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Derived Categories of Toric Varieties II

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1. Introduction

This paper supplements the first part of the series [5], where we considered the semi-orthogonal decompositions of the derived categories with respect to the toric minimal model program. We proved that the semi-orthogonal complements for divisorial contractions and flips have full exceptional collections.

A. Ishii and K. Ueda pointed out that the divisorial extractions, another important class of birational transformations, had not yet been treated. We did not consider toric birational morphisms $f: X \to Y$ between $\mathbf{Q}$-factorial projective toric varieties whose exceptional locus is a prime divisor E such that $K_X + eE = f^*K_Y$ with $e > 0$. We remark that (i) if $e < 0$ then f is a divisorial contraction (which is already treated in [5]) and (ii) if $e = 0$ then f is a log crepant morphism and we have a derived equivalence [4]. So we consider this case in Section 1 and prove that the semi-orthogonal complement again has a full exceptional collection. This result is rather remarkable when one considers that, for the fully faithful embedding functors between derived categories, the directions are opposite in the cases $e > 0$ and $e < 0$.

We also correct certain notation in [5]—namely, in the paragraph before Remark 5.1—in response to a remark of Ishii and Ueda. Thus we write $j_{1*} j_2^*$ instead of j^* because there is no morphism of stacks over a morphism of schemes $D \to X$. However, we do establish a fliplike diagram $\mathcal{D} \leftarrow \tilde{\mathcal{D}} \to \mathcal{X}$; see the text between Lemmas 2 and 3 (to follow).

In Section 2 we answer a question posed by S. Okawa at the Chulalongkorn University conference. We prove that the number of Fourier–Mukai partners of a $\mathbf{Q}$-factorial projective toric variety is finite, confirming a conjecture related to the finiteness conjecture of the minimal models (see [2]).

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2. Divisorial Extraction

We proved in [4] that the minimal model program in the category of toroidal varieties yields semi-orthogonal decompositions of derived categories. In [5] we

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proved in the toric case that the semi-orthogonal complements have full exceptional collections. We extend this result to another case of important toric morphism: a toric divisorial extraction.

THEOREM 1. *Let $\phi\colon X \to Y$ be a birational morphism between $\mathbf{Q}$-factorial projective toric varieties such that the exceptional locus is a prime divisor denoted by D. Let B be an effective torus-invariant $\mathbf{Q}$-divisor on X whose coefficients belong to a set*

$$\left\{\frac{r-1}{r} \mid r \in \mathbf{Z}_{>0}\right\},$$

and let $C = \phi_* B$. Let $\pi_X\colon \mathcal{X} \to X$ and $\pi_Y\colon \mathcal{Y} \to Y$ be the natural morphisms from smooth Deligne–Mumford stacks corresponding to the pairs (X, B) and (Y, C), respectively. Assume that

$$K_X + B < \phi^*(K_Y + C).$$

Then there is a fully faithful triangulated functor

$$\Phi\colon D^b(\mathrm{Coh}(\mathcal{X})) \to D^b(\mathrm{Coh}(\mathcal{Y}))$$

such that the semi-orthogonal complement of the image $\Phi(D^b(\mathrm{Coh}(\mathcal{X})))^\perp$ has a full exceptional collection.

Proof. This is the case not included in [5], where we considered only the case $K_X + B \geq \phi^*(K_Y + C)$. The proof is similar, but we need some additional calculations.

We use the notation of [5] whenever possible and recall that the morphism $\phi\colon X \to Y$ is controlled by an equation

$$a_1 v_1 + \cdots + a_{n+1} v_{n+1} = 0$$

locally over Y; here the v_i are primitive vertices of cones in a decomposition of a cone corresponding to a toric affine open subset U of Y, and the coefficients of the equation a_i are coprime integers. Since ϕ is a divisorial contraction, we have $a_i \geq 0$ except for $i = n+1$. We assume that $a_i > 0$ for $1 \leq i \leq \alpha$, that $a_i = 0$ for $\alpha < i \leq n$, and that $a_{n+1} < 0$.

Let D_i denote the prime divisors on X corresponding to the vertices v_i, and for $1 \leq i \leq n$ let $E_i = \phi(D_i)$ be their images on Y. Note that $D = D_{n+1}$ is the exceptional divisor of ϕ and that the restricted morphism $\bar{\phi}\colon D \to F = \phi(D)$ is a toric Mori fiber space. Let $\mathcal{D}_i$ and $\mathcal{E}_i$ be (respectively) the prime divisors on $\mathcal{X}$ and $\mathcal{Y}$ above D_i and E_i.

We write $B|_{\phi^{-1}(U)} = \sum_{i=1}^{n+1} \frac{r_i - 1}{r_i} D_i$ and $C|_U = \sum_{i=1}^{n} \frac{r_i - 1}{r_i} E_i$. Our assumption that $K_X + B$ is positive for ϕ is expressed by the following inequality:

$$\sum_{i=1}^{n+1} \frac{a_i}{r_i} < 0.$$

The following lemma is [4, Thm. 4.2(4)].

LEMMA 2. *Let $\mathcal{W} = (\mathcal{X} \times_Y \mathcal{Y})^{\tilde{}}$ be the normalized fiber product, and let μ and ν be the projections. Then the functor $\Phi = \nu_* \mu^*\colon D^b(\mathrm{Coh}(\mathcal{X})) \to D^b(\mathrm{Coh}(\mathcal{Y}))$ is fully faithful. Moreover, the invertible sheaves $\mathcal{O}_{\mathcal{X}}\big(\sum_{i=1}^{n+1} k_i \mathcal{D}_i\big)$ for the sequences of integers $k = (k_1, \dots, k_{n+1})$ such that*

$$0 \le -\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} < \sum_{i=1}^{\alpha} \frac{a_i}{r_i}$$

span the triangulated category $D^b(\mathrm{Coh}(\mathcal{X}))$, and

$$\Phi\left(\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i \mathcal{D}_i\right)\right) \cong \mathcal{O}_{\mathcal{Y}}\left(\sum_{i=1}^{n} k_i \mathcal{E}_i\right)$$

for such sequences of integers.

Proof. We recall the proof for the reader's convenience. We have

$$\sum_{i=1}^{n+1} k_i \mu^* \mathcal{D}_i \equiv \sum_{i=1}^{n} k_i \nu^* \mathcal{E}_i + \frac{r_{n+1}}{a_{n+1}} \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \mu^* \mathcal{D}_{n+1}$$

and

$$K_X + \sum_{i=1}^{n} \frac{r_i - 1}{r_i} D_i + D_{n+1} = \phi^*\left(K_Y + \sum_{i=1}^{n} \frac{r_i - 1}{r_i} E_i\right) - \sum_{i=1}^{n} \frac{a_i}{r_i} \frac{1}{a_{n+1}} D_{n+1}.$$

Since

$$\frac{r_{n+1}}{a_{n+1}} \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} < -\frac{r_{n+1}}{a_{n+1}} \sum_{i=1}^{n} \frac{a_i}{r_i},$$

it follows that

$$R^q \nu_* \mu^* \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i \mathcal{D}_i\right) \cong 0$$

for $q > 0$. Moreover, since

$$\frac{r_{n+1}}{a_{n+1}} \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \ge 0$$

we have

$$\nu_* \mu^* \mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1} k_i \mathcal{D}_i\right) \cong \mathcal{O}_{\mathcal{Y}}\left(\sum_{i=1}^{n} k_i \mathcal{E}_i\right).$$

For another such sequence $k' = (k'_1, \dots, k'_{n+1})$,

$$\frac{a_{n+1}}{r_{n+1}} < -\sum_{i=1}^{n} \frac{a_i}{r_i} < -\sum_{i=1}^{n+1} \frac{a_i (k_i - k'_i)}{r_i} < \sum_{i=1}^{n} \frac{a_i}{r_i};$$

hence

$$R^q\nu_*\mu^*\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1}(k_i-k_i')\mathcal{D}_i\right)\cong 0$$

for $q>0$ and

$$\nu_*\mu^*\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1}(k_i-k_i')\mathcal{D}_i\right)\cong \mathcal{O}_{\mathcal{Y}}\left(\sum_{i=1}^{n}(k_i-k_i')\mathcal{E}_i\right).$$

Thus the natural homomorphism

$$\mathrm{Hom}(L',L)\to \mathrm{Hom}(\Phi(L'),\Phi(L'))$$

is bijective for $L=\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1}k_i\mathcal{D}_i\right)$ and $L'=\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1}k_i'\mathcal{D}_i\right)$.

Since $\bigcup_{i=1}^{\alpha}\mathcal{D}_i=\emptyset$, the following Koszul complex is exact:

$$0\to \mathcal{O}_{\mathcal{X}}\left(-\sum_{i=1}^{\alpha}\mathcal{D}_i\right)\to\cdots\to\sum_{i=1}^{\alpha}\mathcal{O}_{\mathcal{X}}(-\mathcal{D}_i)\to\mathcal{O}_{\mathcal{X}}\to 0.$$

Hence the sheaves $\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{n+1}k_i\mathcal{D}_i\right)$ whose coefficients k_i satisfy the assumption of the lemma will generate the same category as the one generated by the sheaves whose coefficients are general. □

We shall construct a full exceptional collection on the semi-orthogonal complement of the image of Φ.

We define a $\mathbf{Q}$-divisor $\bar{B}=\sum_{i=1}^{n}\frac{r_it_i-1}{r_it_i}\bar{D}_i$ for $\bar{D}_i=D_i\cap D$ similarly as in [5, Sec. 5]. Thus we write

$$v_i\equiv t_i\bar{v}_i$$

in $N_D=N_X/\mathbf{Z}v_{n+1}$ for primitive vectors $\bar{v}_i\in N_D$ and $a_it_i=t\bar{a}_i$ such that the $\bar{a}_i$ $(1\le i\le\alpha)$ are coprime integers, where N_X and N_D are lattices corresponding to the respective toric varieties X and D. Here we note that

$$M_D=\{m\in M_X\mid\langle m,v_{n+1}\rangle=0\}$$

for the dual lattices M_X and M_D of N_X and N_D, respectively.

The toric morphism $\bar{\phi}\colon D\to F$ is controlled locally over F by an equation

$$\sum_{i=1}^{n}\bar{a}_i\bar{v}_i=0,$$

where $\bar{a}_i=0$ for $\alpha<i\le n$. We have $D_i|_D=\frac{1}{t_i}\bar{D}_i$ as $\mathbf{Q}$-Cartier divisors.

Let $\pi_D\colon\mathcal{D}\to D$ be the natural morphism from a smooth Deligne–Mumford stack corresponding to the pair $(D,\bar{B})$. Let $\tilde{\mathcal{D}}=(\mathcal{X}\times_X D)\tilde{}$ be the normalized fiber product. Then there are natural morphisms $j_1\colon\tilde{\mathcal{D}}\to\mathcal{D}$ and $j_2\colon\tilde{\mathcal{D}}\to\mathcal{X}$ satisfying

$$j_{1*}j_2^*\mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)\cong\mathcal{O}_{\mathcal{D}}(\bar{\mathcal{D}}_i)$$

for $i = 1, \dots, n$, where $\mathcal{D}_i$ and $\bar{\mathcal{D}}_i$ are (respectively) prime divisors on $\mathcal{X}$ and $\mathcal{D}$ corresponding to D_i and $\bar{D}_i$.

Next we define, similarly as in [5, Sec. 4], a $\mathbf{Q}$-divisor $\bar{C}$ on F by $\bar{C} = \sum_{i=\alpha+1}^{n} \frac{r_i s_i t_i - 1}{r_i s_i t_i} \bar{E}_i$ for $\bar{E}_i = E_i \cap F$. In particular, we write

$$\bar{v}_i \equiv s_i \tilde{v}_i$$

for primitive vectors $\tilde{v}_i$ in the lattice

$$N_F = \frac{N_D}{\sum_{i=1}^{\alpha}(\mathbf{R}v_i \cap N_D)}$$

corresponding to F. Here we observe that

$$M_F = \{m \in M_F \mid \langle m, v_i \rangle = 0,\ i = 1, \dots, \alpha\}$$

for the dual lattices M_F of N_F.

Let $\pi_F \colon \mathcal{F} \to F$ be the natural morphism from a smooth Deligne–Mumford stack corresponding to the pair $(F, \bar{C})$. Let $\tilde{\mathcal{F}} = (\mathcal{Y} \times_Y F)\tilde{}$ be the normalized fiber product. Then there are natural morphisms $j_{F,1} \colon \tilde{\mathcal{F}} \to \mathcal{F}$ and $j_{F,2} \colon \tilde{\mathcal{F}} \to \mathcal{Y}$ satisfying

$$j_{F,1*} j_{F,2}^* \mathcal{O}_{\mathcal{Y}}(\mathcal{E}_i) \cong \mathcal{O}_{\mathcal{F}}(\bar{\mathcal{E}}_i)$$

for $i = \alpha + 1, \dots, n$; here $\mathcal{E}_i$ and $\bar{\mathcal{E}}_i$ are prime divisors on $\mathcal{Y}$ and $\mathcal{F}$ corresponding to E_i and $\bar{E}_i$, respectively. Indeed, we have $E_i|_F = \frac{1}{s_i t_i} \bar{E}_i$ for $\alpha < i \le n$, which is confirmed by the following equalities: $\phi^* E_i = D_i$, $D_i|_D = \frac{1}{t_i} \bar{D}_i$, and $\bar{\phi}^* \bar{E}_i = s_i \bar{D}_i$.

We have an induced morphism $\bar{\psi} \colon \mathcal{D} \to \mathcal{F}$. Since $\bar{\psi}$ is smooth (by [5, Cor. 4.2]), it follows that

$$\bar{\psi}^* \mathcal{O}_{\mathcal{F}}(\bar{\mathcal{E}}_i) \cong \mathcal{O}_{\mathcal{D}}(\bar{\mathcal{D}}_i)$$

for $\alpha < i \le n$.

Lemma 3. *Let $k_1, \dots, k_n$ be integers, and define k_{n+1} by*

$$\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} = 0.$$

If k_{n+1} is not an integer that is divisible by r_{n+1}, then

$$j_{F,1*} j_{F,2}^* \mathcal{O}_{\mathcal{Y}}\left(\sum_{i=1}^{\alpha} k_i \mathcal{E}_i\right) \cong 0.$$

Proof. Because $j_{F,2}^* \mathcal{O}_{\mathcal{Y}}\big(\sum_{i=1}^{\alpha} k_i \mathcal{E}_i\big)$ is an invertible sheaf, its direct image sheaf is either an invertible sheaf or a zero sheaf on $\mathcal{F}$. If it is an invertible sheaf, then its pull-back

$$\bar{\psi}^* j_{F,1*} j_{F,2}^* \mathcal{O}_{\mathcal{Y}}\left(\sum_{i=1}^{\alpha} k_i \mathcal{E}_i\right)$$

is also an invertible sheaf, which should be of the form

$$j_{1*}j_2^*\mathcal{O}_{\mathcal{X}}\left(\sum_{i=1}^{\alpha} k_i\mathcal{D}_i + k_{n+1}\mathcal{D}_{n+1}\right)$$

for an integer k_{n+1} such that $\sum_{i=1}^{n+1}\frac{a_ik_i}{r_i} = 0$. Since the latter sheaf is nonzero if and only if k_{n+1} is divisible by r_{n+1}, the lemma follows. □

Theorem 1 is now a consequence of the following proposition combined with [5, Thm. 1.1].

PROPOSITION 4. (1) *The functor*

$$j_{F,2*}j_{F,1}^*\colon D^b(\mathrm{Coh}(\mathcal{F})) \to D^b(\mathrm{Coh}(\mathcal{Y}))$$

is fully faithful.

Let $D^b(\mathrm{Coh}(\mathcal{F}))_k$ *denote the full subcategory of* $D^b(\mathrm{Coh}(\mathcal{Y}))$ *defined by*

$$D^b(\mathrm{Coh}(\mathcal{F}))_k = j_{F,2*}j_{F,1}^*D^b(\mathrm{Coh}(\mathcal{F})) \otimes \mathcal{O}_{\mathcal{Y}}\left(\sum_{i=1}^{\alpha} k_i\mathcal{E}_i\right)$$

for a sequence of integers $k = (k_1, \dots, k_\alpha)$. *We set* $k_{\alpha+1} = \cdots = k_n = 0$ *when necessary.*

(2) *If*

$$0 < \sum_{i=1}^{n+1}\frac{a_ik_i}{r_i} \le -\sum_{i=1}^{n+1}\frac{a_i}{r_i}$$

for some integer k_{n+1}, *then*

$$\mathrm{Hom}^q(\Phi(D^b(\mathrm{Coh}(\mathcal{X}))), D^b(\mathrm{Coh}(\mathcal{F}))_k) = 0$$

for all q.

(3) *Let* $k' = (k_1', \dots, k_\alpha')$ *be another sequence of integers such that*

$$0 < \sum_{i=1}^{n+1}\frac{a_i(k_i' - k_i)}{r_i} < -\sum_{i=1}^{n+1}\frac{a_i}{r_i}$$

for some integers k_{n+1} *and* k_{n+1}'. *Then*

$$\mathrm{Hom}^q(D^b(\mathrm{Coh}(\mathcal{F}))_k, D^b(\mathrm{Coh}(\mathcal{F}))_{k'}) = 0$$

for all q.

(4) *The subcategories* $\Phi(D^b(\mathrm{Coh}(\mathcal{X})))$ *together with the* $D^b(\mathrm{Coh}(\mathcal{F}))_k$ *for*

$$0 < \sum_{i=1}^{n+1}\frac{a_ik_i}{r_i} \le -\sum_{i=1}^{n+1}\frac{a_i}{r_i}$$

(*for some integers* k_{n+1}) *generate* $D^b(\mathrm{Coh}(\mathcal{Y}))$ *as a triangulated category.*

Proof. (1) We shall prove that the natural homomorphisms

$$\begin{aligned}&\operatorname{Hom}^q(L, L')\\ &\qquad \to \operatorname{Hom}^q(j_{F,2*} j_{F,1}^* L, j_{F,2*} j_{F,1}^* L') \cong \operatorname{Hom}^q(j_{F,1*} j_{F,2}^* j_{F,2*} j_{F,1}^* L, L')\end{aligned}$$

are bijective for all q and for all invertible sheaves L and L' on $\mathcal{F}$.

Let $V = \sum_{i=1}^{\alpha} \mathcal{O}_{\mathcal{Y}}(-\mathcal{E}_i)$. Then we have the following Koszul resolution of $j_{F,2*} j_{F,1}^* \mathcal{O}_{\mathcal{F}}$:

$$0 \to \bigwedge^{\alpha} V \to \cdots \to V \to \mathcal{O}_{\mathcal{Y}} \to j_{F,2*} j_{F,1}^* \mathcal{O}_{\mathcal{F}} \to 0.$$

Thus

$$H_q(j_{F,1*} j_{F,2}^* j_{F,2*} j_{F,1}^* \mathcal{O}_{\mathcal{F}}) \cong \bigoplus_{\#I=q} j_{F,1*} j_{F,2}^* \mathcal{O}_{\mathcal{Y}}\Big(-\sum_{i \in I} \mathcal{E}_i\Big),$$

where the I run through all the subsets of $\{1, \ldots, \alpha\}$ such that $\#I = q$. Since $\sum_{i=1}^{n+1} \frac{a_i}{r_i} < 0$, we have

$$0 < \sum_{i \in I} \frac{a_i}{r_i} < \frac{|a_{n+1}|}{r_{n+1}}$$

for $q \neq 0$. Therefore, by Lemma 3,

$$H_q(j_{F,1*} j_{F,2}^* j_{F,2*} j_{F,1}^* \mathcal{O}_{\mathcal{F}}) \cong 0$$

for such q. The projection formula allows us to conclude that

$$j_{F,1*} j_{F,2}^* j_{F,2*} j_{F,1}^* L \cong L,$$

from which the assertion follows.

(2) By Lemma 2 (or [4, Thm. 4.2(4)]), $\Phi(D^b(\mathrm{Coh}(\mathcal{X}))$ is spanned by invertible sheaves $\mathcal{O}_{\mathcal{Y}}\big(\sum_{i=1}^{n} l_i \mathcal{E}_i\big)$ for

$$0 \le -\sum_{i=1}^{n+1} \frac{a_i l_i}{r_i} < \sum_{i=1}^{\alpha} \frac{a_i}{r_i},$$

where l_{n+1} are some integers. Adding the inequalities yields

$$0 < -\sum_{i=1}^{n+1} \frac{a_i(l_i - k_i)}{r_i} < -\frac{a_{n+1}}{r_{n+1}}.$$

Hence for any invertible sheaf L on $\mathcal{F}$ we have, by Lemma 3,

$$\begin{aligned}&\operatorname{Hom}^q\Big(\mathcal{O}_{\mathcal{Y}}\Big(\sum_{i=1}^{n} l_i \mathcal{E}_i\Big), j_{F,2*} j_{F,1}^* L \otimes \mathcal{O}_{\mathcal{Y}}\Big(\sum_{i=1}^{\alpha} k_i \mathcal{E}_i\Big)\Big)\\ &\qquad \cong \operatorname{Hom}^q\Big(j_{F,1*} j_{F,2}^* \mathcal{O}_{\mathcal{Y}}\Big(\sum_{i=1}^{n} (l_i - k_i) \mathcal{E}_i\Big), L\Big) \cong 0\end{aligned}$$

for all q.

(3) For any fixed subset $I \subset \{1, \dots, \alpha\}$, we set $\varepsilon_i = 1$ or 0 according as whether or not $i \in I$. Then

$$0 > \sum_{i=1}^{n+1} \frac{a_i(k_i - k'_i - \varepsilon_i)}{r_i} > \frac{a_{n+1}}{r_{n+1}}.$$

So for any invertible sheaves L, L' on $\mathcal{F}$ we have, again by Lemma 3,

$$\begin{aligned} &\mathrm{Hom}^q\biggl(j_{F,2*}j_{F,1}^*L \otimes \mathcal{O}_{\mathcal{Y}}\biggl(\sum_{i=1}^{\alpha} k_i\mathcal{E}_i\biggr), j_{F,2*}j_{F,1}^*L' \otimes \mathcal{O}_{\mathcal{Y}}\biggl(\sum_{i=1}^{\alpha} k'_i\mathcal{E}_i\biggr)\biggr) \\ &\qquad \cong \mathrm{Hom}^q\biggl(j_{F,1*}j_{F,2}^*j_{F,2*}j_{F,1}^*L \otimes \mathcal{O}_{\mathcal{Y}}\biggl(\sum_{i=1}^{\alpha}(k_i - k'_i)\mathcal{E}_i\biggr), L'\biggr) \cong 0 \end{aligned}$$

for all q.

(4) Let T be the triangulated subcategory of $D^b(\mathrm{Coh}(\mathcal{Y}))$ that is generated by $\Phi(D^b(\mathrm{Coh}(\mathcal{X})))$ and the $D^b(\mathrm{Coh}(\mathcal{F}))_k$ for

$$0 < \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \le -\sum_{i=1}^{n+1} \frac{a_i}{r_i}$$

for some integers k_{n+1}. We shall prove that T contains all invertible sheaves of the form

$$\mathcal{O}_{\mathcal{Y}}\biggl(\sum_{i=1}^{n} k_i\mathcal{E}_i\biggr).$$

By Lemma 2, T contains invertible sheaves $\mathcal{O}_{\mathcal{Y}}\bigl(\sum_{i=1}^{n} k_i\mathcal{E}_i\bigr)$ if

$$-\sum_{i=1}^{\alpha} \frac{a_i}{r_i} < \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \le 0$$

for some integers k_{n+1}. On the other hand, for an arbitrary sequence of integers $(k_1, \dots, k_n)$ there exists an integer k_{n+1} such that

$$-\sum_{i=1}^{\alpha} \frac{a_i}{r_i} < \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \le -\sum_{i=1}^{n+1} \frac{a_i}{r_i}.$$

The difference between these two intervals is covered by considering the Koszul resolution of the sheaf $j_{F,2*}j_{F,1}^*\mathcal{O}_{\mathcal{F}}$. Indeed, assume that $k = (k_1, \dots, k_{n+1})$ is a sequence of integers such that

$$0 < \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \le -\sum_{i=1}^{n+1} \frac{a_i}{r_i}.$$

If T contains all the invertible sheaves of the form

$$\mathcal{O}_{\mathcal{Y}}\biggl(\sum_{i=1}^{n} l_i\mathcal{E}_i\biggr)$$

such that $l_i = k_i$ or $k_i - 1$, $l_{n+1} = k_{n+1}$, and $\sum_{i=1}^n l_i < \sum_{i=1}^n k_i$, then T also contains $\mathcal{O}_{\mathcal{Y}}\big(\sum_{i=1}^n k_i \mathcal{E}_i\big)$ because $D^b(\mathrm{Coh}(\mathcal{F}))_k$ is contained in T. Hence we obtain the assertion by induction on $\sum_{i=1}^{n+1} k_i$, concluding the proof of Proposition 4. □

If $k = (k_1, \ldots, k_{n+1})$ and $k' = (k'_1, \ldots, k'_{n+1})$ are sequences of integers such that

$$\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} = \sum_{i=1}^{n+1} \frac{a_i k'_i}{r_i},$$

then $j_{F,1*} j^*_{F,2} \mathcal{O}_{\mathcal{Y}}\big(\sum_{i=1}^{\alpha} (k_i - k'_i)\mathcal{E}_i\big)$ either is an invertible sheaf on $\mathcal{F}$ or is 0. In the former case we have $D^b(\mathrm{Coh}(\mathcal{F}))_k = D^b(\mathrm{Coh}(\mathcal{F}))_{k'}$; in the latter case,

$$\mathrm{Hom}^q(D^b(\mathrm{Coh}(\mathcal{F}))_k, D^b(\mathrm{Coh}(\mathcal{F}))_{k'}) = 0$$

for all q. We thus obtain an exceptional collection of the semi-orthogonal complement, which finishes the proof of Theorem 1. □

3. Fourier–Mukai Partners

In the minimal model program we conjectured that, given a smooth projective variety (or, more generally, a smooth projective pair), there exist only finitely many minimal models or log minimal models up to isomorphisms that are birationally equivalent to the given variety. In the derived setting we conjecture that, given a smooth projective variety, there exist only finitely many smooth projective varieties up to isomorphisms whose derived categories are equivalent to the given one [2]. We also expect a more generalized statement to hold for projective varieties with only quotient singularities. We add here one more example that confirms this conjecture.

THEOREM 5. *Let X be a projective $\mathbf{Q}$-factorial toric variety and Y a projective variety that has only quotient singularities. Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth Deligne–Mumford stacks associated to X and Y, respectively. Assume there exists an equivalence of triangulated categories $\Phi\colon D^b(\mathcal{X}) \cong D^b(\mathcal{Y})$. Then Y is also a projective $\mathbf{Q}$-factorial toric variety, and the kernel object for Φ induces a toric birational map $\phi\colon X \dashrightarrow Y$. In particular, Y has only abelian quotient singularities. Moreover, there exist only finitely many such birational maps when X is fixed and Y is varied.*

Proof. Let $E \in D^b(\mathcal{X} \times \mathcal{Y})$ be the kernel of Φ, an object giving the equivalence Φ [3]. Then we have an isomorphism

$$E \otimes p_1^* \omega_{\mathcal{X}} \cong E \otimes p_2^* \omega_{\mathcal{Y}},$$

where p_1 and p_2 are projections.

Let T be the torus contained in X, and let $B = X \setminus T$. We take a general point $y \in \mathcal{Y}$ such that the support of $E_y = \Psi(\mathcal{O}_y)$ contains a point in T, where $\Psi\colon D^b(\mathcal{Y}) \cong D^b(\mathcal{X})$ is the equivalence given by $\Psi(a) = p_{1*}(p_2^* a \otimes E)$. Since $K_X + B \sim 0$, we have an isomorphism

$$E_y \cong E_y \otimes \mathcal{O}_{\mathcal{X}}(-B).$$

It follows that the support of E_y is a point, say $x \in T$, by the same argument as in the proof of [2, Thm. 2.3(2)].

If we take another such point y', then $E_{y'}$ is supported by a different point $x' \in T$ because $\mathrm{Hom}(\mathcal{O}_y, \mathcal{O}_{y'}[p]) = 0$ for all p. Hence the support of E gives a birational map $\phi\colon X \dashrightarrow Y$. Let $Z \subset X \times Y$ be the graph of ϕ. By the first isomorphism we have $q_1^* K_X = q_2^* K_Y$, where $q_1\colon Z \to X$ and $q_2\colon Z \to Y$ are projections (i.e., X and Y are K-equivalent).

We now consider the case where ϕ is not surjective in codimension 1. Let D_Y be a prime divisor on Y whose center on X is not a divisor. Because X and Y are K-equivalent, there exists a crepant divisorial extraction $\alpha\colon X' \to X$ whose exceptional divisor is the prime divisor D_X corresponding to D_Y. Indeed, there exists a toric projective birational morphism $\beta\colon X'' \to X$ such that all divisorial valuations whose log discrepancies are at most 1 appear as prime divisors on X''. We note that β is toric because it is obtained by blowing up a relative minimal model of X, which is toric, at centers that are also toric. Then we can construct α by contracting all exceptional divisors except D_X.

Since α is a crepant toric morphism, we have an equivalence $D^b(\mathcal{X}) \cong D^b(\mathcal{X}')$ for the smooth Deligne–Mumford stack $\mathcal{X}'$ associated to X' [4]. By replacing X with X' if necessary, we may assume that ϕ is surjective in codimension 1.

By the paragraph after Corollary 2.4 in [1], X is a Mori dream space. By [1, Def. 1.10], all birational maps ϕ from X that are surjective in codimension 1 are obtained in the following way. There exist finitely many birational maps $f_i\colon X \dashrightarrow X_i$ corresponding to the chambers of the movable cone $\mathrm{Mov}(X)$, and ϕ coincides with the composition of f_i and a morphism from X_i corresponding to one of the finitely many faces of the nef cone $\mathrm{Nef}(X_i)$. Moreover, all of these birational maps and contraction morphisms are toric. We have thus proved the finiteness of the birational maps ϕ. $\square$

REMARK 6. The same statement (with the same proof) holds more generally if K_X or $-K_X$ supports a big divisor, and if X and its crepant blow-ups are Mori dream spaces.

REMARK 7. This is an important correction. The statement in Theorem 1.1 of [5] (as well as in its Corollaries 5.3 and 6.2)—that the exceptional collections of the derived categories obtained there in Theorems 5.2 and 6.1 consist of sheaves—is wrong because the images of sheaves are not themselves sheaves. The author would like to thank Ludmil Katzarkov for asking me the question.

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