Michigan Math. J. 62 (2013), 227–258

Locally Nilpotent Derivations of Rings with Roots Adjoined

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1. Introduction

Suppose $\mathbf{k}$ is a field of characteristic 0. This paper investigates locally nilpotent derivations of rings of the form $B = R[z]$, where R is a commutative $\mathbf{k}$-domain and $z^n \in R$ for some positive integer n. Such a ring has a natural grading by $\mathbb{Z}_n$, and Theorem 3.1 gives the basic properties of locally nilpotent derivations D of B that are homogeneous relative to this grading. In particular, D is always a quasi-extension of a locally nilpotent derivation δ of R, and $D^2 z = 0$. There are two cases that can occur: (i) $Dz = 0$, in which case $\ker D$ is a free module over $\ker \delta$ of rank n that is generated by the powers of z; or (ii) $Dz \neq 0$, in which case $\ker D = \ker \delta$ and z^n is a local slice of δ. In the first case, for the quotient maps of the corresponding $\mathbb{G}_a$-actions, a generic orbit of Spec(R) will divide into n orbits of Spec(B).

Section 2 reviews some basic theory of locally nilpotent derivations before introducing the *absolute degree* $|f|_R$ of elements of R. In several of our results, it is important to know whether $|f|_R \leq 1$—that is, whether f is in the kernel (or is a local slice) of a nonzero locally nilpotent derivation (LND) of R. Section 3 provides the fundamental method used in the rest of the paper: Corollary 3.1 shows that, if $z^n = f \in R$, then there is a one-to-one correspondence between elements of $\delta \in \mathrm{LND}(R)$ with $\delta^2 f = 0$ and homogeneous elements of $\mathrm{LND}(B)$. This correspondence is given by choosing the appropriate quasi-extension, and it is a useful tool for studying the locally nilpotent derivations of B by looking at those of R. For example, Corollary 3.2 shows that if R is $\mathbb{Z}$-graded, f is $\mathbb{Z}$-homogeneous of degree coprime to n, and $|f|_R \geq 2$, then B is rigid, that is, B has no nonzero locally nilpotent derivations.

Section 4 applies Theorem 3.1 to certain rings that are of transcendence degree 1 over R—namely, rings of the form $B = R[x, y]$, where either $x^m + y^n \in R$ or $x^m y^n \in R$ for relatively prime m and n. In the former case this ring has a natural $\mathbb{Z}_{mn}$-grading; Theorem 4.1 shows that any homogeneous locally nilpotent derivation D of B satisfies $D^2 x = D^2 y = 0$ and that either $Dx = 0$ or $Dy = 0$. In Section 5, we apply this result to the polynomial ring $A^{[n]}$ over an integral domain A containing $\mathbb{Q}$ while assuming that this ring is $\mathbb{Z}$-graded over A. Theorem 5.1 provides numerical criteria which, when satisfied, imply that certain

Received August 2, 2010. Revision received January 24, 2013.

variables of $A^{[n]}$ are either local slices or invariants for *all* homogeneous locally nilpotent derivations.

In Sections 7–9 we apply the theory to Pham–Brieskorn surfaces and threefolds and also to related varieties. Several authors have studied the $\mathbb{G}_a$-actions of these varieties (most notably, Kaliman and Zaidenberg [14] and Kaliman and Makar-Limanov [13]). An important tool in these papers is Mason's theorem, which gives a bound for the degrees of $f, g, h \in K[t]$ (a univariate polynomial ring over a field K) when $f + g + h = 0$. Mason's theorem is useful because, if B is a commutative $\mathbb{Q}$-domain and D is a nonzero locally nilpotent derivation of B, then $B \subset K[t]$; here K is the field of fractions of the kernel of D and $t \in B$ is a local slice of D. Thus, Mason's theorem can be applied to elements of B. Yet the degree bounds that it yields do not settle all cases, and these bounds are weakened when the theorem is generalized to more than three terms. Our results provide additional tools that are used in conjunction with Mason's theorem to show that certain types of Pham–Brieskorn threefolds are rigid. The version of Mason's theorem that we use is presented in Section 6.

Mason published the result that bears his name in 1984 [20], although it originally appeared in the 1981 paper of Strothers [25]. Proofs can also be found in [16, Thm. 7.1] and [22, Thm. 4.3.1]. Several authors have generalized Mason's theorem to more than three terms; see for example [1] and [8].

PRELIMINARIES. We assume throughout that $\mathbf{k}$ is a ground field of characteristic 0 and that rings are commutative. The polynomial ring in n variables over a ring B is denoted by $B^{[n]}$. The field of fractions of $\mathbf{k}^{[n]}$ is denoted $\mathbf{k}^{(n)}$.

By a *degree function* on a ring B we mean a function $\deg\colon B \to \mathbb{Z} \cup \{-\infty\}$ such that, for all $a, b \in B$:

1. $\deg b = -\infty$ if and only if $b = 0$;
2. $\deg(ab) = \deg a + \deg b$; and
3. $\deg(a + b) \le \max\{\deg a, \deg b\}$.

In this paper we consider gradings of rings by a cyclic group Γ. Suppose that the ring B is Γ-graded: $B = \bigoplus_{i\in\Gamma} B_i$. Given $i \in \Gamma$, there exists a unique function $h_i\colon B \to B_i$ such that, for each $f \in B$, $f = \sum_{i\in\Gamma} h_i(f)$. Given $f \in B$, the image $h_i(f)$ is denoted by f_i. Elements of B_i are said to be *homogeneous* and *of degree i*. (Note that, if B is a domain and $\Gamma = \mathbb{Z}$, then the assignment $f \mapsto \max\{i \mid f_i \neq 0\}$ defines a degree function on B.) For R a subring of B, we say that B is Γ*-graded over* R when $R = B_0$. If $\mathbf{k} \subset B$ then, for any Γ-grading of B, we assume that $\mathbf{k} \subset B_0$.

A nonzero derivation $D \in \operatorname{Der}(B)$ is *homogeneous* if and only if (iff) there exists a $d \in \Gamma$ such that $DB_i \subset B_{i+d}$ for all $i \in \Gamma$. In this case, the *degree* of D equals d.

Two Γ-gradings $B = \bigoplus_{i\in\Gamma} B_i$ and $B = \bigoplus_{i\in\Gamma} \tilde{B}_i$ are said to be *equivalent* iff there exists a group isomorphism $\alpha\colon \Gamma \to \Gamma$ such that $\tilde{B}_i = B_{\alpha(i)}$ for each $i \in \Gamma$.

Suppose that B is $\mathbb{Z}$-graded: $B = \bigoplus_{i\in\mathbb{Z}} B_i$. Then, for any integer $n \ge 2$, the natural projection $\pi : \mathbb{Z} \to \mathbb{Z}_n$ induces a $\mathbb{Z}_n$-grading of B. In particular, $B = \bigoplus_{k\in\mathbb{Z}_n} B_k$, where $B_k = \bigoplus_{\pi(i)=k} B_i$.

We say that $f, g \in B$ are *relatively prime* iff $fB \cap gB = fgB$. This notion generalizes the definition of relative primeness for elements in a unique factorization domain. The reader can easily verify the following three properties.

1. If $f, g \in B$ are relatively prime in B, then f and g are relative prime in $B^{[n]}$ for each $n \geq 0$.

When B is an integral domain, the following two properties also hold.

2. If $f, g \in B$ are relatively prime, then f^m and g^n are relatively prime for every $m, n \geq 1$.
3. Let A be a factorially closed subring of B, and let $f, g \in A$. If f and g are relatively prime in B, then f and g are relatively prime in A.

ACKNOWLEDGMENTS. The first author wishes to express his sincere gratitude to the faculty of the Institut de Mathématiques de Bourgogne for their hospitality during June 2009, when research leading to this paper was started. The authors gratefully acknowledge the assistance of Daniel Daigle of the University of Ottawa, who provided us with very helpful comments. The authors also thank the anonymous referee, whose thorough review led to several corrections and improvements in the paper.

2. Locally Nilpotent Derivations

We first recall a few basic definitions and facts concerning locally nilpotent derivations. For a more extensive treatment of the subject, the reader is referred to [12].

Suppose B is a commutative $\mathbf{k}$-algebra. The set of $\mathbf{k}$-derivations of B is denoted by $\mathrm{Der}_{\mathbf{k}}(B)$, and the subalgebra of elements at which $D \in \mathrm{Der}_{\mathbf{k}}(B)$ is nilpotent is

$$\mathrm{Nil}(D) = \{f \in B \mid D^n f = 0 \text{ for } n \gg 0\}.$$

We denote the kernel of D by $\ker D$. An ideal $I \subset B$ is an *integral ideal* for D if $DI \subset I$.

The derivation $D \in \mathrm{Der}_{\mathbf{k}}(B)$ is *locally nilpotent* if and only if $\mathrm{Nil}(D) = B$. The set of locally nilpotent derivations of B is denoted by $\mathrm{LND}(B)$. If A is a subring of B, then

$$\mathrm{LND}_A(B) = \{D \in \mathrm{LND}(B) \mid DA = 0\}.$$

The $\mathbf{k}$-algebra B is said to be *rigid* if and only if $\mathrm{LND}(B) = \{0\}$ and *stably rigid* iff $\mathrm{LND}(B^{[n]}) = \mathrm{LND}_B(B^{[n]})$ for each integer $n \geq 0$.

To each nonzero $D \in \mathrm{LND}(B)$ we associate the function ν_D on B. Namely, for nonzero $f \in B$,

$$\nu_D(f) = \min\{n \in \mathbb{N} \mid D^{n+1} f = 0\} \quad \text{and} \quad \nu_D(0) = -\infty.$$

Any $f \in B$ with $\nu_D(f) = 1$ is called a *local slice* for D.

The subalgebra

$$\mathrm{ML}(B) = \bigcap_{D \in \mathrm{LND}(B)} \ker D$$

is the *Makar-Limanov invariant* of B. We define a new invariant $\mathcal{R}(B)$, the *rigid core* of B, as follows. Set $\mathrm{ML}_0(B) = B$ and, for each $n \geq 1$, define $\mathrm{ML}_n(B) = \mathrm{ML}(\mathrm{ML}_{n-1}(B))$. Then

$$\mathcal{R}(B) = \bigcap_{n\geq 0} \mathrm{ML}_n(B).$$

Note that $\mathcal{R}(B)$ is always a rigid ring and that $\mathcal{R}(B) = B$ if and only if B is rigid.

If B is a domain then, for each $D \in \mathrm{LND}(B)$, the subrings $\ker D$, $\mathrm{ML}(B)$, and $\mathcal{R}(B)$ are factorially closed and $B^* \subset \mathrm{ML}(B)$. In addition, if $D \neq 0$ then ν_D is a degree function and, for each local slice f,

$$B_{Df} = (\ker D)_{Df}[f] = (\ker D)_{Df}^{[1]}.$$

The following lemma will be needed.

LEMMA 2.1. *Let B be a $\mathbf{k}$-domain and let $D \in \mathrm{LND}(B)$. If $t \in B$ and if $\sqrt{tB}$ is an integral ideal of D, then $Dt = 0$.*

Proof. Assume that $D \in \mathrm{LND}(B)$ is nonzero and that $\sqrt{tB}$ is an integral ideal of D for nonzero $t \in B$. If $n = \nu_D(t)$, then $D^n t$ is a nonzero element of $\sqrt{tB} \cap \ker D$. If $m \geq 0$ is such that $(D^n t)^m \in tB$, then $(D^n t)^m \in tB \cap \ker D$, which means that $tB \cap \ker D \neq \{0\}$. Since $\ker D$ is factorially closed in B, it follows that $t \in \ker D$. $\square$

2.1. Homogeneous LNDs

When B is an affine $\mathbb{Z}$-graded $\mathbf{k}$-domain, any $D \in \mathrm{LND}(B)$ decomposes as a finite sum of homogeneous derivations, and also the highest-degree homogeneous summand Δ of D is locally nilpotent. Furthermore, if $Df = 0$ for $f \in B$ then $\Delta F = 0$, where $F \in B$ is the highest-degree homogeneous summand of f. One technique for showing that B is rigid is to show that the only homogeneous locally nilpotent derivation Δ of B is $\Delta = 0$. One loses these kinds of strong properties in passing from $\mathbb{Z}$-gradings to $\mathbb{Z}_n$-gradings.

2.2. Quasi-extensions

Let $D\colon B \to B$ be a derivation of an integral domain B, and let $\delta\colon R \to R$ be a derivation of a subring $R \subset B$. Then D is a *quasi-extension* of δ if there exists a nonzero $t \in B$ such that $Ds = t \cdot \delta s$ for all $s \in R$. One of the main tools we use to study locally nilpotent derivations for rings graded by $\mathbb{Z}_n$ is the following.

LEMMA 2.2 [12, Lemma 5.38]. *Let B be an integral domain containing $\mathbb{Q}$, and let $D\colon B \to B$ be a derivation that is a quasi-extension of a derivation $\delta\colon R \to R$ for some subring R. If $D \in \mathrm{LND}(B)$, then $\delta \in \mathrm{LND}(R)$.*

2.3. Absolute Degree

DEFINITION 2.1. Suppose B is a commutative $\mathbf{k}$-domain. If B is not rigid then, given $f \in B$, the *absolute degree* of f is defined by

$$|f|_B = \min\{\nu_D(f) \mid D \in \mathrm{LND}(B),\ D \neq 0\}.$$

When B is rigid, we define $|f|_B = -\infty$ if $f = 0$ and $|f|_B = \infty$ otherwise.

It should be noted that this same definition was given by Daigle in [5], where he uses the term "LND-degree" in place of "absolute degree". Note also that absolute degree is not a degree function in the standard sense; instead it satisfies the following properties.

1. $|f|_B = -\infty$ if and only if $f = 0$.
2. $|f^m|_B = m|f|_B$ for all integers $m \geq 0$.
3. $|fg|_B \geq |f|_B + |g|_B$ for all $f, g \in B$.
4. $|f + \kappa|_B = |f|_B$ for all $f \in B$ and $\kappa \in \mathrm{ML}(B)$ with $f, f + \kappa \neq 0$.
5. If K is an algebraic extension field of $\mathbf{k}$ and $B_K = K \otimes_{\mathbf{k}} B$ is a domain, then $|f|_B \geq |f|_{B_K}$.

If B is $\mathbb{Z}$-graded and affine, then the absolute degree satisfies two additional properties relative to the grading.

6. Given $f \in B$, if $F \in B$ is the highest-degree homogeneous summand of f then $|f|_B \geq |F|_B$.
7. If $F \in B$ is homogeneous and B is not rigid, then there exists a nonzero homogeneous $D \in \mathrm{LND}(B)$ such that
$$|F|_B = \nu_D(F).$$

The following well-known result is due to Davenport and dates to 1965; see [7] and [16].

THEOREM 2.1. *Let nonzero $u, v \in \mathbf{k}$ and $f, g \in \mathbf{k}[t] = \mathbf{k}^{[1]}$ be given, where f and g are not both constant, together with positive integers l and m. Then, relative to standard degrees in t,*
$$\deg(uf^l - vg^m) \geq \frac{1}{m}(lm - l - m)\deg f + 1$$
unless $uf^l = vg^m$ identically.

We remark that the condition *f and g are not both constant* is missing from Davenport's original formulation of this theorem, but is necessary for the result to be valid. We shall use Davenport's theorem to prove the following result. Part (a) was first given in [19, Lemma 2]; part (b) is new.

THEOREM 2.2. *Let B be a commutative $\mathbf{k}$-domain, and let $D \in \mathrm{LND}(B)$ be nonzero. Suppose $u, v \in \ker D$ and $x, y \in B$ are nonzero and that a and b are integers with $a, b \geq 2$. Assume $ux^a + vy^b \neq 0$.*

(a) *If $D(ux^a + vy^b) = 0$, then $Dx = Dy = 0$.*
(b) *If $D^2(ux^a + vy^b) = 0$ and a and b are not both 2, then $Dx = Dy = 0$.*

Proof. Recall that every element of B may be viewed as a univariate polynomial over the field $K = \mathrm{frac}(\ker D)$, since the localization of B at the nonzero elements of $\ker D$ equals $K[t]$, where t is a local slice of D. In this setting, the degree of $f \in B$ equals $\nu_D(f)$, and elements of B of degree 0 are precisely the nonzero elements of $\ker D$.

Since $a, b \geq 2$, we have $ab - a - b \geq 0$. If $Dx \neq 0$ or $Dy \neq 0$, then Davenport's theorem implies

$$\nu_D(ux^a + vy^b) \geq \frac{1}{b}(ab - a - b)\nu_D(x) + 1 \geq 1, \tag{1}$$

so $D(ux^a + vy^b) \neq 0$. This proves part (a).

For part (b), assume that $a \geq 3$ or $b \geq 3$. Then $ab - a - b \geq 1$.

Now suppose that $Dx \neq 0$; that is, suppose $\nu_D(x) \geq 1$. The first inequality of (1) implies that $\nu_D(ux^a + vy^b) > 1$, but then $\nu_D(ux^a + vy^b) \geq 2$. This means that $D^2(ux^a + vy^b) \neq 0$, which contradicts the hypothesis of part (b). Therefore, $Dx = 0$. After exchanging the roles of x and y, the same argument shows that $Dy = 0$. □

COROLLARY 2.1. *Let $B = \mathbf{k}[x, y] = \mathbf{k}^{[2]}$. If a and b are integers with $a, b \geq 2$ and either $a \neq 2$ or $b \neq 2$, then $|x^a + y^b|_B \geq 2$. Consequently, $D^2(x^a + y^b) \neq 0$ for all nonzero $D \in \mathrm{LND}(B)$.*

Daigle proved the following result for polynomials in three variables.

THEOREM 2.3 [5, Prop. 4.7]. *Let $B = \mathbf{k}[x, y, z] = \mathbf{k}^{[3]}$. Suppose $a, b, c \geq 2$ and at most one of a, b, c equals 2. Then $|x^a + y^b + z^c|_B \geq 2$. Consequently, $D^2(x^a + y^b + z^c) \neq 0$ for all nonzero $D \in \mathrm{LND}(B)$.*

3. Adjoining One Root

In this section, we assume that:

1. R is a $\mathbf{k}$-domain;
2. $f \in R$ and $n \in \mathbb{Z}$ for $n \geq 2$; and
3. $R[z] = R^{[1]}$ and $f + z^n$ is prime (so $f \neq 0$).

Define

$$B = R[z]/(f + z^n).$$

We make the following observation.

LEMMA 3.1. *If $|f|_R \leq 1$, then B is not rigid.*

This result is an immediate consequence of Lemma 3.2 to follow.

3.1. Canonical Quasi-extensions

Given a derivation δ of R, there exists a unique derivation D of B such that

$$Dr = z^{n-1}\delta r \quad (r \in R) \quad \text{and} \quad Dz = -\frac{1}{n}\delta f.$$

We call D the *canonical quasi-extension* of δ to B.

LEMMA 3.2. *If $\delta \in \mathrm{LND}(R)$ and $\delta^2(f) = 0$, then the canonical quasi-extension D of δ is a locally nilpotent derivation of B.*

Proof. Note first that if $r \in R$ then $\delta r = 0$ implies $Dr = 0$. Therefore, $\ker \delta \subset \mathrm{Nil}(D)$. Since $\delta f \in \ker \delta$, it follows that $Dz \in \ker D$, which implies $z \in \mathrm{Nil}(D)$. Since R and z generate B, it will therefore suffice to show that $R \subset \mathrm{Nil}(D)$. This is demonstrated by induction on $N = \nu_\delta(r)$ for $r \in R$. The basis for induction has been established because $\ker \delta \subset \mathrm{Nil}(D)$.

Given $N \geq 1$, assume $s \in \mathrm{Nil}(D)$ whenever $\nu_\delta(s) \leq N - 1$. In particular, if $r \in R$ is such that $\nu_\delta(r) = N$, then $\delta r \in \mathrm{Nil}(D)$ since $\nu_\delta(\delta r) = N - 1$. It follows that

$$Dr = z^{n-1}\delta r \in \mathrm{Nil}(D) \implies r \in \mathrm{Nil}(D).$$

Therefore, $\mathrm{Nil}(D) = B$; that is, D is locally nilpotent. □

REMARK 3.1. The converse to Lemma 3.1 may fail to hold. For example, let $f = x^2 + y^3$ in $\mathbb{C}[x, y] = \mathbb{C}^{[2]}$. Then $|f|_R = 2$ (by the results in Section 2) but the ring

$$B = R[z]/(f + z^2) = \mathbb{C}[x, y, z]/(x^2 + y^3 + z^2)$$

is the coordinate ring of a nonrigid Pham–Brieskorn surface (see Section 7). However, under certain additional assumptions, the condition $|f|_R \geq 2$ becomes equivalent to the condition that B is rigid; see Corollary 3.2.

3.2. $\mathbb{Z}_n$-Gradings

We next observe that B is a free R-module given by

$$B = R + Rz + \cdots + Rz^{n-1}.$$

Given $u \in \mathbb{Z}_n^*$, this decomposition defines a $\mathbb{Z}_n$-grading of B over R for which z is homogeneous of degree u. In particular, let $\pi : \mathbb{Z} \to \mathbb{Z}_n$ be the natural projection and define

$$B_{u\pi(i)} = Rz^i \quad (0 \leq i \leq n-1).$$

Then the $\mathbb{Z}_n$-grading is given by $B = \bigoplus_{k \in \mathbb{Z}_n} B_k$, where $B_0 = R$. Note that we obtain $\phi(n)$ distinct but equivalent $\mathbb{Z}_n$-gradings of B over R in this way. In particular, objects that are homogeneous in one of these gradings remain homogeneous in any other—it is only the degree that varies.

LEMMA 3.3. *Let $D \in \mathrm{Der}_{\mathbf{k}}(B)$ be nonzero and homogeneous of degree d relative to the $\mathbb{Z}_n$-grading just described, and let λ be the unique integer such that $0 \leq \lambda \leq n - 1$ and $\pi(\lambda) = u^{-1}d$. Then there exists a $\delta \in \mathrm{Der}_{\mathbf{k}}(R)$ such that D is a quasi-extension of δ, where $D|_R = z^\lambda \delta$. In addition, if $D \in \mathrm{LND}(B)$ then $\delta \in \mathrm{LND}(R)$.*

Proof. Given $s \in R = B_0$, we have $Ds \in B_d = Rz^\lambda$. The derivation D is not identically zero on R (since B is an algebraic extension of R), so we can choose s such that $Ds \neq 0$. Define $\delta \colon R \to R$ by $\delta s = z^{-\lambda}Ds$. Then δ is a well-defined $\mathbf{k}$-derivation of R, and D is a quasi-extension of δ. Furthermore, if D is locally nilpotent then (by Lemma 2.2) δ is also locally nilpotent. □

3.3. *Main Theorem*

THEOREM 3.1. *Suppose that B is $\mathbb{Z}_n$-graded over R as before, where z is homogeneous and* $\deg z \in \mathbb{Z}_n^*$. *Given homogeneous $D \in \mathrm{LND}(B)$, let $\delta \in \mathrm{LND}(R)$ and $\lambda \in \mathbb{Z}$ be such that $0 \le \lambda \le n-1$ and $D|_R = z^\lambda \delta$. Then the following statements hold*:

(a) $Dz, \delta f \in \ker \delta = R \cap \ker D$;
(b) *if* $Dz \neq 0$ *then*
  (i) $\lambda = n-1$ *and*
  (ii) $\ker D = \ker \delta$.

Proof. Since Dz is homogeneous, $Dz \in Rz^m$ for some m $(0 \le m \le n-1)$. If $m \neq 0$ then z divides Dz, which implies that $Dz = 0$; otherwise, $m = 0$. So $Dz \in R$ in either case.

If $Dz \neq 0$, then $D(z^n) \neq 0$ because $\ker D$ is algebraically closed in B. Hence

$$z^n \in R \implies D(z^n) = nz^{n-1}Dz \in DR \subset Rz^\lambda \implies \lambda = n-1.$$

This proves part (i) of (b).

To prove part (a) we must show that $D^2 z = \delta^2 f = 0$, which is clear if $Dz = 0$. So assume $Dz \neq 0$. Since $Dz \in R$ and $DR \subset Rz^{n-1}$, it follows that $D^2 z \in Rz^{n-1}$. Thus z divides $D^2 z$, which implies $D^2 z = 0$. In addition, since $D(f + z^n) = 0$ we have

$$z^{n-1}\delta f + nz^{n-1}Dz = 0 \implies \delta f = -nDz \in \ker \delta.$$

So part (a) is proved.

To prove part (ii) of (b), let $b \in \ker D$ be homogeneous and write $b = az^k$, where $a \in R$ and $0 \le k \le n-1$. Then $z^k \in \ker D$, which means that $k = 0$. Therefore, $b = a \in R$. Since $\ker D$ is generated as a **k**-algebra by homogeneous elements, it follows that $\ker D \subset R$ when $Dz \neq 0$. By part (a), we have $\ker D = \ker \delta$ in this case, proving part (ii). □

REMARK 3.2. In Theorem 3.1(b), the condition that $\lambda = n-1$ is equivalent to the statement that D is the canonical quasi-extension of δ constructed previously.

COROLLARY 3.1. *Suppose R is a* **k***-domain. Let $f \in R$ and $n \ge 2$ be given, and assume that $B = R[z]/(f + z^n)$ is a domain. Then the following are equivalent*:

(a) $|f|_R \le 1$;
(b) *there exists a nonzero $D \in \mathrm{LND}(B)$ that is homogeneous relative to a $\mathbb{Z}_n$-grading of B over R such that z is homogeneous and* $\deg z \in \mathbb{Z}_n^*$.

Proof. That (b) implies (a) is a consequence of Theorem 3.1(a). Conversely, if $|f|_R \le 1$ then there exists a nonzero $\delta \in \mathrm{LND}(R)$ such that $\delta^2 f = 0$. In this case, Lemma 3.2 implies that the canonical quasi-extension of δ satisfies the conditions of part (b). □

DEFINITION 3.1. Suppose R is a **k**-domain with $\mathbb{Z}$-grading $\mathfrak{g}$, and let $f \in R$ be homogeneous. Given $n \ge 1$, set $d = \gcd(n, \deg f)$, $a = n/d$, and $\psi = \deg f/d$.

Then $(a\mathfrak{g}, \psi)$ will denote the $\mathbb{Z}$-grading of $B = R[z]/(f + z^n)$ which restricts to the $a\mathbb{Z}$-grading $a\mathfrak{g}$ on R and for which z is homogeneous and $\deg z = \psi$. In this case, $(a\mathfrak{g}, \psi)$ induces a $\mathbb{Z}_a$-grading of B over $R[z^a]$ for which $\deg z = \pi(\psi) \in \mathbb{Z}_a^*$.

Corollary 3.2. *Suppose R is a $\mathbb{Z}$-graded affine* $\mathbf{k}$*-domain, $f \in R$ is homogeneous,* $\deg f \neq 0$, *and $n \geq 2$ is an integer relatively prime to* $\deg f$. *Assume that $B = R[z]/(f + z^n)$ is a domain. Then the following are equivalent*:

(a) $|f|_R \geq 2$;
(b) *B is rigid.*

Proof. That (b) implies (a) is a direct consequence of Corollary 3.1.

Conversely, assume that B is not rigid. Let $\mathfrak{g}$ be the given $\mathbb{Z}$-grading of R, and consider the induced $\mathbb{Z}$-grading $\mathfrak{h} = (n\mathfrak{g}, \deg f)$ of B. Then $\mathfrak{h}$ induces a $\mathbb{Z}_n$-grading $\hat{\mathfrak{h}}$ of B over R for which $\deg z \in \mathbb{Z}_n^*$.

Suppose a nonzero $D \in \mathrm{LND}(B)$ is given, and let Δ be the highest-degree homogeneous summand of D relative to $\mathfrak{h}$. (This is where we use the assumption that R is affine.) Then Δ is also nonzero and homogeneous relative to $\hat{\mathfrak{h}}$. By Theorem 3.1, there exists a $\delta \in \mathrm{LND}(R)$ such that Δ is a quasi-extension of δ and $\delta^2 f = 0$. Therefore, $|f|_R \leq 1$. $\square$

Corollary 3.3. *Suppose R is an affine* $\mathbf{k}$*-domain with $\mathbb{Z}$-grading $\mathfrak{g}$, $f \in R$ is homogeneous, and $n \geq 2$ is an integer not dividing* $\deg f$. *Let $d = \gcd(n, \deg f)$, and define the rings*

$$S = R[w]/(f + w^d) \quad \textit{and} \quad B = R[z]/(f + z^n).$$

Assume that B is a domain.

(a) *Let $D \in \mathrm{LND}(B)$ be homogeneous relative to the $\mathbb{Z}$-grading $(a\mathfrak{g}, \psi)$ of B, where $a = n/d$ and $\psi = \deg f/d$; then $D^2 z = 0$.*
(b) *B is rigid if and only if $|w|_S \geq 2$.*

Proof. By hypothesis, $a \geq 2$ and $\gcd(a, \psi) = 1$. We have

$$\begin{aligned} B &= R[z]/(f + z^n) = R[z, w]/(f + w^d, w - z^a) \\ &= (R[w]/(f + w^d))[z]/(w - z^a) = S[z]/(w - z^a). \end{aligned}$$

Observe that, since S is isomorphic to a subring of B, it follows that S is a domain. The $\mathbb{Z}$-grading $(a\mathfrak{g}, \psi)$ of B induces a $\mathbb{Z}_a$-grading of B over $S = R[z^a]$, where z is homogeneous and $\deg z \in \mathbb{Z}_a^*$. The given derivation D that is homogeneous for $(a\mathfrak{g}, \psi)$ is also homogeneous for the $\mathbb{Z}_a$-grading. Now, by Theorem 3.1, $D^2 z = 0$. Thus part (a) is proved.

For part (b), note that the $\mathbb{Z}$-grading $\mathfrak{g}$ of R extends to S if we set $\deg w = \psi$, where w is homogeneous. The equivalence stated in part (b) now follows from Corollary 3.2. $\square$

Corollary 3.4. *Suppose R is a* $\mathbf{k}$*-domain, $\mathfrak{g}$ is a $\mathbb{Z}$-grading of R, and $f \in R$ is homogeneous* $(\deg f \neq 0)$. *Let $n \geq 2$ be relatively prime to* $\deg f$, *and* *assume that* $B = R[z]/(f + z^n)$ *is a domain. Then* $D(\mathrm{ML}(R)) = 0$ *for every* $D \in \mathrm{LND}(B)$ *that is homogeneous relative to the* $\mathbb{Z}$*-grading* $(n\mathfrak{g}, \deg f)$ *of* B.

Proof. The $\mathbb{Z}$-grading induces a $\mathbb{Z}_n$-grading of B over R for which $\deg z \in \mathbb{Z}_n^*$, and D is homogeneous relative to the $\mathbb{Z}_n$-grading. By Theorem 3.1, D is a quasi-extension of $\delta \in \mathrm{LND}(R)$. Since $\delta(\mathrm{ML}(R)) = 0$, it follows that $D(\mathrm{ML}(R)) = 0$. □

The following two lemmas are needed in subsequent sections.

LEMMA 3.4. *Let* R *be a commutative* $\mathbf{k}$*-domain and let*

$$B = R[z]/(f + z^n),$$

where $f \in R, n \geq 1$, *and* B *is a domain. If* $x, y \in R$ *are relatively prime in* R, *then* x *and* y *are also relatively prime in* B.

Proof. Assume that $x, y \in R$ are relatively prime in R. We need to show that $xB \cap yB \subset xyB$.

Let $g, h, k \in B$ be such that $k = xg = yh$. Since B is a free R-module given by

$$B = R + Rz + \cdots + Rz^{n-1},$$

we can write

$$g = g_0 + g_1 z + \cdots + g_{n-1} z^{n-1} \quad \text{and} \quad h = h_0 + h_1 z + \cdots + h_{n-1} z^{n-1},$$

where $g_i, h_i \in R$. Since $xg = yh$, it follows that $xg_i = yh_i$ for each i. By hypothesis, x and y are relatively prime in R. Hence there exist $G_i \in R$, $1 \leq i \leq n$, such that $g_i = yG_i$ for each i. It follows that $g \in yB$ and thus $k \in xyB$. □

LEMMA 3.5. *Assume that*:

- $\mathbf{k}$ *is algebraically closed*;
- $R = \mathbf{k}[x, y, z] = \mathbf{k}^{[3]}$;
- $f \in R$ *is a prime element*;
- $S = R/fR$ *is rigid*; *and*
- $R[w] = R^{[1]}$ *and* $B = R[w]/(f + w^d)$ *is a rational domain over* $\mathbf{k}$ $(d \geq 2)$.

If S *is not rational over* $\mathbf{k}$, *then* $|w|_B \geq 2$.

Proof. We show that $|w|_B \leq 1$ implies that S is rational.

Assume $|w|_B \leq 1$. If $|w|_B = 0$ then $B/wB \cong S$ admits a nontrivial locally nilpotent derivation, contradicting the hypothesis that S is rigid. So $|w|_B = 1$.

Let $D \in \mathrm{LND}(B)$ have w as a local slice, and let $a = Dw$. Set $A = \ker D$ and $K = \mathrm{frac}(A)$, and define the multiplicatively closed set $T = \{1, a, a^2, \ldots\} \subset A$. Then

$$T^{-1}B = T^{-1}A[w] = T^{-1}A^{[1]} \implies \mathbf{k}^{(3)} = \mathrm{frac}(B) = K(w) = K^{(1)}.$$

By the rational cancellation theorem, it follows that $K \cong \mathbf{k}^{(2)}$; see [2, Thm. 1.1].

Let $\pi_s\colon T^{-1}B \to T^{-1}A$ be the Dixmier map for $s = a^{-1}w$ (see [12, 1.1.8, and Princ. 11]). Let $p \in \mathbf{k}^{[4]}$ be such that $Dw = p(x, y, z, w)$. Since π_s is surjective and $\pi_s(w) = 0$, it follows that

$$T^{-1}A = \mathbf{k}[p(X,Y,Z,0)^{-1}, X, Y, Z],$$

where

$$\pi_s(x) = X, \quad \pi_s(y) = Y, \quad \pi_s(z) = Z.$$

Hence $K = \mathbf{k}(X,Y,Z)$, where

$$0 = \pi_s(f + w^d) = f(X,Y,Z).$$

Because f is prime, we conclude that $K \cong \mathrm{frac}(S)$. □

REMARK 3.3. Corollary 3.1 admits the following geometric interpretation when the underlying ring R is affine. In this case, when the conditions of the corollary are satisfied, f is either a kernel element or a local slice for D. Consider the affine variety $X = \mathrm{Spec}(B)$; it is endowed with an action of the cyclic group C_n of order n, and the quotient of this action is $Y = \mathrm{Spec}(R)$. Let $\pi_n \colon X \to Y$ be the quotient map, which is totally ramified over the zero set of f. The homogeneous locally nilpotent derivation D of B induces a $\mathbb{G}_a$-action on X that semi-commutes with the action of C_n. In other words, $\mathbb{G}_a \rtimes C_n$ acts on X. This action induces an action of C_n on the algebraic quotient $X /\!\!/ \mathbb{G}_a$. If f is not in the kernel of D, then the action of C_n on $X /\!\!/ \mathbb{G}_a$ is trivial. Thus the quotient map induces a morphism $Y \to X /\!\!/ \mathbb{G}_a$, which is simply the quotient map of the $\mathbb{G}_a$-action on Y induced by δ (where D is a quasi-extension of δ). Consider a generic $\mathbb{G}_a$-orbit L in X. It is isomorphic to an affine line, and $\pi_n(L)$ is a generic orbit of the action induced by δ on Y. Hence $\pi_n(L)$ corresponds to a ramified covering of an affine line onto an affine line. This is possible only if the map is ramified at exactly one point. Thus a generic orbit of the action of δ intersects the zero set of f at exactly one point. That is, f is a local slice.

4. Adjoining Two Elements

Given relatively prime positive integers m and n, we study rings of the form

$$B = R[x,y],$$

where either

$$x^m + y^n \in R \quad \text{or} \quad x^m y^n \in R.$$

We continue to assume that R is a $\mathbf{k}$-domain.

4.1. Sum of Two Powers

Let $f \in R$ and let relatively prime integers $m, n \geq 2$ be given. For indeterminates x, y over R, define

$$B = R[x,y]/(f + x^m + y^n).$$

In this situation, B is always a domain.

Define the subalgebra $S \subset B$ by

$$S = R[x] \cap R[y] = R[x^m] = R[y^n].$$

Then B is a free S-module given by

$$B = \bigoplus Sx^i y^j \quad (0 \le i \le m-1,\, 0 \le j \le n-1).$$

Given $u, v \in \mathbb{Z}_{mn}^*$, this decomposition defines a $\mathbb{Z}_{mn}$-grading of B over S; here x and y are homogeneous, $\deg x = un$, and $\deg y = vm$. We identify $\mathbb{Z}_{mn}$ with the quotient $\mathbb{Z}/mn\mathbb{Z}$ and view $m, n \in \mathbb{Z}_{mn}$ as the images of $m, n \in \mathbb{Z}$ under the canonical surjection $\mathbb{Z} \to \mathbb{Z}/mn\mathbb{Z}$.

THEOREM 4.1. *Let $D \in \mathrm{LND}(B)$ be homogeneous relative to the given $\mathbb{Z}_{mn}$-grading of B over S. Then*
(a) $D^2 x = D^2 y = 0$ *and*
(b) $Dx = 0$ *or* $Dy = 0$.

Proof. Define the quotient group

$$\Gamma = \mathbb{Z}_{mn}/n\mathbb{Z}_{mn},$$

where $\Gamma \cong \mathbb{Z}_n$, and let $\rho\colon \mathbb{Z}_{mn} \to \Gamma$ be the canonical homomorphism. The given $\mathbb{Z}_{mn}$-grading induces a Γ-grading of B over $R[x]$ in which the degree of y equals $\rho(vm) \in \Gamma^*$, and D is homogeneous relative to this induced grading. Using $R[x]$ in place of R in Theorem 3.1 as well as $f + x^m$ in place of f, it follows that $D^2 y = 0$. By symmetry, also $D^2 x = 0$ and thus part (a) is proved.

Assume that $Dx \neq 0$ and $Dy \neq 0$. Given homogeneous $b \in \ker D$, write $b = \sigma x^i y^j$ for $\sigma \in S$, $0 \le i \le m-1$, and $0 \le j \le n-1$. Then $i = j = 0$, since $\ker D$ is factorially closed. Therefore, $\ker D \subset S$. Since $Dx \in \ker D$ and $Dx \neq 0$, it follows that

$$0 = \deg(Dx) = \deg D + \deg x = \deg D + un \implies \deg D \in n\mathbb{Z}_{mn}.$$

Likewise, $Dy \in \ker D$ and $Dy \neq 0$ together imply $\deg D \in m\mathbb{Z}_{mn}$. Hence

$$\deg D \in m\mathbb{Z}_{mn} \cap n\mathbb{Z}_{mn} = \{0\} \implies \deg x = 0,$$

a contradiction. This proves part (b). □

REMARK 4.1. Theorem 4.1 does not depend on our choice of $f \in R$ and is valid even when $f = 0$.

DEFINITION 4.1. Suppose R is a $\mathbf{k}$-domain with $\mathbb{Z}$-grading $\mathfrak{g}$. Assume that $f \in R$ is homogeneous of degree $\psi \in \mathbb{Z}$ and that $m, n \ge 2$. Then $(mn\mathfrak{g}, n\psi, m\psi)$ will denote the $\mathbb{Z}$-grading of

$$B = R[x, y]/(f + x^m + y^n)$$

which restricts to the $\mathbb{Z}$-grading $mn\mathfrak{g}$ on R and for which x is homogeneous of degree $n\psi$ and y is homogeneous of degree $m\psi$. If m, n, ψ are pairwise relatively prime, then $(mn\mathfrak{g}, n\psi, m\psi)$ induces a $\mathbb{Z}_{mn}$-grading of B over $S = R[x^m] = R[y^n]$ for which $\deg x = \bar{\psi} n$ and $\deg y = \bar{\psi} m$, where $\bar{\psi} \in \mathbb{Z}_{mn}^*$ is the canonical image of ψ in $\mathbb{Z}_{mn}$.

COROLLARY 4.1. *Let R be a $\mathbb{Z}$-graded affine $\mathbf{k}$-domain, let $f \in R$ be nonzero and homogeneous of degree ψ, and let t be an indeterminate over R. Let integers $m, n \geq 2$ be such that m, n, ψ are pairwise relatively prime. If the rings $R[t]/(f + t^m)$ and $R[t]/(f + t^n)$ are rigid domains (equivalently, $R[t]/(f + t^m)$ and $R[t]/(f + t^n)$ are domains and $|f|_R \geq 2$) and if $R[x, y] = R^{[2]}$, then*

$$B = R[x, y]/(f + x^m + y^n)$$

is a rigid domain.

Proof. Let $\mathfrak{g}$ denote the $\mathbb{Z}$-grading of R, and let $S = R[x^m] = R[y^n]$. The $\mathbb{Z}$-grading $\mathfrak{h}$ of B defined by

$$\mathfrak{h} = (mn\mathfrak{g}, n\psi, m\psi)$$

induces a $\mathbb{Z}_{mn}$-grading $\hat{\mathfrak{h}}$ of B over S for which $\deg x = \bar{\psi} n$ and $\deg y = \bar{\psi} m$, where $\bar{\psi} \in \mathbb{Z}_{mn}^*$.

Let $D \in \mathrm{LND}(B)$ be nonzero and homogeneous for $\mathfrak{h}$. We may assume that D is irreducible. Since D is also homogeneous for $\hat{\mathfrak{h}}$, Theorem 4.1 implies that either $Dx = 0$ or $Dy = 0$. Assume that $Dx = 0$, and let $\bar{D}$ denote the quotient derivation on $B/xB = R[y]/(f + y^n)$. Since D is irreducible, it follows that $\bar{D} \neq 0$, which contradicts the hypothesis that $R[y]/(f + y^n)$ is rigid. Similarly, a contradiction is reached if $Dy = 0$.

We conclude that no such D exists; in other words, B is rigid. $\square$

EXAMPLE 4.1. Let $a_0, a_1, \ldots$ be a sequence of pairwise relatively prime integers with $a_i \geq 2$ for each i. We show by induction that the ring

$$B_n = \mathbf{k}[x_0, \ldots, x_n]/(x_0^{a_0} + \cdots + x_n^{a_n})$$

is rigid for each $n \geq 1$. Note first that the ring $\mathbf{k}[x_0, x_1]/(x_0^{e_0} + x_1^{e_1})$ is a rigid domain if $e_0, e_1 \geq 2$ and $\gcd(e_0, e_1) = 1$, since the cuspidal curve defined by this ring is rigid (by an easy application of Corollary 3.2).

Given $n \geq 2$, assume that the ring $\mathbf{k}[x_0, \ldots, x_k]/(x_0^{e_0} + \cdots + x_k^{e_k})$ is a rigid domain whenever $1 \leq k \leq n-1$ and that the integers $e_i \geq 2$ are pairwise relatively prime. Set $R = \mathbf{k}[x_0, \ldots, x_{n-2}]$, and define $f \in R$ by $f = x_0^{a_0} + \cdots + x_{n-2}^{a_{n-2}}$. Then R admits a $\mathbb{Z}$-grading relative to which f is homogeneous of degree $a_0 \cdots a_{n-2}$. By the inductive hypothesis, the rings $R[t]/(f + t^{a_{n-1}})$ and $R[t]/(f + t^{a_n})$ are rigid domains. By Corollary 4.1, B_n is a rigid domain.

The variety $\mathrm{Spec}(B_n)$ is an example of a *Pham–Brieskorn variety* of dimension n. These varieties are discussed in Sections 7 and 8.

4.2. Product of Two Powers

In this section we consider rings of the form

$$B = R[x, y]/(f + x^m y^n),$$

where R is a $\mathbf{k}$-domain, $R[x, y] = R^{[2]}$ $(m, n \geq 1)$ are relatively prime, and $f \in R$ is nonzero. In this situation, B is a domain.

Consider the $\mathbb{Z}$-grading $\mathfrak{g}$ of B over R in which x and y are homogeneous, with

$$\deg x = -n \quad \text{and} \quad \deg y = m.$$

If B_0 is the subalgebra of elements of degree 0 then, since $\gcd(m,n) = 1$, we have $B_0 = R[x^m y^n] = R$.

LEMMA 4.1. *If $D \in \mathrm{LND}(B)$ is $\mathfrak{g}$-homogeneous, then $Dx = 0$ or $Dy = 0$.*

Proof. Consider first the case $\ker D \subset B_0 = R$. Then B_0 is algebraic over $\ker D$ and, since $\ker D$ is algebraically closed in B, it follows that $\ker D = B_0 = R$. But then

$$0 = D(f + x^m y^n) = D(x^m y^n) \implies Dx = Dy = 0,$$

a contradiction since neither x nor y belongs to B_0.

Hence there exists a nonzero homogeneous $h \in \ker D$ such that $\deg h \neq 0$. If $\deg h < 0$ then h can be expressed as a sum of monomials of the form $rx^\alpha y^\beta$, where $r \in R$ and $\alpha > 0$. Thus $h \in xB$, which implies $Dx = 0$. Likewise, if $\deg h > 0$ then h is a sum of monomials of the form $rx^\alpha y^\beta$, where $r \in R$ and $\beta > 0$. Then $h \in yB$, which implies $Dy = 0$. □

THEOREM 4.2. *Suppose that R is an affine $\mathbf{k}$-domain and that $f \in R$ satisfies $|f|_R \geq 2$. Let $m, n \in \mathbb{Z}$ be such that $m, n \geq 2$, $\gcd(m,n) = 1$, and $f + t^m$, $f + t^n \in R[t] = R^{[1]}$ are prime. If $R[x,y] = R^{[2]}$, then the ring*

$$B = R[x,y]/(f + x^m y^n)$$

is rigid.

Proof. Since B is affine, it will suffice to show that if $D \in \mathrm{LND}(B)$ is $\mathfrak{g}$-homogeneous then $D = 0$. Assume to the contrary that $D \in \mathrm{LND}(B)$ is homogeneous and nonzero. Moreover, choose D to be irreducible.

By Lemma 4.1, either $Dx = 0$ or $Dy = 0$. If $Dx = 0$, consider the $\mathbb{Z}_n$-grading of B over R induced by $\mathfrak{g}$. Since $\deg x = 0$ relative to the $\mathbb{Z}_n$-grading, it follows that $B/(x-1)B$ is $\mathbb{Z}_n$-graded and that the quotient derivation $\bar{D}$ induced by D is both nonzero (since D is irreducible) and $\mathbb{Z}_n$-homogeneous. We then have

$$B/(x-1)B = R[x,y]/(f + x^m y^n, x - 1) = R[y]/(f + y^n),$$

where $\deg y = -m$ is a unit of $\mathbb{Z}_n$. By Theorem 3.1, there exists a nonzero $\delta \in \mathrm{LND}(R)$ such that $\delta^2 f = 0$. Yet this contradicts the hypotheses.

In exactly the same way, a contradiction is reached in the case $Dy = 0$. We conclude that the only possibility is $\mathrm{LND}(B) = \{0\}$. □

THEOREM 4.3. *Suppose that R is a rigid affine $\mathbf{k}$-domain and that $f \in R$ with $f \neq 0$. Let m, n be positive integers such that $\gcd(m,n) = 1$ and $f + t^m$, $f + t^n \in R[t] = R^{[1]}$ are prime. If $R[x,y] = R^{[2]}$, then the ring*

$$B = R[x,y]/(f + x^m y^n)$$

is rigid.

Proof. If $m, n \geq 2$ then the claim follows from Theorem 4.2, so assume that $m = 1$ or $n = 1$.

Consider first the case $m = n = 1$:

$$B = R[x, y]/(f + xy).$$

Let $D \in \mathrm{LND}(B)$ be $\mathfrak{g}$-homogeneous and irreducible. By Lemma 4.1, either $Dx = 0$ or $Dy = 0$. If $Dx = 0$ then D induces a nonzero quotient derivation on the ring

$$B/(x-1)B = R[x, y]/(f + y, x - 1) \cong_{\mathbf{k}} R,$$

contradicting the rigidity of R. Therefore, B is rigid when $m = n = 1$.

For the remaining cases, we may assume $n = 1$ and $m \geq 2$. Then

$$B = R[x, y]/(f + x^m y) = R[x, y, z]/(f + yz, z - x^m) = \Omega[x]/(z - x^m),$$

where $\Omega = R[y, z]/(f + yz)$. By our previous results, Ω is rigid.

Define a $\mathbb{Z}$-grading on Ω by $\deg y = -1$ and $\deg z = 1$. Then $z \in \Omega$ is homogeneous, and its degree is relatively prime to $m \geq 2$. Therefore, by Corollary 3.2, B is rigid. □

COROLLARY 4.2. *Let R be an affine* **k***-domain, and let $R[x, y] = R^{[2]}$. Given nonzero $f \in R$, set*

$$B = R[x, y]/(f + xy).$$

Then R is rigid if and only if B is rigid.

Proof. If R is rigid, then B is rigid by Theorem 4.3.

So assume that R is not rigid, and let D be a nonzero element of $\mathrm{LND}(R)$. Then D extends to a locally nilpotent derivation $\hat{D}$ on $R[y, y^{-1}]$ defined by $\hat{D}(y) = 0$ and $\hat{D}|_R = D$. Note that $B = R[x, y] \subset R[y, y^{-1}]$, and the locally nilpotent derivation $y\hat{D}$ of $R[y, y^{-1}]$ restricts to B. □

5. $\mathbb{Z}$-Gradings of Polynomial Rings

For rings of polynomials that are $\mathbb{Z}$-graded over a ground ring, the main theorem (Theorem 3.1) provides the basis for a degree criterion whereby certain variables are either local slices or invariants for *all* homogeneous locally nilpotent derivations. In particular, Theorem 5.1 shows that, from a finite sequence of integers, we can immediately deduce invariant properties of homogeneous derivations that are otherwise difficult to calculate. Our theorem does not assume that the derivation D is an A-derivation but only that the $\mathbb{Z}$-grading of the ring is over A.

Results in this section generalize a result of R. Kolhatkar; Theorem 5.1 was proved in her thesis for the special case where $A = \mathbf{k}$ and $\mathbf{k}$ is algebraically closed [15, Thm. 2.6.3].

THEOREM 5.1. *Suppose A is a commutative* **k***-domain and $B = A[x_1, \ldots, x_n] = A^{[n]}$ is $\mathbb{Z}$-graded over A, where each x_i is homogeneous of degree a_i $(1 \leq i \leq n)$ and $\gcd(a_1, \ldots, a_n) = 1$. Let homogeneous $D \in \mathrm{LND}(B)$ be given.*

(a) *For each i such that* $1 \leq i \leq n$,

$$\gcd(a_1, \dots, \widehat{a_i}, \dots, a_n) \neq 1 \implies D^2 x_i = 0.$$

(b) *For each pair i, j such that* $1 \leq i \leq n$, $1 \leq j \leq n$, *and* $i \neq j$,

$$\gcd(a_1, \dots, \widehat{a_i}, \dots, a_n) \neq 1 \text{ and } \gcd(a_1, \dots, \widehat{a_j}, \dots, a_n) \neq 1 \\ \implies Dx_i = 0 \text{ or } Dx_j = 0.$$

Proof. Let homogeneous $D \in \mathrm{LND}(B)$ be given. For convenience, assume $\gcd(a_1, \dots, a_{n-1}) \neq 1$ and let $p \in \mathbb{Z}$ be a prime dividing each a_j for $j = 1, \dots, n-1$. Set $R = A[x_1, \dots, x_{n-1}, y] = A^{[n]}$, where $\deg x_j = a_j$ $(1 \leq j \leq n-1)$ and $\deg y = pa_n$. Then $B = R[x_n]/(y + x_n^p)$ has a natural $\mathbb{Z}_p$-grading over R such that $\deg x_n = a_n$. Since this is the same $\mathbb{Z}_p$-grading induced by the given $\mathbb{Z}$-grading, it follows that D is homogeneous relative to this $\mathbb{Z}_p$-grading of B. By Theorem 3.1, $D^2 x_n = 0$. This proves part (a).

Under the additional hypotheses of (b) and still assuming that $i = n$, suppose $Dx_n \neq 0$. Then by Theorem 3.1 we have that $D|_R = x_n^{p-1}\delta$ for some $\delta \in \mathrm{LND}(R)$. As a result, $Dx_j = x_n^{p-1}\delta x_j \in \ker D$. Since $x_n \notin \ker D$, it follows that $\delta x_j = 0$. Therefore, $Dx_j = 0$ when $Dx_n \neq 0$. By symmetry, $Dx_n = 0$ when $Dx_j \neq 0$. This proves part (b). □

In the particular case of two variables adjoined to A, Theorem 5.1 gives the following result.

COROLLARY 5.1. *Suppose A is a commutative $\mathbf{k}$-domain and $B = A[x, y] = A^{[2]}$. Assume that B is $\mathbb{Z}$-graded over A, where x, y are homogeneous,* $\deg x = a$, $\deg y = b$, *and* $\gcd(a, b) = 1$. *If* $|a|, |b| \geq 2$ *then, for every homogeneous* $D \in \mathrm{LND}(B)$, *either* $Dx = 0$ *or* $Dy = 0$.

Proof. This is immediately implied by Theorem 5.1(b). □

Part of Kolhatkar's thesis investigates locally nilpotent derivations of polynomial rings $\mathbf{k}[x_1, \dots, x_n]$ ($\mathbf{k}$ algebraically closed) that are homogeneous relative to gradings by an abelian group G, with special interest in the case $G = \mathbb{Z}$. In particular, suppose a linear $\mathbb{Z}$-grading $\mathfrak{g}$ of B is defined by $\deg x_i = a_i \in \mathbb{Z}$ for $i = 1, \dots, n$, where $\gcd(a_1, \dots, a_n) = 1$. Given i, set $\alpha_i = \gcd(a_1, \dots, \widehat{a_i}, \dots, a_n)$. Then the *type* of $\mathfrak{g}$ is defined by

$$\mathrm{type}(\mathfrak{g}) = \#\{i \mid \alpha_i \neq 1\}.$$

In [15, Sec. 2.3.14] it is shown that, if $D \in \mathrm{LND}(B)$ is homogeneous, then

$$\mathrm{rank}(D) + \mathrm{type}(\mathfrak{g}) \leq n + 1.$$

The integer $\mathrm{rank}(D)$ is defined in [12, Sec. 3.2.1] and coincides with $\mathrm{rank}_{\mathbf{k}}(D)$ as defined in what follows.

In order to extend Kolhatkar's result, let A be a commutative $\mathbf{k}$-domain and $B = A[x_1, \dots, x_n] = A^{[n]}$ for $n \geq 2$. Let $\mathfrak{g}$ be a $\mathbb{Z}$-grading of B over A; here each

x_i is homogeneous, $\deg x_i = a_i$ for $i = 1, \ldots, n$, and $\gcd(a_1, \ldots, a_n) = 1$. Given i, set $\alpha_i = \gcd(a_1, \ldots, \widehat{a_i}, \ldots, a_n)$. Then the *type* of $\mathfrak{g}$ over A is defined by

$$\operatorname{type}_A(\mathfrak{g}) = \#\{i \mid \alpha_i \neq 1\}.$$

Similarly, let $D \in \mathrm{LND}_A(B)$; then the *corank* of D over A, denoted $\operatorname{corank}_A(D)$, is the largest integer m such that there exist $y_1, \ldots, y_m \in B$ satisfying

$$y_1, \ldots, y_m \in \ker D \quad \text{and} \quad B = A[y_1, \ldots, y_m]^{[n-m]}.$$

We then define the *rank* of D over A by $\operatorname{rank}_A(D) = n - \operatorname{corank}_A(D)$. Now Theorem 5.1(b) implies the next corollary.

COROLLARY 5.2. *Suppose A is a commutative $\mathbf{k}$-domain, $B = A^{[n]}$, and $\mathfrak{g}$ is a linear $\mathbb{Z}$-grading of B over A. Then, given homogeneous $D \in \mathrm{LND}_A(B)$, we have*

$$\operatorname{rank}_A(D) + \operatorname{type}_A(\mathfrak{g}) \leq n + 1.$$

The following example illustrates that Kolhatkar's bound cannot, in general, be improved.

EXAMPLE 5.1. For even $n \geq 6$, consider the polynomial ring $R = \mathbf{k}[x_1, \ldots, x_n] = \mathbf{k}^{[n]}$ with the standard $\mathbb{Z}$-grading defined by $\deg x_i = 1$ for each i. By [12, Thm. 3.37] there exists a homogeneous $\delta \in \mathrm{LND}(R)$ of rank n and degree 4 such that $\delta^2 x_i = 0$ for each i.

For prime $p \in \mathbb{Z}$, define $B = R[z]/(x_n + z^p) = \mathbf{k}[x_1, \ldots, x_{n-1}, z]$. Then B has a $\mathbb{Z}$-grading $\mathfrak{g}$ defined by $\deg x_i = p$ $(1 \leq i \leq n)$ and $\deg z = 1$. In particular, $\operatorname{type}(\mathfrak{g}) = 1$. Let D be the canonical quasi-extension of δ to B:

$$Dr = z^{p-1}\delta r \quad (r \in R) \quad \text{and} \quad Dz = -\frac{1}{p}\delta x_n.$$

By Lemma 3.2 we have that $D \in \mathrm{LND}(B)$. Note that $\delta x_n \neq 0$ since the rank of δ is n. Therefore, $Dz \neq 0$. It remains to show that the rank of D is n.

Assume $v \in B$ is a variable such that $Dv = 0$, and let L be its linear part; that is, L is the degree-1 summand of v relative to the standard $\mathbb{Z}$-grading of B. Then $L = c_1 x_1 + \cdots c_{n-1} x_{n-1} + c_n z$ for $c_i \in \mathbf{k}$ and, since v is a variable, we have $L \neq 0$.

Now consider the $\mathbb{Z}_p$-grading of B induced by $\mathfrak{g}$. Since D is $\mathbb{Z}_p$-homogeneous and $Dz \neq 0$, Theorem 3.1(b) implies that $\ker D = \ker \delta$; in particular, $v \in \ker \delta$. It follows that $c_n = 0$, since v can support only pth powers of z. Since δ is homogeneous in the standard grading of R, it follows that $L \in \ker \delta$—contradicting the fact that δ has no variable in its kernel. Therefore, the rank of D is n.

6. A Version of Mason's Theorem

Define $T \subset \mathbb{Z}^3$ to be the set of triples (a, b, c) satisfying $a, b, c \geq 2$, where at most one of a, b, c equals 2. Then $T = T_1 \cup T_2$ for subsets T_1 and T_2 defined by

$$T_1 = \{(a, b, c) \in T \mid \gcd(ab, c) = 1 \text{ or } \gcd(ac, b) = 1 \text{ or } \gcd(bc, a) = 1\},$$

$$T_2 = \{(a, b, c) \in T \mid a^{-1} + b^{-1} + c^{-1} \leq 1\}.$$

The following is a version of Mason's theorem.

THEOREM 6.1. *Let B be a commutative* $\mathbf{k}$*-domain, and suppose $x, y, z \in B$ are pairwise relatively prime elements that satisfy $x^a + y^b + z^c = 0$ for integers $a, b, c \geq 2$.*

(a) *If $(a,b,c) \in T_2$ then $\mathbf{k}[x,y,z] \subset \mathcal{R}(B)$, the rigid core of B.*
(b) *If there exists a $D \in \mathrm{LND}(B)$ such that $\nu_D(z) = 1$, then $a = b = 2$.*

Proof. Assume that $\mathbf{k}[x,y,z]$ is not contained in $\mathrm{ML}(B)$ and let $D \in \mathrm{LND}(B)$ be given, where at least one of Dx, Dy, Dz is nonzero.

Clearly, if two of Dx, Dy, Dz are 0 then the third is as well. So consider the case where $Dz = 0$ but $Dx \neq 0$ and $Dy \neq 0$. Then $D(x^a + y^b) = 0$. By Theorem 2.2(a), either $Dx = Dy = 0$ (which contradicts the assumption) or $x^a + y^b = 0$. If $x^a + y^b = 0$ then, since x^a and y^b are relatively prime, it follows that x and y are either 0 or invertible. In this case $Dx = Dy = 0$, which again contradicts the assumption. Therefore, $\nu_D(x)$, $\nu_D(y)$, and $\nu_D(z)$ are positive.

Apply D to the equation $x^a + y^b + z^c = 0$ to obtain

$$\begin{pmatrix} x & y & z \\ aDx & bDy & cDz \end{pmatrix} \begin{pmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and define the matrix

$$A = \begin{pmatrix} x & y & z \\ aDx & bDy & cDz \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\det(A) = bxDy - ayDx = 0$ then the element $k := bxDy = ayDx$ belongs to xyB, since x and y are relatively prime. If $k = xyh$ for $h \in B$ then $bDy = yh$, since B is a domain. But then $Dy = 0$, a contradiction.

Consequently, $\det(A) \neq 0$. We have

$$\begin{aligned} A\begin{pmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{pmatrix} &= z^{c-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \Longrightarrow \det(A)\begin{pmatrix} x^{a-1} \\ y^{b-1} \\ z^{c-1} \end{pmatrix} &= z^{c-1}\,\mathrm{adj}(A)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = z^{c-1}\begin{pmatrix} cyDz - bzDy \\ azDx - cxDz \\ bxDy - ayDx \end{pmatrix}. \end{aligned}$$

Since x, y, and z are pairwise relatively prime in B, it follows that

$$\begin{aligned} z^{c-1} \text{ divides } bxDy - ayDx &\Longrightarrow (c-1)\nu_D(z) \leq \nu_D(x) + \nu_D(y) - 1, \\ y^{b-1} \text{ divides } azDx - cxDz &\Longrightarrow (b-1)\nu_D(y) \leq \nu_D(x) + \nu_D(z) - 1, \\ x^{a-1} \text{ divides } cyDz - bzDy &\Longrightarrow (a-1)\nu_D(x) \leq \nu_D(y) + \nu_D(z) - 1; \end{aligned}$$

note that, in the first line, $bxDy - ayDx = \det(A)$. In order to prove part (a), let $\sigma = \nu_D(x) + \nu_D(y) + \nu_D(z)$. The preceding inequalities show that

$$\nu_D(x) \le \frac{\sigma - 1}{a}, \quad \nu_D(y) \le \frac{\sigma - 1}{b}, \quad \nu_D(z) \le \frac{\sigma - 1}{c}.$$

By addition we obtain

$$\sigma \le \frac{\sigma-1}{a} + \frac{\sigma-1}{b} + \frac{\sigma-1}{c} = (\sigma - 1)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

This implies

$$1 < 1 + \frac{1}{\sigma - 1} \le \frac{1}{a} + \frac{1}{b} + \frac{1}{c};$$

that is, $(a,b,c) \notin T_2$. We have thus established that $\mathbf{k}[x,y,z] \subset \mathrm{ML}(B)$ whenever $(a,b,c) \in T_2$.

Note that x, y, z are pairwise relatively prime in $\mathrm{ML}(B)$, since $\mathrm{ML}(B)$ is factorially closed in B. By induction, $\mathbf{k}[x,y,z] \subset \mathrm{ML}_n(B)$ for each $n \ge 0$. But then $\mathbf{k}[x,y,z] \subset \mathcal{R}(B)$, so part (a) is proved.

For part (b), assume that $\nu_D(z) = 1$. Then the previous inequalities yield

$$1 = \nu_D(z) \ge (b-1)\nu_D(y) - \nu_D(x) + 1 \quad \text{and}$$
$$1 = \nu_D(z) \ge (a-1)\nu_D(x) - \nu_D(y) + 1,$$

which in turn give

$$\nu_D(x) \ge (b-1)\nu_D(y) \quad \text{and} \quad \nu_D(y) \ge (a-1)\nu_D(x).$$

Hence $\nu_D(y) \ge (a-1)(b-1)\nu_D(y)$, which implies $a = b = 2$ since $\nu_D(y) \ne 0$. This proves part (b). □

The next corollary is a well-known analogue of Fermat's last theorem for rational functions; see [23, XIII.1] for a nice historical account.

COROLLARY 6.1. *Suppose $\mathbf{k} \subset B \subset \mathbf{k}^{(m)}$ for some $m \ge 0$. Given $n \ge 3$, if nonzero $a,b,c \in B$ are such that $a^n + b^n + c^n = 0$ then there exist $\lambda, \mu \in \mathbf{k}$ such that $b = \lambda a$ and $c = \mu a$.*

Proof. It suffices to prove the result for the rings $B = \mathbf{k}^{(m)}$, $m \ge 0$. Toward this end, define the following property.

> $P(n)$: *If nonzero $a,b,c \in B$ satisfy $a^n + b^n + c^n = 0$, then there exist $\lambda, \mu \in \mathbf{k}$ such that $b = \lambda a$ and $c = \mu a$.*

We show by induction on m that $\mathbf{k}^{(m)}$ satisfies $P(n)$ for all $m \ge 0$, as the case $m = 0$ is self-evident.

Assume that the fields $\mathbf{k}^{(0)}, \mathbf{k}^{(1)}, \dots, \mathbf{k}^{(m)}$ satisfy $P(n)$ for some $m \ge 0$, and let nonzero $a,b,c \in \mathbf{k}^{(m+1)}$ be such that $a^n + b^n + c^n = 0$. Write

$$a = \frac{a_1(t)}{a_2(t)}, \quad b = \frac{b_1(t)}{b_2(t)}, \quad c = \frac{c_1(t)}{c_2(t)},$$

where $a_i, b_i, c_i \in K[t] = K^{[1]}$ are nonzero $(i = 1,2)$, and $K = \mathbf{k}^{(m)}$. Then

$$(a_1 b_2 c_2)^n + (a_2 b_1 c_2)^n + (a_2 b_2 c_1)^n = 0$$

in the polynomial ring $K[t]$. Write

$$a_1b_2c_2 = d\alpha, \quad a_2b_1c_2 = d\beta, \quad a_2b_2c_1 = d\gamma;$$

here $d, \alpha, \beta, \gamma \in K[t]$ and α, β, γ are relatively prime. Then

$$\alpha^n + \beta^n + \gamma^n = 0$$

in $K[t]$. In particular, α, β, γ are pairwise relatively prime. By Theorem 6.1, $\alpha, \beta, \gamma \in \mathcal{R}(K[t]) = K = \mathbf{k}^{(m)}$. By the inductive hypothesis, there exist $\lambda, \mu \in \mathbf{k}$ such that $\beta = \lambda\alpha$ and $\gamma = \mu\alpha$. As a result,

$$b = \frac{b_1}{b_2} = \frac{d\beta/a_2c_2}{d\alpha/a_1c_2} = \frac{\beta a_1}{\alpha a_2} = \lambda a \quad \text{and} \quad c = \frac{c_1}{c_2} = \frac{d\gamma/a_2b_2}{d\alpha/a_1b_2} = \frac{\gamma a_1}{\alpha a_2} = \mu a.$$

Hence $\mathbf{k}^{(m+1)}$ satisfies $P(n)$.

We conclude by induction that $\mathbf{k}^{(m)}$ satisfies $P(n)$ for all $m \geq 0$. □

REMARK 6.1. Theorem 6.1(a) does not hold for $(a, b, c) \in T_1 - T_2$. The set $T_1 - T_2$ consists of the three elements $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$, and for each of these there exist pairwise relatively prime $x(t), y(t), z(t) \in \mathbf{k}[t]$ such that $x(t)^a + y(t)^b + z(t)^c = 0$ (see [11]). There are even counterexamples if we add the condition that x, y, z should be pairwise algebraically independent. For example, [6] gives $x(u, v), y(u, v), z(u, v) \in \mathbf{k}[u, v] = \mathbf{k}^{[2]}$, which are pairwise algebraically independent such that $x(u, v)^2 + y(u, v)^3 + z(u, v)^5 = 0$. Note also that Theorem 6.1(a) is false unless we assume that x, y, z are pairwise relatively prime. For example, if $\zeta \in \mathbb{C}$ is a primitive ninth root of unity and $\mathbb{C}[t] = \mathbb{C}^{[1]}$, then $t^3 + (\zeta t)^3 + (\zeta^2 t)^3 = 0$.

REMARK 6.2. In [10, Lemma 2], the authors give a result similar to Theorem 6.1(a) for $\mathrm{ML}(B)$ instead of $\mathcal{R}(B)$ but without the assumption that x, y, z are pairwise relatively prime. This means that their result is not correct as stated. However, the elements involved in their applications happen to be pairwise relatively prime and so their subsequent results remain valid.

7. Pham–Brieskorn Surfaces

Given $n \geq 1$ and positive integers a_i, $0 \leq i \leq n$, the corresponding *Pham–Brieskorn variety* is the hypersurface $H \subset \mathbb{A}^{n+1}$ defined by

$$x_0^{a_0} + x_1^{a_1} + \cdots + x_n^{a_n} = 0.$$

These hypersurfaces have been of interest in topology and algebraic geometry for decades; see for example the excellent survey of Seade [24].

In particular, given positive integers a, b, c, the corresponding Pham–Brieskorn surface is defined by

$$x^a + y^b + z^c = 0$$

in $\mathbb{A}^3$ and is denoted $S_{(a,b,c)}$. The coordinate ring of $S_{(a,b,c)}$ is

$$B = \mathbf{k}[x, y, z]/(x^a + y^b + z^c).$$

Note that, by Lemma 3.4, x, y, z are pairwise relatively prime in B.

THEOREM 7.1. *Suppose positive integers a, b, c are given.*

(a) *If $(a,b,c) \in T$, then $S_{(a,b,c)}$ is rigid.*
(b) *If $(a,b,c) \in T_2$, then $S_{(a,b,c)}$ is stably rigid.*

Proof. For any integer $n \geq 0$, suppose $D \in \mathrm{LND}(B^{[n]})$ is given. Since x, y, z are pairwise relatively prime in B, they are also pairwise relatively prime in $B^{[n]}$. If $(a,b,c) \in T_2$, then $Dx = Dy = Dz = 0$ by Theorem 6.1, which means that $DB = 0$. This proves part (b).

Therefore, in order to prove part (a), it suffices to assume that $(a,b,c) \in T_1$ and $n = 0$. In particular, suppose $\gcd(ab, c) = 1$ and define a $\mathbb{Z}$-grading of $R = \mathbf{k}[x, y]$ by

$$\deg x = b, \qquad \deg y = a.$$

Then $f = x^a + y^b$ is a homogeneous element of R of degree ab that is relatively prime to c. Since $|f|_R \geq 2$ by Corollary 2.1, it follows from Corollary 3.2 that B is rigid. This proves part (a). □

COROLLARY 7.1. *Let A be a commutative $\mathbf{k}$-domain, and suppose that $A[x,y,z] = A^{[3]}$. If $(a,b,c) \in T$ and $B = A[x,y,z]/(x^a + y^b + z^c)$, then $\mathrm{LND}_A(B) = \{0\}$.*

Proof. Set $K = \operatorname{frac} A$, and define

$$B' = K \otimes_A B = K[x,y,z]/(x^a + y^b + z^c).$$

By Theorem 7.1(a), B' is rigid. Since any $D \in \mathrm{LND}_A(B)$ extends to B', it follows that $\mathrm{LND}_A(B) = \{0\}$. □

QUESTION. Are the surfaces $S_{(2,3,3)}$, $S_{(2,3,4)}$, and $S_{(2,3,5)}$ stably rigid?

REMARK 7.1. If R is a rigid affine $\mathbf{k}$-domain, then it is known that $\mathrm{ML}(R^{[1]}) = R$. Moreover, if $\dim_{\mathbf{k}} R = 1$ then $\mathrm{ML}(R^{[n]}) = R$ for all $n \geq 0$; that is, R is stably rigid. In general it remains an open question whether $\mathrm{ML}(R^{[2]}) = R$ when $\dim_{\mathbf{k}} R \geq 2$, although some cases were settled by Crachiola in his thesis (see [3; 12] for details).

REMARK 7.2. Theorem 7.1 was essentially proved by Kaliman and Zaidenberg in [14]. They do not explicitly state the stable rigidity of $S = S_{(a,b,c)}$ when $(a,b,c) \in T_2$ but this is contained in their proof, which relies on geometric methods in addition to Mason's theorem. They consider the existence of sufficiently many rational curves on a variety and show that any rational curve on S must pass through the singularity. If there were a nontrivial $\mathbb{G}_a$-action on $V = S \times \mathbb{A}^n$, then any one-dimensional orbit not contained in $\mathrm{Sing}(V)$ must be disjoint from $\mathrm{Sing}(V)$. The projection of this orbit to S is thus either a point or a rational curve not passing through the singularity. Hence the only possibility is that this projection is a point.

8. Pham–Brieskorn Threefolds

In this section we consider several classes of Pham–Brieskorn threefolds, as described in the following result.

THEOREM 8.1. *Given integers $a,b,c,d \geq 2$, the ring*

$$B = \mathbf{k}[x_0,x_1,x_2,x_3]/(x_0^a + x_1^b + x_2^c + x_3^d)$$

is rigid in each of the following cases:

(a) $\gcd(abc,d) = 1$;
(b) $\min\{a,b,c,d\} \geq 8$;
(c) $(a,b,c,d) = (a,3,3,3)$, *where* $a \neq 3$;
(d) $(a,b,c,d) = (2,b,c,d)$, *where* $b,c,d \geq 3$, *b is even,* $\gcd(b,c) \geq 3$, *and* $\gcd(d, \operatorname{lcm}(b,c)) = 2$.

The proof proceeds in several steps.

8.1. Proof of Theorem 8.1(a)

Define a $\mathbb{Z}$-grading of $R = \mathbf{k}[x_0,x_1,x_2]$ by

$$\deg x_0 = bc, \quad \deg x_1 = ac, \quad \deg x_2 = ab.$$

Then $f = x_0^a + x_1^b + x_2^c$ is a homogeneous element of R of degree abc relatively prime to d. Since $|f|_R \geq 2$ by Theorem 2.3, it follows from Corollary 3.2 that B is rigid.

8.1. Proof of Theorem 8.1(b)

The statement is a consequence of the following result, which is due to Bayat and Teimoori.

THEOREM 8.2 [1, Thm. 5]. *Let n_1,n_2,n_3,n_4 be integers with* $\min\{n_1,n_2,n_3,n_4\} \geq 8$. *Then there is no solution in the polynomial ring $\mathbb{C}[t]$ for the equation*

$$f_1(t)^{n_1} + f_2(t)^{n_2} + f_3(t)^{n_3} + f_4(t)^{n_4} = 0$$

with nonconstant pairwise relatively prime $f_i \in \mathbb{C}[t]$.

We first note that the proof of Theorem 8.2 given by Bayat and Teimoori uses nothing special about $\mathbb{C}$—only that the ground field is algebraically closed and of characteristic 0. Thus Theorem 8.2 remains valid when the complex field is replaced by any algebraically closed field of characteristic 0.

Suppose there exists a nonzero $D \in \operatorname{LND}(B)$. Then $Dx_i \neq 0$ for each i, since otherwise the quotient ring B/x_iB is nonrigid for some i. Set $A = \ker D$ and $K = \operatorname{frac}(A)$, and let $t \in B$ be a local slice. We have $B \subset K[t] \subset \bar{K}[t]$, where $\bar{K}$ denotes the algebraic closure of K. Hence the equation

$$f_0(t)^a + f_1(t)^b + f_2(t)^c + f_3(t)^d = 0$$

has a solution $x_i = f_i(t)$ in $K[t] \subset \bar{K}[t]$, where each f_i is nonconstant. By Theorem 8.2, it follows that $f_i(t)$ and $f_j(t)$ have a common root $\xi \in \bar{K}$ for at least one pair $i \neq j$. Let $g(t) \in K[t]$ be the minimal polynomial of ξ over K. Then $\deg_t g(t) \geq 1$, and there exist $p(t), q(t) \in K[t]$ such that

$$f_i(t) = g(t)p(t) \quad \text{and} \quad f_j(t) = g(t)q(t). \tag{2}$$

Therefore, $q(t)x_i = p(t)x_j$ in $K[t]$. Choose nonzero $\alpha \in A$ such that $P(t) := \alpha p(t)$ and $Q(t) := \alpha q(t)$ belong to $A[t]$. Then, in B, we have

$$Q(t)x_i = P(t)x_j.$$

Since x_iB is a prime ideal of B not containing x_j, it follows that $P(t) \in x_iB$. From the first equation of (2) we conclude that $\deg_t g(t) = 0$, a contradiction. Therefore, no such D can exist.

8.3. Proof of Theorem 8.1(c)

In order to prove part (c), consider the Pham–Brieskorn threefold Y with coordinate ring

$$\Upsilon = \mathbf{k}[Y] = \mathbf{k}[\theta_0, \theta_1, \theta_2, \theta_3]/(\theta_0^3 + \theta_1^3 + \theta_2^3 + \theta_3^3).$$

This Y is called the *affine Fermat cubic threefold* and is the affine cone over the famous Fermat cubic surface X. It is well known that X is rational; hence Y is also rational. Here Υ is naturally $\mathbb{Z}$-graded, where each θ_i is homogeneous of degree 1.

Lemma 8.1. $|\theta_0|_\Upsilon \geq 2$.

Proof. If $|\theta_0|_\Upsilon = 0$ then there exists an irreducible $D \in \mathrm{LND}(\Upsilon)$ with $D\theta_0 = 0$. But then $D \bmod (\theta_0\Upsilon)$ is a nonzero locally nilpotent derivation on $\Upsilon/\theta_0\Upsilon$, which by Theorem 7.1 is a rigid ring. Therefore, $|\theta_0|_\Upsilon \geq 1$.

Assume that $|\theta_0|_\Upsilon = 1$, and let $D \in \mathrm{LND}(\Upsilon)$ have θ_0 as a local slice. Let $L = \mathrm{frac}(\Upsilon)$ and $K = \mathrm{frac}(\ker D)$. If $\pi_s \colon \Upsilon \to K$ is the Dixmier map for $s = \theta_0/D\theta_0$, then $\ker \pi_s = \theta_0\Upsilon_{D\theta_0} \cap \Upsilon$; see [12, 1.1.8, Princ. 11]. In particular, $\pi_s(\theta_0) = 0$ and so

$$0 = \pi_s(\theta_0^3 + \theta_1^3 + \theta_2^3 + \theta_3^3) = \pi_s(\theta_1)^3 + \pi_s(\theta_2)^3 + \pi_s(\theta_3)^3.$$

This is an equation in K, and $\mathbf{k} \subset K \subset L = \mathbf{k}^{(3)}$.

If $\pi_s(\theta_i) = 0$ for $i \in \{1, 2, 3\}$ then $\theta_i \in \ker \pi_s$, which implies that $(D\theta_0)^n\theta_i \in \theta_0\Upsilon$ for some $n \geq 0$. Note that $\theta_0\Upsilon$ is a prime ideal of Υ not containing θ_1, θ_2, or θ_3. Therefore, $n \geq 1$ and $D\theta_0 \in \theta_0\Upsilon$. But this implies $D\theta_0 = 0$, a contradiction.

As a result, $\pi_s(\theta_i) \neq 0$ for $i = 1, 2, 3$. By Corollary 6.1, there exists a $\lambda \in \mathbf{k}$ such that $\pi_s(\theta_2) = \lambda\pi_s(\theta_1)$. It follows that

$$\begin{aligned} \pi_s(\theta_2 - \lambda\theta_1) = 0 &\implies \theta_2 - \lambda\theta_1 \in \ker \pi_s \\ &\implies (D\theta_0)^n(\theta_2 - \lambda\theta_1) \in \theta_0\Upsilon \text{ for some } n \geq 0. \end{aligned}$$

But $\theta_0\Upsilon$ is a prime ideal of Υ not containing $\theta_2 - \lambda\theta_1$. Hence $n \geq 1$ and $D\theta_0 \in \theta_0\Upsilon$—again a contradiction. It must therefore be the case that $|\theta_0|_\Upsilon \geq 2$. □

To complete the proof of Theorem 8.1(c), let $m \geq 2$ be given. Since $|\theta_0|_\Upsilon \geq 2$ by Lemma 8.1, it follows from Corollary 3.2 that the ring

$$\Upsilon[z]/(\theta_0 - z^m) = \mathbf{k}[z, \theta_1, \theta_2, \theta_3]/(z^{3m} + \theta_1^3 + \theta_2^3 + \theta_3^3)$$

is rigid.

On the other hand, if $a \geq 2$ is an integer not divisible by 3, then Theorem 8.1(a) indicates that the ring

$$\mathbf{k}[x_0, x_1, x_2, x_3]/(x_0^a + x_1^3 + x_2^3 + x_3^3)$$

is rigid.

8.4. *Proof of Theorem 8.1(d)*

Set $R = \mathbf{k}[x_0, x_1, x_2] = \mathbf{k}^{[3]}$, and define $f \in R$ by $f = x_0^2 + x_1^b + x_2^c$. Consider the following (nonrigid) Pham–Brieskorn threefold with coordinate ring:

$$W = R[w]/(f + w^2) = \mathbf{k}[x_0, x_1, x_2, w]/(x_0^2 + x_1^b + x_2^c + w^2).$$

Assume first that $\mathbf{k}$ is algebraically closed. We established in Theorem 7.1 that the ring $S = R/fR$ is rigid. Now we will show by the following lemma that the hypotheses imply that S is nonrational over $\mathbf{k}$.

LEMMA 8.2. *Let S be the ring $\mathbf{k}[x_0, x_1, x_2]/(x_0^2 + x_1^b + x_2^c)$, where b is even and* $\gcd(b, c) \geq 3$. *Then S is irrational.*

Proof. Let r be the greatest common divisor of b and c. Suppose that $b = \beta r$ and $c = \gamma r$ for integers β and γ. Choose $n, m \in \mathbb{Z}$ such that $n\beta + m\gamma = 1$. Consider the generators $x_0, x_1, x_2 \in S$. Observe that $\mathrm{frac}(S) = \mathbf{k}(x_0, x_1, x_2) = \mathbf{k}(X, Y, Z)$, where

$$X = x_0/x_1^{b/2}, \quad Y = x_2^\gamma/x_1^\beta, \quad Z = x_1^m x_2^n.$$

Since $x_0^2 + x_1^b + x_2^c = 0$, we have that $X^2 + 1 + Y^r = 0$. Thus $\mathbf{k}(x_0, x_1, x_2) = K(Z) = K^{(1)}$, where $K = \mathbf{k}(X, Y)$. If $r \geq 3$, then K is the field of a hyperelliptic curve. In particular, it is not rational and so $K(Z)$ is not rational. □

Since W is rational over $\mathbf{k}$, it follows from Lemma 3.5 that $|w|_W \geq 2$. Therefore, $|w|_W \geq 2$ without the assumption that $\mathbf{k}$ is algebraically closed (see Section 2.3).

We remark that R has a $\mathbb{Z}$-grading for which f is homogeneous and $\deg f = \mathrm{lcm}(b, c)$. In addition, d does not divide $\deg f$ because $\gcd(d, \deg f) = 2 < d$. Since $|w|_W \geq 2$, it follows from Corollary 3.3(b) that $B = R[x_3]/(f + x_3^d)$ is rigid.

This completes the proof of Theorem 8.1.

EXAMPLE 8.1. Let $k \geq 2$ be an integer not divisible by 3. Then Theorem 8.1(d) implies that B is rigid when $(a, b, c, d) = (2, 3, 6, 2k)$.

EXAMPLE 8.2. It is possible to glean additional cases from Theorem 8.1. The preceding example shows that the ring

$$S' = \mathbf{k}[x_0, x_1, x_2, x_3]/(x_0^2 + x_1^3 + x_2^6 + x_3^{2k})$$

is rigid when $k \geq 2$ is not divisible by 3. In particular, $|x_0|_{S'} \geq 2$. Define $R = \mathbf{k}[x_1, x_2, x_3]$ and $f \in R$ by $f = x_1^3 + x_2^6 + x_3^{2k}$. Then R has a $\mathbb{Z}$-grading relative to which $\deg f = 6k$. If $m \geq 2$ is any integer such that $\gcd(m, 3k) = 1$, then it follows from Corollary 3.3(b) that the ring

$$R[z]/(f + z^{2m}) = \mathbf{k}[z, x_1, x_2, x_3]/(z^{2m} + x_1^3 + x_2^6 + x_3^{2k})$$

is rigid.

REMARK 8.1. Lemma 8.2 is a special case of a more general result concerning the rationality of Pham–Brieskorn surfaces. When $\mathbf{k} = \mathbb{C}$, it is well known for which values of a, b, c the surface V defined by the equation $x_0^a + x_1^b + x_2^c = 0$ in $\mathbb{C}^3$ is rational (see e.g. [14; 21]). The idea is to use that V has a natural $\mathbb{C}^*$-action and that the quotient of $V \setminus \{(0, 0, 0)\}$ is a smooth curve (denoted by L). Since V is birationally equivalent to the product of L and $\mathbb{C}$, one checks whether the conditions for L to be rational are satisfied.

For the proof of Theorem 8.1(d) we need only consider the case when $a = 2$. In that case, Lemma 8.2 establishes a sufficient condition for V to be irrational.

Note also that, in [5, Cor. 4.10], Daigle gives conditions on integer triples (a, b, c) which imply that the weighted projective plane curve defined by the polynomial $x_0^a + x_1^b + x_2^c$ is not rational over any algebraically closed field of characteristic 0.

REMARK 8.2. An important open question is whether the affine Fermat cubic threefold Y defined by

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

is rigid. We address also the following cubic threefold, which is the affine cone over a singular cubic surface.

THEOREM 8.3. *If B is the ring*

$$B = \mathbf{k}[x, y, z, t]/(x^3 + y^3 + xyz + t^3),$$

then $\mathrm{ML}(B) = \mathbf{k}$.

Proof. Let $R = \mathbf{k}[x, y, z] = \mathbf{k}^{[3]}$, and define $r \in R$ by $r = x^3 + y^3 + xyz$. According to [12, Sec. 5.5.2], there exists a sequence $\delta_n \in \mathrm{LND}(R)$ such that $\delta_n r \in \ker \delta_n$ but $\delta_n r \neq 0$ and such that $\ker \delta_m \cap \ker \delta_n = \mathbf{k}$ when $n \geq m + 2$. Let D_n be the canonical quasi-extension of δ_n for each n. By Lemma 3.2, the derivations D_n are locally nilpotent and $\ker D_n = \ker \delta_n$ for each n. We conclude that $\mathrm{ML}(B) = \mathbf{k}$. $\square$

It is worth mentioning that the same argument shows that every fiber $x^3 + y^3 + xyz + t^3 = \lambda$ for $\lambda \in \mathbf{k}$ has trivial Makar-Limanov invariant.

9. Pham–Brieskorn Surfaces with Parameter

In this section we consider threefolds defined by rings of the form

$$B = R[x, y, z]/(ux^a + vy^b + wz^c),$$

where $R = \mathbf{k}[t] = \mathbf{k}^{[1]}$ and $u, v, w \in R$ satisfy $\gcd(u, v, w) = 1$. These threefolds may be viewed as Pham–Brieskorn surfaces with a parameter introduced, in the sense that $\mathbf{k}(t) \otimes_{\mathbf{k}} B$ is the coordinate ring of a Pham–Brieskorn surface over the function field $\mathbf{k}(t)$.

We first need the following result.

THEOREM 9.1. *Given integers a, b, c with $a, b, c \geq 2$, the rings*

$$A = \mathbf{k}[y,z]/(y^b + z^c) \quad \textit{and} \quad B = \mathbf{k}[x,y,z]/(x^a y^b + z^c)$$

are rigid.

Our proof requires the following lemma.

LEMMA 9.1. *Let a, b, c be positive integers. Then* $\gcd(a,b,c) = 1$ *if and only if $x^a y^b + z^c$ is irreducible in* $\mathbf{k}[x,y,z]$.

Proof. Let $F = x^a y^b + z^c$, and set $d = \gcd(a,b)$. Let $\alpha = a/d$ and $\beta = b/d$. Then $F = (x^\alpha y^\beta)^d + z^c$ and $\gcd(a,b,c) = \gcd(c,d)$. It is easy to see that F is reducible if $\gcd(c,d) \geq 2$.

Now suppose that $\gcd(c,d) = 1$. Choose $u, v \in \mathbb{Z}$ such that $au + bv = d$, and define the map $\varphi\colon \mathbf{k}[x,y,z] \to \mathbf{k}[t,t^{-1},z]$ by

$$\varphi(x) = t^u, \quad \varphi(y) = t^v, \quad \varphi(z) = z.$$

Note that if $P \in \mathbf{k}[x,y,z]$ is monic in z and $P \notin \mathbf{k}[x,y]$, then $\deg_z \varphi(P) = \deg_z P$.

Suppose that $F = GH$ for $G, H \in \mathbf{k}[x,y,z]$. We may assume that G and H are monic in z. Then $\varphi(F) = t^d + z^c = gh$, where $g = \varphi(G)$ and $h = \varphi(H)$. Since $\gcd(c,d) = 1$, we have that $t^d + z^c$ is irreducible in $\mathbf{k}[t,z]$ and thus also in $\mathbf{k}[t,t^{-1},z]$. Therefore, either g or h is constant. It follows that either $\deg_z G = 0$ or $\deg_z H = 0$. Given the particular form of F, this suffices to conclude that either $G \in \mathbf{k}$ or $H \in \mathbf{k}$. $\square$

Proof of Theorem 9.1. We may assume that the ground field $\mathbf{k}$ is algebraically closed. Note that A and B are reduced rings. In addition, the curve $Z = \operatorname{Spec}(A)$ is irreducible if and only if $\gcd(b,c) = 1$, and by Lemma 9.1 the surface $X = \operatorname{Spec}(B)$ is irreducible iff $\gcd(a,b,c) = 1$.

To prove the assertion for A, set $d = \gcd(b,c)$. If $d = 1$ then Z is called a *cuspidal plane curve*; in this case, for $S = \mathbf{k}[y]$ we have

$$|y^b|_S = b|y|_S = b \geq 2.$$

Moreover, y is homogenous of degree 1 in the standard $\mathbb{Z}$-grading of S. It then follows from Corollary 3.2 that A is rigid in this case.

If $d \geq 2$, then Z is a either a union of distinct cuspidal curves (when $b/d \geq 2$ and $c/d \geq 2$) or a union of distinct lines (when $b/d = 1$ or $c/d = 1$). The irreducible components $Z_1, \ldots, Z_d$ comprised by Z intersect in a single point Q, and this point is fixed by any $\mathbb{G}_a$-action. Moreover, any $\mathbb{G}_a$-action on Z restricts to each of the irreducible components Z_i (see [4, Prop. 1.4]). Hence any $\mathbb{G}_a$-action on Z restricts to the complement $Z_i - Q$ for each i. Recall, however, that $Z_i - Q = \mathbf{k}^*$ for each i and that $\mathbf{k}^*$ is rigid. We therefore conclude that A is rigid in this case as well.

In order to prove the assertion for B, consider first the case $\gcd(a,b,c) = 1$. Let $D \in \mathrm{LND}(B)$ be given. If $D^2(x^a y^b) = 0$ then $a, b \geq 2$ implies that xy divides the image $D(x^a y^b)$, which is in the kernel of D. But then $Dx = Dy = 0$, which implies $D = 0$. Therefore, $|x^a y^b|_B \geq 2$. In addition, since $\gcd(a,b,c) = 1$, there exists a $\mathbb{Z}$-grading of $R = \mathbf{k}[x,y]$ such that $x^a y^b$ is homogeneous and c is relatively prime to $\deg(x^a y^b)$. By Corollary 3.2, we conclude that B is rigid.

For the general case, set $e = \gcd(a,b,c)$ and assume $e \geq 2$. Then $u^a v^b + w^c$ is a reducible polynomial in $\mathbf{k}[u,v,w] = \mathbf{k}^{[3]}$, where each prime factor appears with multiplicity 1. As before, any $\mathbb{G}_a$-action on X restricts to each of the irreducible components $X_1, \dots, X_e$ of X. In addition, any $\mathbb{G}_a$-action on X must fix the intersection $Y = \bigcap_{1 \leq i \leq e} X_i$, which is a union of two distinct but intersecting lines (since $e \geq 2$). Therefore, any $\mathbb{G}_a$-action on X restricts to the complement $X_i - Y$ for each i.

The component X_i has the form $X_i = \mathrm{Spec}(B_i)$, where

$$B_i = \mathbf{k}[x,y,z]/(x^{a/e} y^{b/e} + \lambda_i z^{c/e}) \quad (\lambda_i \in \mathbf{k}^*).$$

If $a/e, b/e, c/e \geq 2$ then, by what we have already shown, each B_i is rigid and so any $\mathbb{G}_a$-action on X is trivial. Otherwise, $a/e = 1$, $b/e = 1$, or $c/e = 1$, which means that each component X_i is isomorphic to a Danielewski surface (possibly a plane). Let i be given, $1 \leq i \leq e$. Since $Y \subset X_i$ is defined by $xy = z = 0$, we see that $X_i - Y$ is isomorphic to $\mathbf{k}^* \times \mathbf{k}^*$, which is a rigid variety. Therefore, the only $\mathbb{G}_a$-action on X is trivial. □

9.1. The Case of One Coefficient with Parameter

THEOREM 9.2. *Let integers $a, b, c, d \geq 2$ be given, where b and c are not both 2. Define*

$$B = \mathbf{k}[t,x,y,z]/(t^d x^a + y^b + z^c),$$

and set $e = \gcd(a,d)$. Then B is rigid in each of the following cases:

(i) $e = 1$;
(ii) $(e,b,c) \in T$;
(iii) $e = 2$, $a \neq 2$, *and* $d \neq 2$.

Proof. Set $R = \mathbf{k}[y,z]$ and $f = y^b + z^c$. Let m, n be positive integers such that $a = em$ and $d = en$.

(i) Since b and c are not both 2, Corollary 2.1 implies that $|f|_R \geq 2$. By Theorem 4.2, we conclude that B is rigid in this case.

(ii), (iii) We have

$$B = R[t,x]/(f + t^d x^a) = R[t,x,v]/(f + v^e, v - t^n x^m) = S[t,x]/(v - t^n x^m),$$

where

$$S = R[v]/(f + v^e) = \mathbf{k}[v,y,z]/(v^e + y^b + z^c).$$

If $|v|_S = 1$ then Theorem 6.1(b) implies that $b = c = 2$, a contradiction. If $|v|_S = 0$ then the quotient ring R/fR is not rigid, which contradicts Theorem 9.1. Therefore, $|v|_S \geq 2$.

If $e \neq 2$ or if $e = 2$ and $b, c \geq 3$, then S is rigid by Theorem 7.1. It follows from Theorem 4.3 that B is rigid in this case. Otherwise, $e = 2$ and $m, n \geq 2$. Since $|v|_S \geq 2$, it follows from Theorem 4.2 that B is rigid in this case as well. □

REMARK 9.1. When $a, b, c, d \geq 2$, all other cases are nonrigid if we assume that $i \in \mathbf{k}$ and $i^2 = -1$. These cases are defined by polynomials of the form

$$t^d x^a + y^2 + z^2 \quad \text{or} \quad t^{2k} x^2 + y^2 + z^c \ \ (k \geq 1),$$

which are irreducible polynomials. For the second case, let $R = \mathbf{k}[t, x, y] = \mathbf{k}^{[3]}$ and $f = t^{2k}x^2 + y^2$. It is easy to check that $|f|_R = 1$, so Lemma 3.1 implies nonrigidity in this case. It should be noted that, when $d = 1$, the hypersurface $tx^a + y^b + z^c = 0$ is nonrigid for all positive integers a, b, c. We also have the following statement.

THEOREM 9.3. *Given an integer $n \geq 1$, define the ring*

$$B = \mathbf{k}[t, x, y, z]/(t^2x^2 + y^2 + z^n).$$

If $i = \sqrt{-1}$ belongs to $\mathbf{k}$, then $\mathrm{ML}(B) = \mathbf{k}$.

Proof. If $n = 1$, then $B = \mathbf{k}^{[3]}$ and the result is clear. So assume $n \geq 2$.

Let $R = \mathbf{k}[t, x, y]$, and define $f \in R$ by $f = t^2x^2 + y^2$. If $u = y + itx$ and $v = y - itx$, then $R = \mathbf{k}[t, x, u] = \mathbf{k}[t, x, v]$ and $f = uv$. Define $\delta_1, \delta_2 \in \mathrm{LND}(R)$ by

$$\delta_1(t) = 1,\ \delta_1(x) = \delta_1(u) = 0 \quad \text{and} \quad \delta_2(x) = 1,\ \delta_2(t) = \delta_2(u) = 0.$$

Then $\delta_1^2 f = \delta_2^2 f = 0$. Let D_1 and D_2 be the canonical quasi-extensions of δ_1 and δ_2, respectively. By Lemma 3.2, D_1 and D_2 are locally nilpotent. We have

$$\ker D_1 = \ker \delta_1 = \mathbf{k}[x, u] \quad \text{and} \quad \ker D_2 = \ker \delta_2 = \mathbf{k}[t, u],$$

so $\mathrm{ML}(B) \subset \mathbf{k}[u]$. By symmetry, $\mathrm{ML}(B) \subset \mathbf{k}[v]$ as well. Therefore, $\mathrm{ML}(B) = \mathbf{k}$. □

9.2. The Case of Two Coefficients with Parameter

THEOREM 9.4. *Suppose $a, b, c, d, e \geq 2$, and set*

$$B = \mathbf{k}[t, x, y, z]/(t^d x^a + t^e y^b + z^c).$$

(a) $t \in \mathrm{ML}(B)$.
(b) *If $(a, b, c) \in T$, then B is rigid.*

Proof. Since $d, e \geq 2$, the singular locus of $X = \mathrm{Spec}(B)$ consists of the union of the hypersurface $Y \subset X$ defined by the ideal $\sqrt{tB} = (t, z)$ and the line defined by $x = y = 0$. Therefore, any $\mathbb{G}_a$-action on X restricts to Y, which means that $\sqrt{tB}$ is an integral ideal for each $D \in \mathrm{LND}(B)$. By Lemma 2.1, it follows that $Dt = 0$ for every $D \in \mathrm{LND}(B)$. This proves part (a).

For part (b), assume $(a, b, c) \in T$. If $\mathrm{LND}(B) \neq \{0\}$ then $|t|_B = 0$, which implies that the quotient $B/(t-1)B$ is nonrigid. Since this would contradict Theorem 7.1, it follows that B is rigid. □

REMARK 9.2. Without the hypothesis that $d, e \geq 2$ in this theorem, we do not know whether B is rigid when $(a, b, c) \in T$. Some of these cases can be settled by using the following lemma in combination with Corollary 3.2. It should be noted that, when $d = e = 1$, the hypersurface $tx^a + ty^b + z^c = 0$ is nonrigid for all positive integers a, b, c.

LEMMA 9.2. *Let $R = \mathbf{k}[t, x, y] = \mathbf{k}^{[3]}$ and let $f \in R$ be given by*

$$f = t^d x^a + t^e y^b,$$

where $d, e \geq 1$, $a, b \geq 2$, a and b are not both 2, and d and e are not both 1. Then $|f|_R \geq 2$.

Proof. We may assume without loss of generality that $d \geq e$. Suppose $|f|_R \leq 1$, and choose irreducible $D \in \mathrm{LND}(R)$ with $|f|_R = \nu_D(f)$. If $Dt = 0$ then $\nu_D(f) \geq 2$ by Theorem 2.2, a contradiction. Hence $Dt \neq 0$ and so $\nu_D(t) \geq 1$. We have

$$1 \geq \nu_D(f) = e\nu_D(t) + \nu_D(t^{d-e}x^a + y^b),$$

which implies

$$e = \nu_D(t) = 1 \quad \text{and} \quad \nu_D(t^{d-e}x^a + y^b) = 0.$$

Therefore, by hypothesis, $d \geq 2$.

Since D is irreducible, it induces a nonzero locally nilpotent derivation θ on the quotient ring $\bar{R} = \mathbf{k}[t, x, y]/(t^{d-1}x^a + y^b)$. In particular, $\bar{R}$ is not rigid. If $d \geq 3$ then Lemma 9.1 implies that $\bar{R}$ is rigid, a contradiction. Therefore, $d = 2$.

We thus have $f = tg$ for $g = tx^a + y^b$, where $D^2 t = 0$ and $Dg = 0$. This implies $\theta^2 t = 0$. Since $\mathrm{ML}(\bar{R}) = \mathbf{k}[x]$ it follows that $\theta x = 0$ (see [17; 18]), and $\theta y \neq 0$ (for otherwise $\theta = 0$). But then

$$1 \geq \nu_\theta(t) = \nu_\theta(tx^a) = \nu_\theta(y^b) = b\nu_\theta(y) \geq b \geq 2,$$

a contradiction. Therefore, $|f|_R \geq 2$. $\square$

Kaliman and Makar-Limanov [13] consider the complex threefolds defined by

$$t^{m(n-1)}x^n - t^{m(k-1)}y^k + z^\ell = 0,$$

where $m \geq 1$, $n > k \geq 2$, $\gcd(n, k) = 1$, and $\ell \geq 2$. They show that such a threefold is rigid unless $k = l = 2$ and m is even. Our framework provides an alternate proof over any field $\mathbf{k}$ of characteristic 0 (and without the assumption $\gcd(n, k) = 1$).

COROLLARY 9.1 (cf. [13, Prop. 10.1]). *Let*

$$B = \mathbf{k}[t, x, y, z]/(t^{m(n-1)}x^n - t^{m(k-1)}y^k + z^\ell),$$

where $\ell \geq 2$, $m \geq 1$, $n \geq k \geq 2$, and n and k are not both 2. Then B is rigid except when $k = \ell = 2$ and m is even.

Proof. Set $\gamma = \gcd(\ell, m)$. Consider first the case $\gamma = 1$, and define a $\mathbb{Z}$-grading on $R = \mathbf{k}[t, x, y] = \mathbf{k}^{[3]}$ by

$$(\deg t, \deg x, \deg y) = (-1, m, m).$$

Let $f \in R$ be defined by $f = t^{m(n-1)}x^n - t^{m(k-1)}y^k$. Then $f \in R$ is homogeneous of degree m and ℓ is relatively prime to m. Lemma 9.2 and Corollary 3.2 now imply that B is rigid.

So assume that $\gamma \geq 2$. Then

$$m \geq \gamma \geq 2 \implies m(n-1) \geq m(k-1) \geq 2.$$

By Theorem 9.4, if $(\ell, n, k) \in T$ then B is rigid. Otherwise, the triple (ℓ, n, k) equals $(2, 2, N)$, $(2, N, 2)$, or $(N, 2, 2)$ for some $N \geq 2$. Because $k = \min\{k, n\}$, we conclude that $k = 2$ and therefore $n \neq 2$. Thus, $\ell = \gamma = 2$ and 2 divides m. $\square$

10. Concluding Remarks

REMARK 10.1. Let B be an affine $\mathbf{k}$-domain with $\dim_{\mathbf{k}} B \geq 2$. Given a nonzero $f \in B$, we have

$$B/fB \text{ is rigid} \implies |f|_B \geq 1.$$

However, the converse of this is generally false, as seen from the following result (due to Derksen, Kutzschebauch, and Winkelmann). The ground field for this result is the field of complex numbers.

THEOREM 10.1 [9, Thm. 1]. *There exist a smooth irreducible hypersurface $H \subset \mathbb{A}^5$ and an algebraic action μ of the additive group $(\mathbb{C}, +)$ on H such that, for all $t \neq 0$, there exists neither an algebraic nor a holomorphic automorphism φ of $\mathbb{A}^5$ with $\varphi|_H = \mu(t)$.*

It follows that the natural map

$$\mathrm{LND}_f(B) \to \mathrm{LND}(B/fB)$$

is not generally surjective even when B is a polynomial ring.

REMARK 10.2. We have focused on rings $R[z]$ such that $z^n \in R$. There are three related classes of rings that naturally suggest themselves for similar investigation:

1. rings of the form $R[z]$, where z is integral over R;
2. rings of the form $R[z]$, where $z \in \mathrm{frac}(R)$; and
3. rings of the form $R[x, y]$, where $xy \in R$.

Note that a ring of the third type is the extended Rees algebra of a principal ideal of R and that some results for these rings are given in Section 4.

REMARK 10.3. One of the main ideas of this paper is to develop a systematic way of studying $\mathbb{G}_a$-actions that semi-commute with an action of a cyclic group. More precisely, in Sections 3 and 4 we consider a cyclic group action on a variety where the quotient map is a ramified covering space of a specific form. In the first case (adjoining one root), the ramification is always total. In the second case (adjoining two elements), there are critical points of different ramification indices. It would be of interest to use the ideas developed here to study in general the case of finite abelian group actions; we should like to find results analogous to Theorems 3.1 and 4.1 concerning kernels and local slices of homogeneous derivations.

References

[1] M. Bayat and H. Teimoori, *A generalization of Mason's theorem for four polynomials,* Elem. Math. 59 (2004), 23–28.

[2] A. Belov and J.-T. Yu, *On low-dimensional cancellation problems,* J. Algebra 319 (2008), 2235–2242.

[3] A. Crachiola, *On the AK-invariant of certain domains,* Ph.D. thesis, Wayne State Univ., Detroit, MI, 2004.

[4] A. Crachiola and S. Maubach, *Rigid rings and Makar-Limanov techniques,* preprint, 2010, arXiv:1005.4949v1.

[5] D. Daigle, *Polynomials* $f(X, Y, Z)$ *of low LND-degree,* Affine algebraic geometry: The Russell festschrift (Providence, Montreal), CRM Proc. Lecture Notes, 54, pp. 21–34, Amer. Math. Soc., Providence, RI, 2011.

[6] H. Darmon and A. Granville, *On the equations* $z^m = F(x, y)$ *and* $Ax^p + By^q = Cz^r$, Bull. London Math. Soc. 27 (1995), 513–543.

[7] H. Davenport, *On* $f(t)^3 - g(t)^2$, Norske Vid. Selsk. Forh. (Trondheim) 38 (1965), 86–87.

[8] M. de Bondt, *Homogeneous Keller maps,* Ph.D. thesis, Radboud Univ. Nijmegen, 2010.

[9] H. Derksen, F. Kutzschebauch, and J. Winkelmann, *Subvarieties of* $\mathbb{C}^n$ *with non-extendable authomorphisms,* J. Reine Angew. Math. 508 (1999), 213–235.

[10] D. Finston and S. Maubach, *The automorphism group of certain factorial threefolds and a cancellation problem,* Israel J. Math. 163 (2008), 369–381.

[11] H. Flenner and M. Zaidenberg, *Rational curves and rational singularities,* Math. Z. 244 (2003), 549–575.

[12] G. Freudenburg, *Algebraic theory of locally nilpotent derivations,* Encyclopaedia Math. Sci., 136, Springer-Verlag, Berlin, 2006.

[13] S. Kaliman and L. Makar-Limanov, *AK-invariant of affine domains,* Affine algebraic geometry (Osaka), pp. 231–255, Osaka Univ. Press, 2007.

[14] S. Kaliman and M. Zaidenberg, *Miyanishi's characterization of the affine 3-space does not hold in higher dimensions,* Ann. Inst. Fourier (Grenoble) 50 (2000), 1649–1669.

[15] R. Kolhatkar, *Homogeneous locally nilpotent derivations and affine ML-surfaces,* Ph.D. thesis, Univ. Ottawa, 2009.

[16] S. Lang, *Algebra,* 3rd ed., Addison-Wesley, Reading, MA, 1993.

[17] L. Makar-Limanov, *On groups of automorphisms of a class of surfaces,* Israel J. Math. 69 (1990), 250–256.

[18] ———, *On the group of automorphisms of a surface* $x^n y = p(z)$, Israel J. Math. 121 (2001), 113–123.

[19] ———, *Again* $x + x^2 y + z^2 + t^3 = 0$, Contemp. Math., 369, pp. 177–182, Amer. Math. Soc., Providence, RI, 2005.

[20] R. C. Mason, *Equations over function fields,* Lecture Notes in Math., 1068, pp. 149–157, Springer-Verlag, Berlin, 1984.

[21] P. Orlik and P. Wagreich, *Algebraic surfaces with* k^**-actions,* Acta Math. 138 (1977), 43–81.

[22] V. V. Prasolov, *Polynomials,* Algorithms Comput. Math., 11, Springer-Verlag, Berlin, 2004.
[23] P. Ribenboim, *13 lectures on Fermat's last theorem,* Springer-Verlag, New York, 1979.
[24] J. Seade, *On the topology of isolated singularities in analytic spaces,* Progr. Math., 241, Birkhäuser, Basel, 2006.
[25] W. W. Strothers, *Polynomial identities and Hauptmodulen,* Quart. J. Math. Oxford Ser. (2) 32 (1981), 349–370.

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