delta \leq 0$, we assume that (N^n, h) is simply connected, and if $\delta > 0$, we assume that $\phi(M)$ is contained in a convex ball of radius less than or equal to $\pi/4\sqrt{\delta}$. Let T be a $(1, 1)$-tensor on M which is divergence-free and symmetric and let L_T be the associated elliptic* *operator defined on (M^m, g) as above. Then, we have*

$$\lambda_{1,T} \leq \frac{\sup_M |H_T|^2 + \sup_M \delta(\operatorname{tr}(T))^2}{\inf_M \operatorname{tr}(T)},$$

and if equality holds, then $\phi(M)$ is contained in a geodesic sphere.

When (N^n, h) is a simply connected space form and $T = T_r$, we deduce from this Theorem an estimate of $\lambda_{1,r}$ without the convexity assumption. In fact, we have

Corollary 1 *Let (M^m, g) be a compact, connected, orientable m-dimensional Riemannian manifold $(m \geq 2)$, immersed in a simply connected space form $(N^{m+1}(c), h)$ $(c = 0, -1, +1)$. Assume, if $c = 1$, that $\phi(M)$ is contained in a ball of radius $\pi/4$. If $H_{r+1} > 0$ for $r \in \{0, \ldots, m-1\}$, then we have*

$$\lambda_{1,r} \leq (m-r)\binom{m}{r} \frac{\sup_M H_{r+1}^2 + \sup_M (cH_r^2)}{\inf_M H_r},$$

and equality holds if and only if $\phi(M)$ is a geodesic sphere.

This last corollary has just been obtained independently by Alencar, do Carmo and Marques ([3]).

When $|H_T|$ is constant, we show a different estimate which is usefull in the proof of stability results; indeed, we have

Theorem 2 *Let (M^m, g) be a compact, connected, m-dimensional Riemannian manifold $(m \geq 2)$ and let ϕ be an isometric immersion of (M^m, g) in an n-dimensional complete Riemannian manifold (N^n, h) of sectional curvature bounded above by δ. If $\delta \leq 0$, we assume that (N^n, h) is simply connected, and if $\delta > 0$, we assume that $\phi(M)$ is contained in a convex ball of radius less than or equal to $\pi/4\sqrt{\delta}$. Let T be a $(1, 1)$-tensor on M which is divergence-free and symmetric and let L_T be the associated elliptic operator defined on (M^m, g) as above. Then, we have*

$$\lambda_{1,T} \leq \sup_M (|H_T||H| + \delta \operatorname{tr}(T)),$$

and if equality holds then $\phi(M)$ is contained in a geodesic sphere. Here, H is the classical mean curvature, i.e. $H = (1/m)\sum_{i=1}^m B(e_i, e_i)$.

As a consequence, we have

Corollary 2 *Let (M^m, g) be a compact, connected, orientable m-dimensional Riemannian manifold ($m \geq 2$), immersed in a space form $(N^{m+1}(c), h)$ ($c = 0, -1, +1$). Assume that, if $c = 1$, $\phi(M)$ is contained in a ball of radius $\pi/4$. If for $r \in \{0, \ldots, m-1\}$, H_{r+1} is a positive constant, then we have*

$$\lambda_{1,r} \leq \sup_M \left((m-r) \binom{m}{r} (H_{r+1} H_1 + c H_r) \right),$$

and equality holds if and only if $\phi(M)$ is a geodesic sphere.

This paper is organized as follows: the first part deals with the proofs of these theorems and corollaries. In the second part, we give a theorem (see Theorem 3) on the stability problem of hypersurfaces of constant r-th mean curvature in a space form, which is a consequence of a generalization of the Corollary 2 for Schrödinger operators of the form $L_r + q$.

The results of this paper were announced in the note [13].

2. Proofs of the results

Let (M^m, g) be a compact, connected m-dimensional Riemannian manifold isometrically immersed by ϕ in an n-dimensional Riemannian manifold (N^n, h) which sectional curvature is bounded by δ. The manifold M is endowed with a symmetric positive definite $(1, 1)$-tensor T of free divergence. The associated operator L_T defined by $L_T(u) = -\mathbf{div}(T \nabla^M u)$ is self-adjoint and elliptic, and we denote by $\lambda_{1,T}$ its first nonzero eigenvalue.

Let $p_0 \in N$ and $\exp_{p_0}$ the exponential map at this point. We consider $(x_i)_{1 \leq i \leq n}$ the normal coordinates of N centered at p_0 and for all $x \in N$, we denote by $r(x) = d(p_0, x)$, the geodesic distance between p_0 and x on (N^n, h). If $\delta > 0$, we assume that $\phi(M)$ lies in a convex ball around p_0 of radius less than or equal to $\pi/2\sqrt{\delta}$.

Let s_δ and c_δ be the functions defined by

$$s_\delta(r) = \begin{cases} \dfrac{1}{\sqrt{\delta}} \sin \sqrt{\delta} r & \text{if} \quad \delta > 0 \\ r & \text{if} \quad \delta = 0 \\ \dfrac{1}{\sqrt{|\delta|}} \sinh \sqrt{|\delta|} r & \text{if} \quad \delta < 0, \end{cases}$$

and

$$c_\delta(r) = \begin{cases} \cos\sqrt{\delta}r & \text{if} \quad \delta > 0 \\ 1 & \text{if} \quad \delta = 0 \\ \cosh\sqrt{|\delta|}r & \text{if} \quad \delta < 0. \end{cases}$$

We remark that $c_\delta^2 + \delta s_\delta^2 = 1$, $s'_\delta = c_\delta$ and $c'_\delta = -\delta s_\delta$.

In the sequel, we denote respectively by ∇^M and ∇^N, the gradients associated to (M^m, g) and (N^n, h). It is easy to see that the coordinates of $Z = s_\delta(r)\nabla^N r$ in the normal local frame are $\big((s_\delta(r)/r)x_i\big)_{1\le i\le n}$. Furthermore, the tangential and normal projection of a vector field X on the tangent bundle and the normal bundle to $\phi(M)$ will be denoted by X^t and X^n respectively.

We recall now some facts and properties of the exponential map. Let U, $V \in T_{p_0}N$ and $x \in N$. If we set $X = \exp_{p_0}^{-1}(x)$, then, we have

$$\sum_{i\le n} h_x\big(\nabla^N x_i, \big(d\exp_{p_0}\big)_X (U)\big) \\ \times h_x\big(\nabla^N x_i, \big(d\exp_{p_0}\big)_X (V)\big) = h_{p_0}(U, V). \quad (3)$$

On the other hand, $\exp_{p_0}$ is a radial isometry (Gauss lemma), that is, for each x of N, we have

$$h_x\big(\big(d\exp_{p_0}\big)_X (X), \big(d\exp_{p_0}\big)_X (U)\big) = h_{p_0}(X, U). \quad (4)$$

First, we begin by proving some lemmas.

Lemma 1 *For each x of M, we have*

$$\sum_{1\le i\le n} g_x\left(T\nabla^M\left(\frac{s_\delta(r)}{r}x_i\right), \nabla^M\left(\frac{s_\delta(r)}{r}x_i\right)\right) \\ \le \operatorname{tr}(T) - \delta g_x(TZ^t, Z^t), \quad (5)$$

and equality holds if (N^n, h) has constant sectional curvature δ.

Proof. We compute the left hand side of (5). Since

$$\nabla^M\left(\frac{s_\delta(r)}{r}x_i\right) = \frac{rc_\delta(r) - s_\delta(r)}{r^2}(\nabla^M r)x_i + \frac{s_\delta(r)}{r}\nabla^M x_i,$$

we have

$$\sum_{i=1}^n g_x\left(T\nabla^M\left(\frac{s_\delta(r)}{r}x_i\right), \nabla^M\left(\frac{s_\delta(r)}{r}x_i\right)\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \left(\frac{rc_\delta(r) - s_\delta(r)}{r^2} x_i \right)^2 g_x(T\nabla^M r, \nabla^M r) \\
&+ 2 \sum_{i=1}^{n} \frac{rc_\delta(r) - s_\delta(r)}{r^2} \frac{s_\delta(r)}{r} x_i g_x(T\nabla^M r, \nabla^M x_i) \\
&+ \sum_{i=1}^{n} \frac{s_\delta^2(r)}{r^2} g_x(T\nabla^M x_i, \nabla^M x_i).
\end{aligned}
$$

Using the fact that $\sum_{i=1}^n x_i \nabla^M x_i = r\nabla^M r$, we deduce

$$
\begin{aligned}
&\sum_{i=1}^{n} g_x \left(T\nabla^M \left(\frac{s_\delta(r)}{r} x_i \right), \nabla^M \left(\frac{s_\delta(r)}{r} x_i \right) \right) \\
&= \frac{s_\delta^2(r)}{r^2} \sum_{i=1}^{n} g_x(T\nabla^M x_i, \nabla^M x_i) \\
&+ \left[\frac{(rc_\delta(r) - s_\delta(r))^2}{r^2} + 2 \frac{rc_\delta(r) - s_\delta(r)}{r^2} s_\delta(r) \right] g_x(T\nabla^M r, \nabla^M r).
\end{aligned}
$$

After an easy computation and noting that $Z^t = s_\delta(r)\nabla^M r$, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n} g_x \left(T\nabla^M \left(\frac{s_\delta(r)}{r} x_i \right), \nabla^M \left(\frac{s_\delta(r)}{r} x_i \right) \right) \\
&= \frac{s_\delta^2(r)}{r^2} \sum_{i=1}^{n} g_x(T\nabla^M x_i, \nabla^M x_i) \\
&+ \left(1 - \frac{s_\delta^2(r)}{r^2} \right) g_x(T\nabla^M r, \nabla^M r) - \delta g_x(TZ^t, Z^t). \qquad (6)
\end{aligned}
$$

Since T is a positive symmetric $(1, 1)$-tensor, we can define a natural positive symmetric $(1, 1)$-tensor $\sqrt{T}$. Indeed, if $(e_i)_{1 \le i \le m}$ is an orthonormal basis at x which diagonalizes T in such a way that $T = \operatorname{diag}(\mu_1, \ldots, \mu_m)$, then $\sqrt{T}$ is defined at x by $\sqrt{T} = \operatorname{diag}(\sqrt{\mu_1}, \ldots, \sqrt{\mu_m})$.

Now let $(e_i)_{1 \le i \le m}$ be an orthonormal frame at x such that $\sqrt{T} e_m$ lies in the direction of $\nabla^M r$ and let e_m^* be a unit vector orthogonal to $\nabla^N r$ such that $\sqrt{T} e_m = \lambda \nabla^N r + \mu e_m^*$. Then (6) becomes

$$
\sum_{i=1}^{n} g_x \left(T\nabla^M \left(\frac{s_\delta(r)}{r} x_i \right), \nabla^M \left(\frac{s_\delta(r)}{r} x_i \right) \right)
$$

$$
\begin{aligned}
&= \frac{s_\delta^2(r)}{r^2} \sum_{i=1}^{n} \sum_{j=1}^{m} h_x(\nabla^N x_i, \sqrt{T} e_j)^2 \\
&\qquad + \left(1 - \frac{s_\delta^2(r)}{r^2}\right) g_x(T\nabla^M r, \nabla^M r) - \delta g_x(TZ^t, Z^t) \\
&= \frac{s_\delta^2(r)}{r^2} \sum_{i=1}^{n} \sum_{j=1}^{m} h_x(\nabla^N x_i, \sqrt{T} e_j)^2 \\
&\qquad + \frac{s_\delta^2(r)}{r^2} \sum_{i=1}^{n} \left(h_x(\nabla^N x_i, \lambda \nabla^N r) + h_x(\nabla^N x_i, \mu e_m^*)\right)^2 \\
&\qquad + \left(1 - \frac{s_\delta^2(r)}{r^2}\right) |\sqrt{T} \nabla^M r|_x^2 - \delta g_x(TZ^t, Z^t). \qquad (7)
\end{aligned}
$$

Now, by setting $v_j = \sqrt{T} e_j - h(\sqrt{T} e_j, \nabla^M r) \nabla^N r$ for all $j \leq m-1$, we rewrite the first term of the right hand side of (7) as

$$
\begin{aligned}
&\frac{s_\delta^2(r)}{r^2} \sum_{i=1}^{n} \sum_{j=1}^{m} h_x(\nabla^N x_i, \sqrt{T} e_j)^2 \\
&= \frac{s_\delta^2(r)}{r^2} \sum_{i=1}^{n} \sum_{j=1}^{m} \Big(h_x(\nabla^N x_i, v_j) + h_x(\sqrt{T} e_j, \nabla^M r) h_x(\nabla^N x_i, \nabla^N r)\Big)^2 \\
&= \frac{s_\delta^2(r)}{r^2} \sum_{i=1}^{n} \sum_{j=1}^{m} h_x(\nabla^N x_i, v_j)^2 \\
&\qquad + \frac{s_\delta^2(r)}{r^2} \sum_{i=1}^{n} \sum_{j=1}^{m} h_x(\sqrt{T} e_j, \nabla^M r)^2 h_x(\nabla^N x_i, \nabla^N r)^2 \\
&+2 \frac{s_\delta^2(r)}{r^2} \sum_{i=1}^{n} \sum_{j=1}^{m} h_x(\sqrt{T} e_j, \nabla^M r) h_x(\nabla^N x_i, v_j) h_x(\nabla^N x_i, \nabla^N r). \qquad (8)
\end{aligned}
$$

We compute each term of the right hand side of (8). Using the standard Jacobi field estimates (cf. for instance, Corollary 2.8, p. 153 of [20]), we have, for all v orthogonal to $\nabla^N r$,

$$
\frac{s_\delta^2(r)}{r^2} \left| \left(d(\exp_{p_0}^{-1})\right)_x (v) \right|_{p_0}^2 \leq |v|_x^2, \qquad (9)
$$

with equality if N has a constant sectional curvature δ. Since v_j is orthogonal to $\nabla^N r$ ($j = 1, \ldots, m-1$), and by applying successively (3) and (9), we obtain

$$\frac{s_\delta^2(r)}{r^2}\sum_{i=1}^{n}\sum_{j=1}^{m} h_x(\nabla^N x_i,\, v_j)^2 = \frac{s_\delta^2(r)}{r^2}\sum_{j=1}^{m-1}\left|\left(d\exp_{p_0}^{-1}\right)_x(v_j)\right|_{p_0}^2$$
$$\leq \sum_{j=1}^{m-1}|v_j|_x^2 = \sum_{j=1}^{m-1}|\sqrt{T}e_j|_x^2 - \sum_{j=1}^{m-1} h_x(\sqrt{T}e_j,\, \nabla^M r)^2. \tag{10}$$

Moreover, from (3) and (4), we have, for all v orthogonal to $\nabla^N r$,

$$\begin{aligned}
&\sum_{i=1}^{n} h_x(\nabla^N x_i,\, v) h_x(\nabla^N x_i,\, \nabla^N r)\\
&= h_{p_0}\left(\left(d(\exp_{p_0}^{-1}\right)_x(v), \left(d\exp_{p_0}^{-1}\right)_x(\nabla^N r)\right)\\
&= h_{p_0}\left(\left(d\exp_{p_0}^{-1}\right)_x(v),\, \frac{X}{r}\right)\\
&= h_x(v,\, \nabla^N(r)) = 0.
\end{aligned} \tag{11}$$

Hence, the last term of the right hand side of (8) vanishes identically. By substituting (10) in (8), and noting that $\sum_{i=1}^{n} h_x(\nabla^N x_i,\, \nabla^N r)^2 = 1$ by (3), we find

$$\begin{aligned}
&\frac{s_\delta^2(r)}{r^2}\sum_{i=1}^{n}\sum_{j=1}^{m} h_x(\nabla^N x_i,\, \sqrt{T}e_j)^2\\
&\leq \sum_{j=1}^{m-1}|\sqrt{T}e_j|_x^2 + \left(\frac{s_\delta^2(r)}{r^2} - 1\right)\sum_{j=1}^{m-1} h_x(\sqrt{T}e_j,\, \nabla^M r)^2\\
&= \operatorname{tr}(T) - |\sqrt{T}e_m|_x^2 + \left(\frac{s_\delta^2(r)}{r^2} - 1\right)\sum_{j=1}^{m-1} h_x(\sqrt{T}e_j,\, \nabla^M r)^2.
\end{aligned} \tag{12}$$

Furthermore, from (9) and (11), we deduce that

$$\begin{aligned}
&\frac{s_\delta^2(r)}{r^2}\sum_{i=1}^{n}\left(h_x(\nabla^N x_i,\, \lambda\nabla^N r) + h_x(\nabla^N x_i,\, \mu e_m^*)\right)^2\\
&= \frac{s_\delta^2(r)}{r^2}\lambda^2\sum_{i=1}^{n} h_x(\nabla^N x_i,\, \nabla^N r)^2 + \frac{s_\delta^2(r)}{r^2}\mu^2\sum_{i=1}^{n} h_x(\nabla^N x_i,\, e_m^*)^2\\
&\leq \lambda^2\frac{s_\delta^2(r)}{r^2} + \mu^2.
\end{aligned} \tag{13}$$

Finally, by substituting (12) and (13) into (7), we get

$$\begin{aligned}
&\sum_{i=1}^{n} g_x\left(T\nabla^M\left(\frac{s_\delta(r)}{r}x_i\right), \nabla^M\left(\frac{s_\delta(r)}{r}x_i\right)\right)\\
&\leq \mathrm{tr}(T) - |\sqrt{T}e_m|_x^2\\
&\quad + \left(\frac{s_\delta^2(r)}{r^2} - 1\right)\sum_{j=1}^{m-1} h_x(\sqrt{T}e_j,\, \nabla^M r)^2 + \lambda^2\frac{s_\delta^2(r)}{r^2} + \mu^2\\
&\quad + \left(1 - \frac{s_\delta^2(r)}{r^2}\right) g_x(\sqrt{T}\nabla^M r,\, e_m)^2\\
&\quad + \left(1 - \frac{s_\delta^2(r)}{r^2}\right)\sum_{i=1}^{m-1} g_x(\sqrt{T}\nabla^M r,\, e_i)^2 - \delta g_x(TZ^t,\, Z^t)\\
&= \mathrm{tr}(T) - |\sqrt{T}e_m|_x^2 + \lambda^2\frac{s_\delta^2(r)}{r^2} + \mu^2\\
&\quad + \left(1 - \frac{s_\delta^2(r)}{r^2}\right) g_x(\sqrt{T}\nabla^M r,\, e_m)^2 - \delta g_x(TZ^t,\, Z^t).
\end{aligned}$$

Here we have

$$g_x(\sqrt{T}\nabla^M r,\, e_m) = h_x(\sqrt{T}e_m,\, \nabla^N r) = \lambda$$

and

$$\lambda^2 + \mu^2 = |\sqrt{T}e_m|_x^2,$$

which yield the desired inequality, and if $(N^n,\, h)$ is of constant sectional curvature, all the inequalities above are in fact equalities. $\square$

Now, we will prove

Lemma 2 *For all symmetric divergence-free positive definite* $(1,\, 1)$*-tensors* T *on* M*, we have*

$$\mathbf{div}_M(TZ^t) \geq (\mathrm{tr}(T))c_\delta + h(Z,\, H_T),$$

and if T *is the identity and* $(N^n,\, h)$ *has a constant sectional curvature equal to* δ*, then equality holds.*

Proof. We use the same local frame as in the proof of Lemma (1) and we compute $\mathbf{div}_M(TZ^t)$ in this frame by using the fact that T is a free divergence tensor (i.e., $\sum_{i=1}^n \left(\nabla^M_{e_i}T\right)e_i = 0$.)

$$\begin{aligned}
\mathbf{div}_M\big(TZ^t\big) &= \sum_{i=1}^{n} g_x\big(\nabla^M_{e_i}\big(TZ^t\big),\, e_i\big)\\
&= \sum_{i=1}^{n} g_x\big(\big(\nabla^M_{e_i}T\big)Z^t,\, e_i\big)\\
&= \sum_{i=1}^{n} g_x\big(\nabla^M_{e_i}Z^t,\, Te_i\big)\\
&= \sum_{i=1}^{n} h_x\big(\nabla^N_{e_i}Z^t,\, Te_i\big)\\
&= \sum_{i=1}^{n} h_x\big(\nabla^N_{e_i}Z,\, Te_i\big) - \sum_{1\le i\le m} h_x\big(\nabla^N_{e_i}Z^n,\, Te_i\big)\\
&= \sum_{i=1}^{n} h_x\big(\nabla^N_{e_i}Z,\, Te_i\big) + \sum_{1\le i\le m} h_x\big(Z,\, B(Te_i,\, e_i)\big)\\
&= \sum_{i=1}^{n} h_x\big(\nabla^N_{e_i}Z,\, Te_i\big) + h_x(Z,\, H_T).
\end{aligned} \tag{14}$$

Now, we want to estimate $\sum_{i=1}^{n} h_x\big(\nabla^N_{e_i}Z,\, Te_i\big)$. We first have

$$\begin{aligned}
&\sum_{i=1}^{m} h_x\big(\nabla^N_{e_i}Z,\, Te_i\big)\\
&= \sum_{1\le i\le m} h_x\big(\nabla^N_{e_i}\big(s_\delta\nabla^N r\big),\, Te_i\big)\\
&= c_\delta h_x\big(\nabla^N r,\, T\big(\nabla^N r\big)^t\big) + s_\delta \sum_{i=1}^{m} h_x\big(\nabla^N_{e_i}\nabla^N r,\, Te_i\big)\\
&= c_\delta h_x\big(T\big(\nabla^N r\big)^t,\, \big(\nabla^N r\big)^t\big) + s_\delta \sum_{i=1}^{m} h_x\big(\nabla^N_{\sqrt{T}e_i}\nabla^N r,\, \sqrt{T}e_i\big)
\end{aligned} \tag{15}$$

Using the standard Jacobi field estimates (see Lemma 2.9 p. 153 of [20]), we can find a lower bound of the last term of (15). Indeed, we have for all vector ξ orthogonal to $\nabla^N r$ at x, the inequality

$$h_x(\nabla^N_\xi \nabla^N r,\, \xi) \ge \frac{c_\delta}{s_\delta}|\xi|^2_x,$$

and equality holds if N has constant sectional curvature δ. Thus, we have

$$\begin{aligned}
&\sum_{i=1}^{m} h_x\big(\nabla^N_{\sqrt{T}e_i}\nabla^N r,\, \sqrt{T}e_i\big)\\
&= \sum_{i=1}^{m-1} h_x\big(\nabla^N_{\sqrt{T}e_i}\nabla^N r,\, \sqrt{T}e_i\big) + h_x\big(\nabla^N_{\sqrt{T}e_m}\nabla^N r,\, \sqrt{T}e_m\big)\\
&\geq \frac{c_\delta}{s_\delta}\sum_{i=1}^{m-1}\big|\sqrt{T}e_i\big|_x^2 + \mu^2 h_x\big(\nabla^N_{e_m^*}\nabla^N r,\, e_m^*\big)\\
&\geq \frac{c_\delta}{s_\delta}\sum_{i=1}^{m-1}\big|\sqrt{T}e_i\big|_x^2 + \mu^2\frac{c_\delta}{s_\delta},
\end{aligned}$$

and inserting this inequality into (15), we obtain

$$\sum_{i=1}^{n} h_x\big(\nabla^N_{e_i}Z,\, Te_i\big) \geq c_\delta\big|\sqrt{T}(\nabla^N r)^t\big|_x^2 + c_\delta\sum_{i=1}^{m-1}\big|\sqrt{T}e_i\big|_x^2 + \mu^2 c_\delta. \quad (16)$$

Now we have

$$\begin{aligned}
\lambda^2 &= h_x\big(\sqrt{T}e_m,\, \nabla^N r\big)^2 = h_x\big(\sqrt{T}e_m,\, (\nabla^N r)^t\big)^2\\
&= h_x\big(e_m,\, \sqrt{T}(\nabla^N r)^t\big)^2\\
&\leq \big|\sqrt{T}(\nabla^N r)^t\big|_x^2
\end{aligned}$$

and if T is the identity, equality holds in this last inequality. Furthermore, it is easy to verify that

$$\lambda^2 + \mu^2 = \big|\sqrt{T}e_m\big|_x^2.$$

Thus, inequality (15) becomes

$$\begin{aligned}
\sum_{i=1}^{n} h_x\big(\nabla^N_{e_i}Z,\, Te_i\big) &\geq c_\delta\lambda^2 + c_\delta\sum_{i=1}^{m-1}\big|\sqrt{T}e_i\big|_x^2 + \mu^2 c_\delta\\
&= \operatorname{tr}(T)c_\delta,
\end{aligned}$$

and inserting this last inequality into (14), we complete the proof of Lemma 2. □

Lemma 3 *We have*

$$\delta\int_M g_x(TZ^t,\, Z^t)\,dv_g \geq \int_M \operatorname{tr}(T)c_\delta^2\,dv_g - \int_M |H_T| s_\delta c_\delta\,dv_g.$$

Proof. If $\delta \neq 0$, then

$$\begin{aligned}\delta \int_M g(TZ^t, Z^t)\, dv_g =& \frac{1}{\delta}\int_M g\big(T\nabla^M c_\delta(r), \nabla^M c_\delta(r)\big)\, dv_g\\ =& -\frac{1}{\delta}\int_M \mathbf{div}_M\big(T\nabla^M c_\delta(r)\big)c_\delta(r)\, dv_g\\ =& \int_M \mathbf{div}_M(TZ^t)c_\delta\, dv_g\\ \geq& \int_M c_\delta^2\, \mathrm{tr}(T)\, dv_g - \int_M |H_T|s_\delta c_\delta\, dv_g,\end{aligned}$$

where the last inequality follows from Lemma 2. Moreover, if $\delta = 0$, then $c_\delta(r) = 1$ and we have

$$0 = \int_M \mathbf{div}_M(TZ^t)c_\delta\, dv_g \geq \int_M c_\delta^2\, \mathrm{tr}(T)\, dv_g - \int_M |H_T|s_\delta c_\delta\, dv_g.$$

This concludes the proof. □

We are in position to give the proof of our results.

Proof of Theorem 1. Let $p_0 \in N$ and $r(x) = d(p_0, x)$, where $r(x)$ is the geodesic distance between p_0 and x. We will use $(s_\delta(r)/r)x_i$ as test functions in the variational characterization of $\lambda_{1,T}$ but the mean of these functions must be zero. For this purpose, we use a standard argument used by Chavel and Heintze ([14] and [8]). Indeed, let Y be a vector field defined by

$$Y_q = \int_M \frac{s_\delta\big(d(q, p)\big)}{d(q, p)} \exp_q^{-1}(p)\, dv_g(p) \in T_qN,\ q \in M,$$

From the fixed point theorem of Brouwer, there exists a point $p_0 \in N$ such that $Y_{p_0} = 0$ and consequently, for a such p_0, $\int_M \big(s_\delta(r)/r\big)x_i\, dv_g = 0$. But for $\delta > 0$, we must assume $\phi(M)$ is contained in a ball of radius $\pi/4\sqrt{\delta}$. Indeed, in this case $\phi(M)$ lies in a ball of center p_0 (the point p_0 such that $Y_{p_0} = 0$) with a radius less or equal to $\pi/2\sqrt{\delta}$ (this hypothesis is necessary in the proof of the preceding Lemmas). It follows from the above and the variational characterization of $\lambda_{1,T}$, that

$$\begin{aligned}\lambda_{1,T}\int_M s_\delta^2(r)\, dv_g =& \lambda_{1,T}\int_M |Z|^2\, dv_g\\ =& \lambda_{1,T}\int_M \sum_{i=1}^n \left(\frac{s_\delta(r)}{r}x_i\right)^2 dv_g\end{aligned}$$

$$
\begin{aligned}
&\leq \int_M \sum_{i=1}^n L_T\left(\frac{s_\delta(r)}{r}x_i\right)\frac{s_\delta(r)}{r}x_i\,dv_g\\
&= \int_M \sum_{i=1}^n g\left(T\nabla^M\left(\frac{s_\delta(r)}{r}x_i\right), \nabla^M\left(\frac{s_\delta(r)}{r}x_i\right)\right)dv_g
\end{aligned}
$$

and using Lemmas 1 and 3, we obtain

$$
\begin{aligned}
\lambda_{1,T}\int_M s_\delta^2\,dv_g &\leq \int_M \operatorname{tr}(T)\,dv_g - \delta\int_M g(TZ^t,\, Z^t)\,dv_g\\
&\leq \int_M \operatorname{tr}(T)\,dv_g - \int_M \operatorname{tr}(T)c_\delta^2\,dv_g + \int_M |H_T|s_\delta c_\delta\,dv_g\\
&\leq \delta\int_M \operatorname{tr}(T)s_\delta^2\,dv_g + \sup_M |H_T|\int_M s_\delta c_\delta\,dv_g\\
&\leq \delta\int_M \operatorname{tr}(T)s_\delta^2\,dv_g\\
&\quad + \sup_M |H_T|\sup_M\left(\frac{1}{\operatorname{tr}(T)}\right)\int_M \operatorname{tr}(T)s_\delta c_\delta\,dv_g.
\end{aligned}
$$

Furthermore, from Lemma 2, we deduce

$$
\begin{aligned}
\int_M \operatorname{tr}(T)s_\delta c_\delta\,dv_g &\leq \int_M s_\delta^2|H_T|\,dv_g + \int_M s_\delta\mathbf{div}_M(TZ^t)\,dv_g\\
&= \int_M s_\delta^2|H_T|\,dv_g + \int_M \mathbf{div}_M(s_\delta TZ^t)\,dv_g\\
&\qquad - \int_M g(\nabla^M s_\delta,\, TZ^t)\,dv_g\\
&= \int_M s_\delta^2|H_T|\,dv_g - \int_M c_\delta s_\delta g(\nabla^M r,\, T\nabla^M r)\,dv_g.
\end{aligned}
$$

Since c_δ and s_δ are positive functions (because for $\delta > 0$, $\phi(M) \subset B(p_0, \pi/2\sqrt{\delta})$), we deduce that

$$
\int_M \operatorname{tr}(T)s_\delta c_\delta\,dv_g \leq \int_M s_\delta^2|H_T|\,dv_g,
$$

and if equality holds, then $\phi(M)$ lies in a geodesic sphere. Finally, we have

$$
\begin{aligned}
\lambda_{1,T}\int_M s_\delta^2\,dv_g &\leq \delta\int_M \operatorname{tr}(T)s_\delta^2\,dv_g + \frac{\sup_M |H_T|}{\inf_M \operatorname{tr}(T)}\int_M |H_T|s_\delta^2\,dv_g\\
&\leq \left(\sup_M(\delta\operatorname{tr}(T)) + \frac{\sup_M |H_T|^2}{\inf_M \operatorname{tr}(T)}\right)\int_M s_\delta^2\,dv_g,
\end{aligned}
$$

and this completes the proof of Theorem 1. □

Proof of Theorem 2. We assume now that $|H_T|$ is constant. Then from the first step of the preceding proof, it follows that

$$\lambda_{1,T}\int_M s_\delta^2\,dv_g \le \int_M (\delta\,\mathrm{tr}(T))s_\delta^2\,dv_g + |H_T|\int_M s_\delta c_\delta\,dv_g.$$

Now applying Lemma 2 to the identity, we get

$$\int_M \mathbf{div}(Z^t)s_\delta\,dv_g \ge m\int_M \left(c_\delta s_\delta + h(\nabla^N r, H)s_\delta^2\right)\,dv_g,$$

and an easy computation gives

$$\int_M \mathbf{div}(Z^t)s_\delta\,dv_g = -\int_M s_\delta c_\delta|\nabla^M r|^2\,dv_g \le 0,$$

From this, we deduce

$$\int_M s_\delta c_\delta\,dv_g \le \int_M |H|s_\delta^2\,dv_g,$$

thus

$$\lambda_{1,T}\int_M s_\delta^2\,dv_g \le \int_M (|H_T||H| + \delta\,\mathrm{tr}(T))s_\delta^2\,dv_g,$$

which completes the proof. □

Proof of Corollaries 1 *and* 2. Let $(M^m,\, g)$ be a compact, connected and orientable m-dimensional Riemannian manifold ($m \ge 2$) isometrically immersed by ϕ in a simply connected space form $N^{m+1}(c)$ ($c = 0$, 1 or -1 respectively for $\mathbb{R}^{m+1}$, $\mathbb{S}^{m+1}$ or $\mathbb{H}^{m+1}$) and let A be the Weingarten operator associated to the second fundamental form of the immersion. When $c \le 0$, assumptions of Theorems 1 and 2 are trivially verified. For $c = 1$, we assume that $\phi(M)$ lies in a ball of radius $\pi/4$. Since $H_{r+1} > 0$ with $\phi(M)$ contained in a hemisphere when $c = 1$, then L_r is elliptic ([6]). Finally, under these hypotheses, the corollaries follow from the Theorems by applying them to the special (1, 1)-tensors T_r defined in the introduction and by using the following relations: $\mathrm{tr}(T_r) = (m-r)\binom{m}{r}H_r$ and $\mathrm{tr}(AT_r) = (m-r)\binom{m}{r}H_{r+1}$ ([17]).

Furthermore, from Theorems, if inequalities expressed in corollaries are equalities, then $\phi(M)$ is a geodesic sphere. Conversely, if $\phi(M)$ is a geodesic variations F and for a constant κ,

$$\int_M (-H_{r+1} + \kappa) f \, dv_g = 0.$$

Thus, M is a constant $(r+1)$-th mean curvature hypersurface if and only if, ϕ is a critical point of A_r, with constant balance volume, and in this case,

$$A_r''(0) = \int_M (L_r(f) + qf) \, f \, dv_g, \tag{20}$$

where we put $q = k(r+1)H_{r+2} - m\big(k(r)/(r+1)\big)H_1 H_{r+1} - ck(r)H_r$. We give now definition for the stability of hypersurfaces with constant r-th mean curvature H_{r+1} following [4] and [6].

Definition 1 Let (M^m, g) be an orientable compacte hypersurface of $(N^{m+1}(c), h)$ with H_{r+1} constant. Then (M^m, g) is said to be H_{r+1}-*stable* if $A_r''(0) \geq 0$ for all variations such that $V(t) = 0$.

From Theorem 2, we have the following theorem

Theorem 3 *Let (M^m, g) be an orientable compact Riemannian manifold of dimension $m \geq 2$ and ϕ an isometric immersion of (M^m, g) in $\mathbb{H}^{m+1}$. Then, if H_{r+1} is a nonnegative constant, then M is H_{r+1}-stable if and only if $\phi(M)$ is a geodesic sphere.*

Remark 2 Note that Alencar, do Carmo and Rosenberg have proved this stability result for hypersurfaces of $\mathbb{R}^{m+1}$ ([4] and [5]). Barbosa and Colares extend it to hypersurfaces of $\mathbb{H}^{m+1}$ and of an open hemisphere of $\mathbb{S}^{m+1}$, but without using estimates of the eigenvalues of the second variation operator ([6]). In [11] and [12], we proved independently a stability result for convex hypersurfaces of $\mathbb{H}^{m+1}$ and $\mathbb{S}^{m+1}$, by using an upper bound of the second eigenvalue of the second variation operator (of A_r).

Proof of Theorem 3. A straightforward computation shows that the geodesic spheres are H_{r+1}-stable. In fact, such spheres are totally umbilical. This implies that $H_r = H_1^r$ and

$$L_r = \binom{m-1}{r} H_1^r \Delta.$$

Variations $(F_t)_t$ for which $V(t) \equiv 0$ are the one satisfying that $\int_M f \, dv_g = 0$

([6]). For such variations, we have from (20):

$$\begin{aligned} A_r''(0) &= \binom{m-1}{r} H_1^r \int_M \left(f\Delta f - m(H_1^2+c)f^2\right) dv_g \\ &\geq \binom{m-1}{r} H_1^r \int_M \left(\lambda_1 - m(H_1^2+c)\right) f^2 \, dv_g = 0, \end{aligned}$$

where λ_1 denotes the first nonzero eigenvalue of the Laplacian. This proves the stability of the geodesic spheres. Conversely, suppose that ϕ is H_{r+1}-stable. This implies that $A_r''(0) \geq 0$ for all variations (F_t) such that $V(t) \equiv 0$, and from (20), we have

$$\int_M (L_r + q)(f) f \, dv_g \geq 0$$

for any smooth function f on M such that $\int_M f \, dv_g = 0$. Hence, by the min-max principle, we deduce that

$$\lambda_2(L_r + q) \geq 0,$$

and from the inequality (17) of the Remark 1, we have

$$\begin{aligned} &\lambda_2(L_r + q) \int_M s_\delta^2 \, dv_g \\ &\leq \int_M \left(k(r) H_{r+1} H_1 + k(r+1) H_{r+2} - m\frac{k(r)}{r+1} H_1 H_{r+1}\right) s_\delta^2 \, dv_g, \end{aligned}$$

and consequently

$$0 \leq \int_M \left(k(r) H_{r+1} H_1 + k(r+1) H_{r+2} - m\frac{k(r)}{r+1} H_1 H_{r+1}\right) s_\delta^2 \, dv_g.$$

Now, using the fact that $H_{r+2} \leq H_1 H_{r+1}$, with equality at umbilical points ([4]), we obtain

$$\begin{aligned} &k(r) H_{r+1} H_1 + k(r+1) H_{r+2} - m\frac{k(r)}{r+1} H_1 H_{r+1} \\ &\leq \left(k(r) + k(r+1) - m\frac{k(r)}{r+1}\right) H_1 H_{r+1}, \end{aligned}$$

and it is easy to verify that

$$k(r) + k(r+1) - m\frac{k(r)}{r+1} = 0.$$

Thus finally, we get

$$\int_M (H_{r+2} - H_{r+1}H_1)s_\delta^2\, dv_g = 0,$$

hence M is totally umbilical and then it is a geodesic sphere. □

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Univalent functions with missing Taylor coefficients

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Abstract. For $n \geq 2$, let $\mathcal{U}(\lambda)$ denote the class of all analytic functions f in the unit disc Δ of the form

$$f(z) = z + a_{n+1}z^{n+1} + \cdots$$

satisfying the condition

$$f'(z)\ \frac{z}{f(z)}^{\ 2} - 1\ < \lambda, \quad z \in \Delta.$$

In this paper, among other results, we find condition on λ so that each function in $\mathcal{U}(\lambda)$ is starlike, strongly starlike or convex of some order. In addition, we discuss the mapping properties of the integral operator

$$[I(f)](z) = \frac{c}{z^{c-1}} \int_0^z \frac{\zeta^c}{f(\zeta)}\, d\zeta, \quad c > 0.$$

Key words: Univalent, starlike and convex functions, and integral transform.

1. Introduction

Let $\mathcal{H}$ denote the class of all functions f analytic in the unit disc $\Delta = \{z\colon |z| < 1\}$. For $n \geq 1$, a positive integer, let

$$\mathcal{A}_n = \left\{ f \in \mathcal{H}\colon f(z) = z + \sum_{k=1}^{\infty} a_{n+k} z^{n+k} \right\}$$

with $\mathcal{A}_1 = \mathcal{A}$, where $\mathcal{A}$ is referred to as the normalized analytic functions in the unit disc. A function $f \in \mathcal{A}$ is called starlike in Δ if $f(\Delta)$ is starlike with respect to the origin. The class of all starlike functions is denoted by $\mathcal{S}^* \equiv \mathcal{S}^*(0)$. For $\alpha < 1$, we define

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A}\colon \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \Delta \right\}$$

and is called the class of all starlike functions of order α. Clearly, $\mathcal{S}^*(\alpha) \subsetneq \mathcal{S}^*$ for $0 < \alpha < 1$. For $0 < \alpha \leq 1$, a function $f \in \mathcal{A}$ is called strongly starlike

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