Hokkaido Mathematical Journal Vol. 38 (2009) p. 97–110

Relative Tchebychev hypersurfaces which are also translation hypersurfaces

Thomas Binder

(Received June 11, 2007; Revised September 12, 2007)

Abstract. The class of relative Tchebychev hypersurfaces extends the class of affine hyperspheres in Blaschke geometry. This paper aims at finding examples for relative Tchebychev hypersurfaces among the translation hypersurfaces. A complete local classification is given. Specializing from relative to centroaffine geometry, the list reduces to paraboloids.

Key words: Centroaffine geometry, relative geometry, Tchebychev[1] hypersurfaces, translation hypersurfaces

1. Introduction

Measuring area in relative geometry is not straightforward due to the existence of several volume forms. The two main notions of area relate to the induced volume form ω and the volume form $\widehat{\omega}$ of the relative metric. The ω-area is meaningful in Euclidean and Blaschke geometry, with the classical Euler-Lagrange equation $H = 0$ in both cases.

However, in centroaffine geometry, there are no hypersurfaces with extremal ω-area. Considering the $\widehat{\omega}$-area instead yields a meaningful Euler-Langrange equation, namely trace $\widehat{\nabla}^c T^c = 0$, where $\widehat{\nabla}^c$ and T^c denote the Levi-Civita connection of the centroaffine metric and the centroaffine Tchebychev vector field, respectively. This result is due to Wang [16].

Thus, $\widehat{\nabla}^c T^c$ formally replaces the shape operator when comparing this equation to Euclidean or Blaschke geometry, the main difference being the different volume measure. Centroaffine hypersurfaces where $\widehat{\nabla}^c T^c$ is a multiple of the identity are called centroaffine Tchebychev hypersurfaces ([8], [9]). This analogy was pursued further by a generalization to relative geometry: based on Codazzi relations, [7] introduces the concept of a relative Tchebychev hypersurface. A global result was obtained in dimension $n = 2$

2000 Mathematics Subject Classification : Primary 53A15.

[1] For consistency we retain the transliteration hitherto used in connection with this hypersurface class. The correct transliteration is *Chebyshev*.

in [10].

In the Blaschke geometry, the relative Tchebychev hypersurfaces are exactly the affine hyperspheres. The latter class alone is so large that classifications succeed only under strong additional assumptions. We therefore restrict the search to translation hypersurfaces. Translation hypersurfaces provide a large class of examples which is easily accessible. Due to this fact, they were studied intensively; see e.g. [2], [3], [4], [5], [13], [15].

Section 2 introduces terminology from relative geometry; for a detailed introduction to the subject see [12] or [14]. Section 3 derives the Blaschke invariants of a translation hypersurface. The last Section 4 gives a local classification of relative Tchebychev hypersurfaces which are also translation hypersurfaces. This result is contained in the author's dissertation [1].

2. Relative geometry and Tchebychev hypersurfaces

Throughout this article we assume that $n \geq 2$. A hypersurface immersion $\mathfrak{x} : M \to \mathbb{R}^{n+1}$ of an n-dimensional manifold into real affine space is called a relative hypersurface, if there is a transversal field $\mathfrak{y}$ such that $\overline{\nabla}_p \mathfrak{y}$ has its image in $d\mathfrak{x}(T_pM)$, where $\overline{\nabla}$ is the canonical flat affine connection on $\mathbb{R}^{n+1}$. For any vector fields u and v tangent to M we have the decompositions

$$\overline{\nabla}_u d\mathfrak{x}(v) = d\mathfrak{x}(\nabla_u v) + h(u,v)\mathfrak{y}, \quad d\mathfrak{y}(u) = -d\mathfrak{x}(Su).$$

Such an $\mathfrak{y}$ is called a relative normal. The regularity of the symmetric tensor field h is independent of the choice of $\mathfrak{y}$, it is a property of $\mathfrak{x}$ only. In the regular case, we call $\mathfrak{x}$ a non-degenerate hypersurface and h the relative metric induced by $\mathfrak{y}$. From now on we will always assume that $\mathfrak{x}$ is non-degenerate.

∇ is a torsion-free Ricci-symmetric affine connection called the induced connection. S is called the shape operator. Its trace $nH := \operatorname{trace} S$ is the mean curvature. A relative hypersphere is a hypersurface with $S = H \,\mathrm{id}$ and is called proper if $H \neq 0$, and improper if $H = 0$.

Let $\widehat{\nabla}$, $\widehat{\mathrm{Hess}}$, Δ denote the Levi-Civita connection, the Hessian and the Laplacian regarding h, respectively. Define the difference tensor C by $C(u,v) := \nabla_u v - \widehat{\nabla}_u v$. The Tchebychev vector field T is obtained from C by

$$nh(T, u) := \operatorname{trace}\{v \mapsto C(v, u)\}.$$

By $^\flat$ we denote the operation of lowering an index with respect to h. The conormal connection ∇^* is given implicitly by

$$w(h(u, v)) = h(\nabla_w u, v) + h(u, \nabla^*_w v).$$

The conormal $\mathfrak{Y}$ corresponding to $\mathfrak{y}$ is the dual vector field satisfying

$$\langle \mathfrak{Y}, d\mathfrak{x}\rangle = 0 \qquad \text{and} \qquad \langle \mathfrak{Y}, \mathfrak{y}\rangle = 1,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing of $(\mathbb{R}^{n+1})^*$ and $\mathbb{R}^{n+1}$. Normals and conormals are in a bijective correspondence for non-degenerate hypersurfaces; we talk about a relative normalization of $\mathfrak{x}$ provided either one is given. Any two relative normals $\mathfrak{y}_1$, $\mathfrak{y}_2$ of $\mathfrak{x}$ are related by a function $\varphi \in C^\infty(M)$ such that $\mathfrak{Y}_1 = \pm e^\varphi \mathfrak{Y}_2$ for the corresponding conormals. Finally, the relative support function is defined by $\rho := -\langle \mathfrak{Y}, \mathfrak{x}\rangle$.

The Blaschke normal $\mathfrak{y}^e$ is unique up to sign; it is characterized by $T = 0$ up to a constant factor. Relative hyperspheres with respect to the Blaschke geometry are called affine hyperspheres.

On a non-degenerate hypersurface the position vector is transversal up to a nowhere dense set. Hence, by continuity, one can define the centroaffine normal $\mathfrak{y}^c := -\mathfrak{x}$. It is characterized by $S = \mathrm{id}$. A proper relative sphere is the underlying hypersurface with its centroaffine normal up to a constant factor. Henceforth Blaschke and centroaffine invariants will be marked by ^{e} and ^{c}, respectively.

A hypersurface $\mathfrak{x}$ with relative normal $\mathfrak{y}$ is called a relative Tchebychev hypersurface if the relative Tchebychev operator $L := \frac{1}{2}S - \frac{n}{n+2}\widehat{\nabla}T$ is a multiple of the identity. We have the following characterization.

Lemma *Let $\mathfrak{x} : M \to \mathbb{R}^{n+1}$ be a hypersurface with relative normal $\mathfrak{y}$. The transition to the Blaschke normalization is given by $\mathfrak{Y} = e^\varphi \mathfrak{Y}^e$, where $\varphi = \log \frac{\rho}{\rho^e}$, and $\mathfrak{Y}$ is the relative conormal belonging to $\mathfrak{y}$. The following statements are equivalent*:

(i) $\mathfrak{x}$ *is a relative Tchebychev hypersurface with respect to* $\mathfrak{y}$,

(ii) $\widehat{\nabla}\tilde{C}^\flat$ *is totally symmetric, where*

$$\tilde{C}(u, v) := C(u, v) - \frac{n}{n+2}\big(T^\flat(u)v + T^\flat(v)u + h(u, v)T\big)$$

is the traceless part of C.

(iii) $\widehat{\mathrm{Hess}}\varphi - \frac{1}{n}(\Delta\varphi)h = -(S^\flat - Hh)$ *in terms of the relative geometry, and*

(iv) $S^{\flat e} - H^e h^e = d\varphi(C^e(\cdot,\cdot))$ *in terms of the Blaschke geometry.*

It is surprising that the order of the differential equation in terms of φ reduces from two to one when reformulating the problem in the Blaschke geometry.

Proof. (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) was shown in [7]. The equivalence of (iii) and (iv) can be deduced using change of relative normalization formulas

$$\begin{aligned}
\widehat{\mathrm{Hess}}\varphi(u,v) &= \widehat{\mathrm{Hess}}^e\varphi(u,v) - d\varphi(u)d\varphi(v) + \frac{1}{2}\|\mathrm{grad}^e\varphi\|_e^2 h^e(u,v),\\
\Delta\varphi &= e^{-\varphi}\left(\Delta^e\varphi + \frac{n-2}{2}\|\mathrm{grad}^e\varphi\|_e^2\right),\\
S^\flat(u,v) &= S^{\flat e}(u,v) - h^e(\nabla^e_u \mathrm{grad}^e\varphi, v) + d\varphi(u)d\varphi(v),\\
H &= e^{-\varphi}\left(H^e - \frac{1}{n}\Delta^e\varphi + \frac{1}{n}\|\mathrm{grad}^e\varphi\|_e^2\right),
\end{aligned}$$

where $\|\cdot\|_e^2$ is taken with respect to h^e. Plugging these into (iii) and simplifying we get

$$\widehat{\mathrm{Hess}}^e\varphi(u,v) = -S^{\flat e}(u,v) + h^e(\nabla^e_u \mathrm{grad}^e\varphi, v) - H^e h^e(u,v).$$

Rewrite the middle term on the right-hand side as

$$h^e(\nabla^e_u \mathrm{grad}^e\varphi, v) = u(h^e(\mathrm{grad}^e\varphi, v)) - h^e(\mathrm{grad}^e\varphi, {\nabla^*}^e_u v) = \mathrm{Hess}^{*e}\varphi(u,v),$$

where Hess^* denotes the Hessian with respect to ∇^*. We conclude the proof by

$$\begin{aligned}
d\varphi(C^e(u,v)) &= d\varphi(\widehat{\nabla}^e_u v - {\nabla^*}^e_u v) = (\widehat{\mathrm{Hess}}^e - \mathrm{Hess}^{*e})(u,v)\\
&= -(S^{\flat e} + H^e h^e)(u,v).
\end{aligned}$$

□

Remark 1 (i) Any hyperquadric is relative Tchebychev with respect to an arbitrary relative normal. Note that hyperquadrics are characterized by $C^e = 0$, cf. [14].

(ii) In the Blaschke geometry, the Tchebychev hypersurfaces are exactly the affine hyperspheres.

(iii) A hypersurface is centroaffine Tchebychev if and only if T^c is a conformal vector field with respect to h^c: In centroaffine geometry, the definition translates to

$$\widehat{\nabla}^c_u T^c = \lambda u \qquad \text{for all } u,$$

which characterizes T^c as a closed conformal vector field on the Riemannian manifold (M, h^c). Note that T is always closed; in centroaffine terms we have $T^c = \frac{n+2}{2n}\mathrm{grad}^c \log|\rho^e|$. We can rewrite the PDE in (iii) of the Lemma as

$$\widehat{\mathrm{Hess}}^c \log|\rho^e| - \frac{1}{n}(\Delta^c \log|\rho^e|)h^c = 0, \tag{1}$$

which is known from Riemannian geometry (see [6]). The PDE (1) implies that locally strongly convex Tchebychev hypersurfaces with complete centroaffine metric are conformally flat. Using results from Riemannian geometry [8] gives a classification for this case, which provides examples besides the ones studied here.

3. Translation hypersurfaces

The immersion $\mathfrak{x} : M \to \mathbb{R}^{n+1}$ is called a translation hypersurface if it admits a decomposition into the sum of n planar curves. Precisely, any translation hypersurface can be parametrized locally as

$$\mathfrak{x} : (u^1, \ldots, u^n) \mapsto \left(\alpha_1(u^1), \ldots, \alpha_n(u^n), \sum_{l=1}^{n} \beta_l(u^l)\right),$$

where α_i and β_i are differentiable functions. Here and in the following, we do not use the Einstein summation convention. We will now remove the freedom in the parametrization. To calculate the Blaschke invariants we use the fact that $h^e = |\det\Lambda|^{-\frac{1}{n+2}}\Lambda$ where $\Lambda_{ij} := \det(\mathfrak{x}_1, \ldots, \mathfrak{x}_n, \mathfrak{x}_{ij})$. We get

$$\Lambda_{ii} = \alpha_1'\alpha_2'\cdots\alpha_{i-1}'\alpha_{i+1}'\cdots\alpha_n'(\alpha_i'\beta_i'' - \beta_i'\alpha_i'')$$

for $1 \leq i \leq n$ and $\Lambda_{ij} = 0$ for $i \neq j$. In order to obtain a conformal parametrization of h^e we assume

$$\alpha_i' \beta_i'' - \beta_i' \alpha_i'' = \varepsilon_i \alpha_i' \quad \text{or equivalently,} \quad \beta_i'' - \beta_i' \frac{\alpha_i''}{\alpha_i'} = \varepsilon_i$$

for some $\varepsilon_i = \pm 1$ from now on. Define a function γ by

$$e^\gamma := |\alpha_1' \dots \alpha_n'|^{\frac{2}{n+2}}.$$

To keep notation short, we write $\gamma_i := \partial_i \gamma$ from now on. Then

$$\gamma_i = \frac{2}{n+2} \frac{\alpha_i''}{\alpha_i'}.$$

Note that all second order mixed partials of γ vanish. The Blaschke metric can be written as

$$h^e = \varepsilon e^\gamma \sum_{l=1}^n \varepsilon_l du^l \otimes du^l.$$

The choice of $\varepsilon = \pm 1$ corresponds to fixing the orientation of the Blaschke normal. Applying the conformal change formula for a Levi-Civita connection we get $\widehat{\nabla}^e$:

$$\widehat{\nabla}_i^e \partial_i = \frac{1}{2} \gamma_i \partial_i - \frac{1}{2} \varepsilon_i \sum_{l \neq i} \varepsilon_l \gamma_l \partial_l, \qquad \widehat{\nabla}_i^e \partial_j = \frac{1}{2} (\gamma_i \partial_j + \gamma_j \partial_i),$$

where $i \neq j$. Using the identity $n\mathfrak{y}^e = \Delta^e \mathfrak{x}$ we get that the Blaschke normal is then given by

$$\mathfrak{y}^e = \frac{2\varepsilon}{n+2} e^{-\gamma} \left(\varepsilon_1 \alpha_1'', \dots, \varepsilon_n \alpha_n'', \sum_{l=1}^n \varepsilon_l \beta_l'' - \frac{n-2}{2} \right). \tag{2}$$

It is now easy to guess the Blaschke conormal and to compute the support function:

$$\begin{aligned} \mathfrak{Y}^e &= \varepsilon e^{\gamma}\Big(-\frac{\beta_1'}{\alpha_1'},\dots,-\frac{\beta_n'}{\alpha_n'},1\Big),\\ \rho^e &= -\langle \mathfrak{Y}^e,\mathfrak{x}\rangle = \varepsilon e^{\gamma}\sum_{l=1}^{n}\frac{\alpha_l'\beta_l-\alpha_l\beta_l'}{\alpha_l'}. \end{aligned} \tag{3}$$

Using (2) we can calculate the remaining invariants of the Blaschke geometry:

$$\nabla_i^e\partial_i = \frac{n}{2}\gamma_i\partial_i - \varepsilon_i\sum_{l\neq i}\varepsilon_l\gamma_l\partial_l, \qquad \nabla_i^e\partial_j = 0,$$

$$S_{ii}^e = -\Big(\gamma_i' + \frac{n}{2}\gamma_i^2\Big), \qquad S_{ij}^e = \gamma_i\gamma_j,$$

$$C^e(\partial_i,\partial_i) = \frac{n-1}{2}\gamma_i\partial_i - \frac{1}{2}\varepsilon_i\sum_{l\neq i}\varepsilon_l\gamma_l\partial_l, \quad C^e(\partial_i,\partial_j) = -\frac{1}{2}(\gamma_i\partial_j+\gamma_j\partial_i),$$

where $1\leq i,j\leq n$ and $i\neq j$ in all cases.

Remark 2 The Blaschke metric of a translation hypersurface is conformally flat. What is the relative normal associated to the flat metric? Transforming $\mathfrak{Y} = e^{-\gamma}\mathfrak{Y}^e$, a short calculation yields $\mathfrak{y} = (0,\dots,0,\varepsilon)$. Hence for any translation hypersurface there exists a relative normal such that it is an improper relative sphere with flat relative metric with respect to that normal. Using this special relative normal for a graph hypersurface, the relative metric is called Calabi metric.

Example 1 (Quadrics) The quadric condition $C^e = 0$ translates into $\gamma = \text{const}$, i.e. $\gamma_i = 0$ for all $1\leq i\leq n$. Up to translation and affine equivalences we get $\alpha_i(u^i) = u^i$ and $\beta_i(u^i) = \frac{1}{2}\varepsilon_i(u^i)^2$ for each i, which leads to the paraboloids $\mathfrak{x}^{n+1} = \sum_{l=1}^n \varepsilon_l(\mathfrak{x}^l)^2$.

Example 2 Assume γ is a linear function but $\gamma\neq\text{const}$. Then there is some index $1\leq i\leq n$ such that $\gamma_j = \frac{n+2}{2}a_j = \text{const}$ for all $1\leq j\leq i$ and $a_j\in\mathbb{R}\backslash\{0\}$ and $\gamma_j = 0$ for all $i<j\leq n$. Up to translations and affine equivalences we get $\alpha_j(u^j) = e^{a_ju^j}$ and $\beta_j(u^j) = d_je^{a_ju^j} - \frac{\varepsilon_j}{a_j}u^j$ for $j\leq i$. More affine transformations lead to

$$\frac{\varepsilon_1}{a_1^2}\log \mathfrak{x}^1 + \cdots + \frac{\varepsilon_i}{a_i^2}\log \mathfrak{x}^i + \frac{1}{2}\varepsilon_{i+1}(\mathfrak{x}^{i+1})^2 + \cdots + \frac{1}{2}\varepsilon_n(\mathfrak{x}^n)^2 = \mathfrak{x}^{n+1}.$$

We abbreviate this hypersurface by $D_1(a_1, \ldots, a_i, 0, \ldots, 0)$.

Example 3 Suppose γ is a quadratic function. Then for $1 \leq i \leq n$ we can write $\gamma_i(u^i) = \frac{n+2}{4}a_i u^i$ with $a_i \in \mathbb{R}$ after a translation in the parameters. Since γ is quadratic, we may assume $a_i \neq 0$ for a particular i in the following. In this case, $\alpha_i'(u^i) = e^{a_i(u^i)^2}$ and $\beta_i'' - 2a_i u^i \beta_i' = \varepsilon_i$. In the generic case, this cannot be integrated elementarily; integration involves the error function $\int e^{-\xi^2}d\xi$. We write $D_2(a_1, \ldots, a_n)$ for this hypersurface.

Example 4 (Affine hyperspheres) In this context, affine spheres were studied by Manhart [11]. For a classification, we exclude quadrics (Example 1). Then $S_{ij}^e = 0$ for $i \neq j$ demands that exactly one γ_i is nonzero, let this be γ_1. Hence $S^e = 0$ and γ_1 satisfies $\gamma_1' + \frac{n}{2}\gamma_1^2 = 0$ while $\gamma_2 = \cdots = \gamma_n = 0$. Solving for γ_1 and applying a parameter translation we get $\gamma_1(u^1) = \frac{2}{nu^1}$. Integrating further yields $\alpha_1(u^1) = (u^1)^{\frac{2n+2}{n}}$ and $\beta_1(u^1) = c(u^1)^{\frac{2n+2}{n}} - \varepsilon_1\frac{n}{4}(u^1)^2$, where $c \in \mathbb{R}$ and where we have omitted translation constants. The result is the non-quadric improper affine hypersphere

$$x^{n+1} = -\varepsilon_1\frac{n}{4}(x^1)^{\frac{n}{n+1}} + cx^1 + \frac{1}{2}\sum_{l=2}^{n}\varepsilon_l(x^l)^2.$$

4. Classification

Theorem *Let $\mathfrak{x} : M \to \mathbb{R}^{n+1}$ be a relative Tchebychev hypersurface with respect to the conormal $\mathfrak{Y} = e^{\varphi}\mathfrak{Y}^e$. Suppose $\mathfrak{x}$ is also a translation hypersurface. Then $\mathfrak{x}$ is affinely equivalent to an open part of one of the following hypersurfaces:*

- (i) *The paraboloids from Example* 1 *with an arbitrary relative normal,*
- (ii) *$D_1(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in \mathbb{R}$ and at least two a_i are nonzero, with the relative normal described in Remark* 2*,*
- (iii) *$D_2(\varepsilon_1 a, \ldots, \varepsilon_n a)$, where $n \geq 3$ and $a \in \mathbb{R}\backslash\{0\}$, with the relative normal described in Remark* 2*,*
- (iv) *in dimension $n \geq 3$ the hypersurfaces*

$$x^{n+1} = \varepsilon_1 \operatorname{sgn}(1-c)(x^1)^{\frac{2}{1+c}} + \varepsilon_2 (x^2)^{\frac{2}{1-c}} + \sum_{l=3}^{n} \varepsilon_l (x^l)^2, \quad or$$

$$x^{n+1} = \varepsilon_1 |x^1| \log|x^1| + \varepsilon_2 e^{x^2} + \sum_{l=3}^{n} \varepsilon_l (x^l)^2,$$

where $1 \neq c > 0$ is a constant and where the relative normal is determined up to a constant factor,

(v) *in dimension $n \geq 2$ the class of hypersurfaces $x^{n+1} = f(x^1) + \sum_{l=2}^{n} \varepsilon_l (x^l)^2$, where f is an arbitrary differentiable function on the real line, with a relative normal which is determined by f up to a constant factor,*

(vi) *the class of surfaces given by*

$$\gamma_1'^{\,2} = \frac{c}{4}\gamma_1^4 + \tilde{c}_1 \gamma_1^2 + \bar{c}_1 \quad and \quad \gamma_2'^{\,2} = \frac{c}{4}\gamma_2^4 + \tilde{c}_2 \gamma_2^2 + \bar{c}_2,$$

where $c, \tilde{c}_i, \bar{c}_i \in \mathbb{R}$, $i = 1, 2$. Again, the relative normal is determined up to a constant factor.

Proof. Rewriting the Tchebychev condition in (iv) of the Lemma in terms of translation hypersurfaces gives

$$\begin{aligned} (1-n)\varepsilon_i \left(\frac{2}{n}\gamma_i' + \gamma_i^2 \right) &+ \sum_{l \neq i} \varepsilon_l \left(\frac{2}{n}\gamma_l' + \gamma_l^2 \right) \\ &= (n-1)\varepsilon_i \gamma_i \varphi_i - \sum_{l \neq i} \varepsilon_l \gamma_l \varphi_l, \end{aligned} \tag{4}$$

$$-2\gamma_i \gamma_j = \gamma_i \varphi_j + \gamma_j \varphi_i, \tag{5}$$

where $1 \leq i, j \leq n$ and $i \neq j$. Based on dimension and on the number of vanishing partials of γ we distinguish some cases.

Case A Suppose $n \geq 2$ and $\gamma_i = 0$ for all $1 \leq i \leq n$. Then we get Class (i).

Case B Suppose that $\gamma_1 \neq 0$ whereas $\gamma_3, \ldots, \gamma_n = 0$ (γ_2 is not yet restricted). Then from (5) for $1 < i \leq n$ we can express

$$\varphi_i = -\gamma_i\left(2 + \frac{\varphi_1}{\gamma_1}\right). \tag{6}$$

Obviously, $\varphi = \varphi(u^1, u^2)$. For $n \geq 3$ and $j \notin \{1, i\}$, by inserting (6) into (5) we get $\gamma_i\gamma_j(1+\frac{\varphi_1}{\gamma_1}) = 0$, which is by assumption always true. The remaining equations of (4) are

$$-(n-1)\varepsilon_1\left(\frac{2}{n}\gamma_1' + \gamma_1^2\right) + \varepsilon_2\left(\frac{2}{n}\gamma_2' + \gamma_2^2\right) = (n-1)\varepsilon_1\gamma_1\varphi_1 - \varepsilon_2\gamma_2\varphi_2, \tag{7}$$

$$-(n-1)\varepsilon_2\left(\frac{2}{n}\gamma_2' + \gamma_2^2\right) + \varepsilon_1\left(\frac{2}{n}\gamma_1' + \gamma_1^2\right) = (n-1)\varepsilon_2\gamma_2\varphi_2 - \varepsilon_1\gamma_1\varphi_1, \tag{8}$$

$$\varepsilon_1\left(\frac{2}{n}\gamma_1' + \gamma_1^2\right) + \varepsilon_2\left(\frac{2}{n}\gamma_2' + \gamma_2^2\right) = -(\varepsilon_1\gamma_1\varphi_1 + \varepsilon_2\gamma_2\varphi_2), \tag{9}$$

where we have set $i = 1$ and $i = 2$ in (7) and (8), respectively. For $i \geq 3$, Equation (4) always looks like (9).

Case B1 First suppose $n \geq 2$ and $\gamma_2 = 0$, then we obtain $\varphi = \varphi(u^1)$ and $\varphi_1 = -(\frac{2}{n}\gamma_1' + \gamma_1^2)/\gamma_1$. Integration yields $\varphi = -\frac{2}{n}\log|\gamma_1| - \gamma + c$. Setting $f := \beta_1 \circ \alpha_1^{-1}$ we get Class (v). In this subcase any hypersurface is relative Tchebychev with respect to a normalization which is unique up to scaling.

Case B2 If $n \geq 3$ and $\gamma_2 \neq 0$, then the linear system (7)–(9) has the unique solution $\varphi_i = -(\frac{2}{n}\gamma_i' + \gamma_i^2)/\gamma_i$, $i = 1, 2$. Inserting this into (5) we see this is only possible for a solution of

$$\gamma_1' = -\tilde{c}\gamma_1^2, \qquad \gamma_2' = \tilde{c}\gamma_2^2 \qquad \text{for some } \tilde{c} \in \mathbb{R}. \tag{10}$$

Without loss of generality assume $\tilde{c} \geq 0$. If $\tilde{c} = 0$, we get Class (ii). Otherwise, the solutions of (10) are $\gamma_1(u^1) = 1/(\tilde{c}u^1 + \tilde{c}_1)$ and $\gamma_2(u^2) = -1/(\tilde{c}u^2 + \tilde{c}_2)$. By a parameter translation we may assume $\tilde{c}_1 = 0 = \tilde{c}_2$. Setting $c := \frac{n+2}{2\tilde{c}}$ we get

$$\text{for } c \neq 1 \quad \alpha_1 = |u^1|^{1+c}, \quad \beta_1 = \frac{\varepsilon_1}{2(1-c)}(u^1)^2 + \frac{c_1}{1+c}|u^1|^{1+c},$$

$$\alpha_2 = |u^2|^{1-c}, \quad \beta_2 = \frac{\varepsilon_2}{2(1+c)}(u^2)^2 + \frac{c_2}{1-c}|u^2|^{1-c},$$

$$\text{and for } c=1 \qquad \alpha_1 = |u^1|^{1+c}, \quad \beta_1 = \frac{\varepsilon_1}{2}(u^1)^2\log|u^1| + \frac{2c_1-\varepsilon_1}{4}(u^1)^2,$$
$$\alpha_2 = \log|u^2|, \quad \beta_2 = \frac{\varepsilon_2}{4}(u^2)^2 + c_2\log|u^2|,$$

where $c_1, c_2 \in \mathbb{R}$. After more affine transformations we arrive at the hypersurfaces in Class (iv).

Case C Let $n \geq 3$ and without loss of generality $\gamma_1, \gamma_2, \gamma_3 \neq 0$. Setting $i = 2$ and $j = 3$ in Case B it follows that $\varphi_1 = -\gamma_1$, hence $\varphi_i = -\gamma_i$ for any i. This implies $\varphi = -\gamma + c$ for some $c \in \mathbb{R}$. The remaining system (4) reads

$$(1-n)\varepsilon_i\gamma_i' + \sum_{l\neq i}\varepsilon_l\gamma_l' = 0, \qquad 1 \leq i \leq n.$$

This is a linear system in the derivatives of γ, with zero determinant and rank $n-1$. The trivial solution gives Class (ii), while the non-zero constant solutions for γ_i' lead to Class (iii).

Case D Finally, assume $n = 2$ and both $\gamma_1, \gamma_2 \neq 0$. We have the system of PDEs

$$-\varepsilon_1(\gamma_1' + \gamma_1^2) + \varepsilon_2(\gamma_2' + \gamma_2^2) = \varepsilon_1\gamma_1\varphi_1 - \varepsilon_2\gamma_2\varphi_2,$$
$$-2\gamma_1\gamma_2 = \gamma_1\varphi_2 + \gamma_2\varphi_1.$$

Consider this as a linear system in the partials of φ. It has determinant $\varepsilon_1\gamma_1^2 + \varepsilon_2\gamma_2^2$. If this is zero, then we get $\gamma_1 = \pm\gamma_2 = \text{const}$ and $\varepsilon_1 = -\varepsilon_2$, which leads to Class (ii). Suppose the determinant is not zero. Then we obtain

$$\varphi_1 = \gamma_1\left(\frac{\varepsilon_2\gamma_2' - \varepsilon_1\gamma_1'}{\varepsilon_1\gamma_1^2 + \varepsilon_2\gamma_2^2} - 1\right), \tag{11}$$

$$\varphi_2 = \gamma_2\left(\frac{\varepsilon_1\gamma_1' - \varepsilon_2\gamma_2'}{\varepsilon_1\gamma_1^2 + \varepsilon_2\gamma_2^2} - 1\right). \tag{12}$$

Consequently, the Lie bracket for φ reads

$$\left(\varepsilon_1\gamma_1''\gamma_2 - \varepsilon_2\gamma_1\gamma_2''\right)\left(\varepsilon_1\gamma_1^2 + \varepsilon_2\gamma_2^2\right) = 2\gamma_1\gamma_2\left({\gamma_1'}^2 - {\gamma_2'}^2\right).$$

Taking the second derivative $\partial_1\partial_2$ we can separate variables; the result is

$$\gamma_i'''\gamma_i - \gamma_i''\gamma_i' = c\gamma_i'\gamma_i^3, \quad c \in \mathbb{R},\ i = 1, 2.$$

We can integrate this once as

$$\gamma_i'' = \frac{c}{2}\gamma_i^3 + \tilde{c}_i\gamma_i, \quad \tilde{c}_i \in \mathbb{R},\ i = 1, 2.$$

By integrating the latter equation once more we obtain Class (vi). $\square$

Note that the affine hyperspheres from Example 4 are contained in Class (v) of the Theorem. Finally we will show that, except for the paraboloids from Example 1, there are no other centroaffine Tchebychev hypersurfaces contained in the list of the Theorem.

The centroaffine normalization is characterized by $e^{-\varphi} = |\rho^e|$. By inserting (3) we observe that if $\partial_i(\varphi + \gamma) = 0$ for some i, then also

$$0 = \partial_i \frac{\alpha_i'\beta_i - \alpha_i\beta_i'}{\alpha_i'} = \frac{\beta_i^2}{\alpha_i'}\partial_i \frac{\alpha_i}{\beta_i}. \tag{13}$$

The latter would imply that $\alpha_i = c\beta_i$, which is impossible. Thus, centroaffine Tchebychev hypersurfaces are not contained in Cases B and C. It remains to examine Case D. Assume there is a centroaffine Tchebychev surface contained in Class (vi) of the Theorem. Taking derivatives $\partial_1\partial_2$ in $e^{-\varphi} = |\rho^e|$ we obtain $\partial_1\partial_2 e^{-(\gamma+\varphi)} = 0$, or equivalently,

$$(\gamma_1 + \varphi_1)(\gamma_2 + \varphi_2) = \gamma_{21} + \varphi_{21}.$$

Eliminating the derivatives of $\gamma + \varphi$ using (11) and (12) yields

$$\begin{aligned} \frac{-\gamma_1\gamma_2(\varepsilon_1\gamma_1' - \varepsilon_2\gamma_2')^2}{(\varepsilon_1\gamma_1^2 + \varepsilon_2\gamma_2^2)^2} &= \partial_2 \frac{\gamma_1(\varepsilon_1\gamma_1' - \varepsilon_2\gamma_2')}{\varepsilon_1\gamma_1^2 + \varepsilon_2\gamma_2^2} \\ &= \frac{-\varepsilon_2\gamma_1\gamma_2\big(2\gamma_2'(\varepsilon_2\gamma_2' - \varepsilon_1\gamma_1') - \frac{\gamma_2''}{\gamma_2}(\varepsilon_1\gamma_1^2 + \varepsilon_2\gamma_2^2)\big)}{(\varepsilon_1\gamma_1^2 + \varepsilon_2\gamma_2^2)^2}. \end{aligned} \tag{14}$$

Taking derivatives $\partial_1\partial_2$ of the latter equation we get that $\gamma_1'\gamma_2' = 0$ is a necessary condition. Without loss of generality we may assume $\gamma_2 = a =$

const $\neq 0$. Inserting this into (14) it follows that also γ_1 = const. From (11) and (12) it is now clear that $\varphi + \gamma$ = const, which gives a contradiction when applying the same argument as in (13) for $i = 1$ or 2.

References

[1] Binder T., *Two Codazzi problems for relative surfaces.* PhD thesis, Fak. II, Inst. f. Math., TU Berlin, 2002. Shaker Verlag Aachen 2002, ISBN 3-8265-9938-1.

[2] Binder T., *Projectively flat affine surfaces.* J. Geom. **79** (2004), 32–45.

[3] Gigena S., *The Pick invariant for decomposable hypersurfaces.* Math. Notae **37** (1993/94), 87–104, in Spanish.

[4] Gigena S. and Joaquín D., *On affine hypersurfaces with constant sectional curvature.* Math. Notae **39** (1997/98), 61–86.

[5] Gigena S. and Joaquín D., *Affine decomposable hypersurfaces which are locally symmetric.* Math. Notae **40** (1999/02), 95–113.

[6] Kühnel W. and Rademacher H.-B., *Essential conformal fields in pseudo-Riemannian geometry.* J. Math. Pures Appl. **74** (1995), 453–481.

[7] Li A.M., Liu H.L., Schwenk-Schellschmidt A., Simon U. and Wang C.P., *Cubic form methods and relative Tchebychev hypersurfaces.* Geom. Dedicata **66** (1997), 203–221.

[8] Liu H.L., Simon U. and Wang C.P., *Conformal structure in affine geometry: complete Tchebychev hypersurfaces.* Abh. Math. Sem. Univ. Hamburg, **66** (1996), 249–262.

[9] Liu H.L. and Wang C.P., *The centroaffine Tchebychev operator.* Result. Math. **27** (1995), 77–92.

[10] Liu H.L. and Wang C.P., *Relative Tchebychev surfaces in* A^3. Kyushu J. Math. **50**(2) (1996), 533–540.

[11] Manhart F., Uneigentliche Affinsphären, die Regelflächen oder Rückungsflächen sind. In *Geom. Proc. Congr. Thessaloniki 1987*, pages 106–113, 1988.

[12] Nomizu K. and Sasaki T., *Affine Differential Geometry.* Cambridge University Press, 1994.

[13] Pabel H., *Translationsflächen in der äquiaffinen Differentialgeometrie.* J. Geom. **40**(1/2) (1991), 148–164.

[14] Simon U., Schwenk-Schellschmidt A. and Viesel H., Introduction to the affine differential geometry of hypersurfaces, Lecture Notes. Science University of Tokyo, 1991. Distribution: TU Berlin, ISBN 3 7983 1529 9.

[15] Verstraelen L. and Vrancken L., *Affine variation formulas and affine min-*

imal surfaces. Michigan Math. J. **36** (1989), 77–93.

[16] Wang C., *Centroaffine minimal hypersurfaces in* $\mathbb{R}^{n+1}$. Geom. Dedicata **51** (1994), 63–74.

Grosse Burgstr. 15
D-23552 Lübeck, Germany
E-mail: tbinder@math.tu-berlin.de