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Double Covers of EPW-Sextics

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0. Introduction

EPW-sextics are defined as follows. Let V be a 6-dimensional complex vector space. Choose a volume form $\mathrm{vol}\colon \bigwedge^6 V \xrightarrow{\sim} \mathbb{C}$ and equip $\bigwedge^3 V$ with the symplectic form

$$(\alpha, \beta)_V := \mathrm{vol}(\alpha \wedge \beta). \tag{0.0.1}$$

Let $\mathbb{LG}\big(\bigwedge^3 V\big)$ be the symplectic Grassmannian parameterizing Lagrangian subspaces of $\bigwedge^3 V$; of course, $\mathbb{LG}\big(\bigwedge^3 V\big)$ does not depend on the choice of volume form. Let $F \subset \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ be the subvector bundle with fiber

$$F_v := \Big\{\alpha \in \bigwedge^3 V \mid v \wedge \alpha = 0\Big\} \tag{0.0.2}$$

over $[v] \in \mathbb{P}(V)$. Observe that $(\cdot,\cdot)_V$ is zero on F_v and that $2\dim(F_v) = 20 = \dim \bigwedge^3 V$; hence F is a Lagrangian subvector bundle of the trivial symplectic vector bundle on $\mathbb{P}(V)$ with fiber $\bigwedge^3 V$. Next choose $A \in \mathbb{LG}\big(\bigwedge^3 V\big)$. Let

$$F \xrightarrow{\lambda_A} \Big(\bigwedge^3 V/A\Big) \otimes \mathcal{O}_{\mathbb{P}(V)} \tag{0.0.3}$$

be the composition of the inclusion $F \subset \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$ followed by the quotient map. Since $\operatorname{rk} F = \dim(V/A)$, the determinant of λ_A makes sense. Let

$$Y_A := V(\det \lambda_A).$$

A straightforward computation gives that $\det F \cong \mathcal{O}_{\mathbb{P}(V)}(-6)$ and hence $\det \lambda_A \in H^0(\mathcal{O}_{\mathbb{P}(V)}(6))$. It follows that if $\det \lambda_A \neq 0$ then Y_A is a sextic hypersurface. As is easily checked, $\det \lambda_A \neq 0$ for generic $A \in \mathbb{LG}\big(\bigwedge^3 V\big)$ (note that there exist "pathological" As such that $\lambda_A = 0$; e.g., $A = F_{v_0}$). An *EPW-sextic* (after Eisenbud, Popescu, and Walter [5]) is a sextic hypersurface in $\mathbb{P}^5$ that is projectively equivalent to Y_A for some $A \in \mathbb{LG}\big(\bigwedge^3 V\big)$. Let Y_A be an EPW-sextic. One can construct a coherent sheaf ξ_A on Y_A and a multiplication map $\xi_A \times \xi_A \to \mathcal{O}_{Y_A}$ that gives $\mathcal{O}_{Y_A} \oplus \xi_A$ the structure of an $\mathcal{O}_{Y_A}$-algebra; this is known to experts (see [3]), and we will give the construction in Section 1.2. The *double EPW-sextic* associated to A is $X_A := \mathrm{Spec}(\mathcal{O}_{Y_A} \oplus \xi_A)$; we let $f_A\colon X_A \to Y_A$ be the structure morphism. In [12] we considered X_A for generic A and proved that it is a hyper-Kähler deformation of $(\mathrm{K3})^{[2]}$ (the blow-up of the diagonal in the symmetric square of a K3

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surface). In this paper we analyze X_A for A varying in a codimension-1 subset of $\mathbb{LG}(\bigwedge^3 V)$. In order to state our main results, we shall introduce some notation.

Given $A \in \mathbb{LG}(\bigwedge^3 V)$, we let

$$Y_A(k) = \{[v] \in \mathbb{P}(V) \mid \dim(A \cap F_v) = k\}, \tag{0.0.4}$$

$$Y_A[k] = \{[v] \in \mathbb{P}(V) \mid \dim(A \cap F_v) \geq k\}. \tag{0.0.5}$$

Thus $Y_A(0) = (\mathbb{P}(V) \setminus Y_A)$ and $Y_A = Y_A[1]$. Double EPW-sextics come with a natural polarization; we let

$$\mathcal{O}_{X_A}(n) := f_A^* \mathcal{O}_{Y_A}(n), \quad H_A \in |\mathcal{O}_{X_A}(1)|. \tag{0.0.6}$$

The following closed subsets of $\mathbb{LG}(\bigwedge^3 V)$ play a key role:

$$\Sigma := \{A \in \mathbb{LG}(\textstyle\bigwedge^3 V) \mid \exists W \in \mathrm{Gr}(3, V) \text{ s.t. } \textstyle\bigwedge^3 W \subset A\}, \tag{0.0.7}$$

$$\Delta := \{A \in \mathbb{LG}(\textstyle\bigwedge^3 V) \mid Y_A[3] \neq \emptyset\}. \tag{0.0.8}$$

A straightforward computation (see [15]) gives that Σ is irreducible of codimension 1. A similar computation (see Proposition 2.2) gives that Δ is irreducible of codimension 1 and distinct from Σ. Now let

$$\mathbb{LG}(\textstyle\bigwedge^3 V)^0 := \mathbb{LG}(\textstyle\bigwedge^3 V) \setminus \Sigma \setminus \Delta. \tag{0.0.9}$$

Then $\mathbb{LG}(\bigwedge^3 V)^0$ is open dense in $\mathbb{LG}(\bigwedge^3 V)$. In [12] we proved that if $A \in \mathbb{LG}(\bigwedge^3 V)^0$ then X_A is a hyper-Kähler (HK) 4-fold that can be deformed to $(K3)^{[2]}$; we also showed that the family of polarized HK 4-folds (X_A, H_A) for A varying in $\mathbb{LG}(\bigwedge^3 V)^0$ is locally complete. Three other explicit locally complete families of projective HK manifolds of dimension greater than 2 are known (see [2; 4; 8; 9]). In all the examples the HK manifolds are deformations of the Hilbert square of a K3; they are distinguished by the value of the Beauville–Bogomolov form on the polarization class (it equals 2 in the case of double EPW-sextics and equals 6, 22, and 38 in the other cases). Here we shall analyze X_A for $A \in \Delta$, usually assuming that $A \notin \Sigma$. Let $A \in (\Delta \setminus \Sigma)$. We will prove the following results.

(1) $Y_A[3]$ is a finite set and equals $Y_A(3)$. If A is generic in $(\Delta \setminus \Sigma)$, then $Y_A(3)$ is a singleton.

(2) One may associate to $[v_0] \in Y_A(3)$ a K3 surface $S_A(v_0) \subset \mathbb{P}^6$ of genus 6 that is well-defined up to projectivities. Conversely, the generic K3 surface of genus 6 is projectively equivalent to $S_A(v_0)$ for some $A \in (\Delta \setminus \Sigma)$ and $[v_0] \in Y_A(3)$.

(3) The singular set of X_A is equal to $f_A^{-1} Y_A(3)$. There is a single $p_i \in X_A$ mapping to $[v_i] \in Y_A(3)$, and the cone of X_A at p_i is isomorphic to the cone over the set of incident couples $(x, r) \in \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ (i.e., $\mathbb{P}(\Omega_{\mathbb{P}^2})$). Thus we have two standard small resolutions of a neighborhood of p_i in X_A, one with fiber $\mathbb{P}^2$ over p_i and the other with fiber $(\mathbb{P}^2)^\vee$. Making a choice ε of local small resolution at each p_i yields a resolution $X_A^\varepsilon \to X_A$ with the following properties: (a) there is a birational map $X_A^\varepsilon \dashrightarrow S_A(v_i)^{[2]}$ such that the pull-back

of a holomorphic symplectic form on $S_A(v_i)^{[2]}$ is a symplectic form on X_A^{ε}; and (b) if $S_A(v_i)$ contains no lines (by (2), this condition holds for generic A), then there exists a choice of ε such that X_A^{ε} is isomorphic to $S_A(v_i)^{[2]}$.

(4) For a sufficiently small open (classical topology) $\mathcal{U} \subset \big(\mathbb{LG}\big(\bigwedge^3 V\big) \setminus \Sigma\big)$ containing A, the family of double EPW-sextics parameterized by $\mathcal{U}$ has a simultaneous resolution of singularities (no base change) with fiber X_A^{ε} over A (for an arbitrary choice of ε).

We remark that if $Y_A(3)$ has more than one point then we do not expect all the small resolutions to be projective (i.e. Kähler). Items (1)–(4) should be compared with known results on cubic 4-folds. Recall that if $Z \subset \mathbb{P}^5$ is a smooth cubic hypersurface then the variety $F(Z)$ parameterizing lines in Z is a HK 4-fold that can be deformed to $(\mathrm{K3})^{[2]}$; also, the primitive weight-4 Hodge structure of Z is isomorphic (after a Tate twist) to the primitive weight-2 Hodge structure of $F(Z)$ (see [2]).

Let $D \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the prime divisor parameterizing singular cubics, and let $Z \in D$ be generic. The following results are well known.

(1′) $\operatorname{sing} Z$ is a finite set.

(2′) Given $p \in \operatorname{sing} Z$, the set $S_Z(p) \subset F(Z)$ of lines containing p is a K3 surface of genus 4; conversely, the generic genus-4 K3 surface is isomorphic to $S_Z(p)$ for some Z and $p \in \operatorname{sing} Z$.

(3′) $F(Z)$ is birational to $S_Z(p)^{[2]}$.

(4′) After a local base change of order 2 ramified along D, the period map extends across Z.

Items (1′)–(3′) are analogous to (1)–(3). Although (4′) also is analogous to (4), there is an important difference—namely, the need for a base change of order 2. Note that items (3) and (4) prove our previously mentioned theorem that if $A \in \mathbb{LG}\big(\bigwedge^3 V\big)^0$ then X_A is a HK deformation of $(\mathrm{K3})^{[2]}$ (given that, by a straightforward parameter count, the family of polarized double EPW-sextics is locally complete). The proof given in this paper is independent of the one in [12]. Beyond giving a new proof of an “old” theorem, results (1)–(4) show that: (a) away from Σ, the period map is regular and lifts (locally) to the relevant classifying space; and (b) the value at $A \in (\Delta \setminus \Sigma)$ may be identified with the period point of the Hilbert square $S_A(v_0)^{[2]}$. We remark that in [14] we proved that the period map is as well-behaved as possible at the generic $A \in (\Delta \setminus \Sigma)$; however, we did not have the exact statement about X_A^{ε} and we had no statement about an arbitrary $A \in (\Delta \setminus \Sigma)$.

The paper is organized as follows. After summarizing our notation, in Section 1 we give formulas that describe double EPW-sextics locally. Although these formulas are known, we go through the proofs for lack of a suitable reference. We will also perform the local computations needed to prove item (4). In Section 2 we perform standard computations involving Δ. In Section 3 we will prove items (1) and (4) as well as the statements in item (3) that do not involve the K3 surface $S_A(v_0)$. In Section 4 we prove item (2) and the remaining statement of item (3).

Finally, Section 5 contains auxiliary results on 3-dimensional linear sections of $\mathrm{Gr}(3,\mathbb{C}^5)$.

NOTATION AND CONVENTIONS. Throughout the paper, V is a 6-dimensional complex vector space.

Let W be a finite-dimensional complex vector space. The span of a subset $S \subset W$ is denoted by $\langle S\rangle$. Let $S \subset \bigwedge^q W$. The *support of* S is the smallest subspace $U \subset W$ such that $S \subset \mathrm{im}\big(\bigwedge^q U \to \bigwedge^q W\big)$, and we denote it by $\mathrm{supp}(S)$; if $S = \{\alpha\}$ is a singleton, we let $\mathrm{supp}(\alpha) = \mathrm{supp}(\{\alpha\})$ (so if $q = 1$ then $\mathrm{supp}(\alpha) = \langle\alpha\rangle$). We define the support of a set of symmetric tensors analogously. For $\alpha \in \bigwedge^q W$ or $\alpha \in \mathrm{Sym}^d W$, the *rank of* α is the dimension of $\mathrm{supp}(\alpha)$. An element of $\mathrm{Sym}^2 W^\vee$ may be viewed either as a symmetric map or as a quadratic form; we denote the former by $\tilde{q}, \tilde{r}, \dots$ and the latter by $q, r, \dots$.

Let $M = (M_{ij})$ be a $d \times d$ matrix with entries in a commutative ring R. We let $M^c = (M^{ij})$ be the matrix of cofactors of M; that is, $M^{i,j}$ is $(-1)^{i+j}$ multiplied by the determinant of the matrix obtained from M by deleting its ith row and jth column. We recall the following interpretation of M^c. Suppose that $f\colon A \to B$ is a linear map between free R-modules of rank d and that M is the matrix associated to f by the choice of bases $\{a_1,\dots,a_d\}$ and $\{b_1,\dots,b_d\}$ of A and B, respectively. Then $\bigwedge^{d-1} f$ may be viewed as a map

$$\bigwedge^{d-1} f\colon A^\vee \otimes \bigwedge^d A \cong \bigwedge^{d-1} A \to \bigwedge^{d-1} B \cong B^\vee \otimes \bigwedge^d B. \tag{0.0.10}$$

(Here $A^\vee := \mathrm{Hom}(A, R)$ and similarly for $B^\vee$.) The matrix associated to $\bigwedge^{d-1} f$ by the choice of base $\{a_1^\vee \otimes (a_1 \wedge \cdots \wedge a_d), \dots, a_d^\vee \otimes (a_1 \wedge \cdots \wedge a_d)\}$ and of base $\{b_1^\vee \otimes (b_1 \wedge \cdots \wedge b_d), \dots, b_d^\vee \otimes (b_1 \wedge \cdots \wedge b_d)\}$ is equal to M^c.

Let W be a finite-dimensional complex vector space. We will adhere to pre-Grothendieck conventions, so $\mathbb{P}(W)$ is the set of 1-dimensional vector subspaces of W. Given a nonzero $w \in W$, we denote the span of w by $[w]$ rather than $\langle w\rangle$; this is in line with standard notation. Suppose that $T \subset \mathbb{P}(W)$. Then $\langle T\rangle \subset \mathbb{P}(W)$ is the *projective span of* T—that is, the intersection of all linear subspaces of $\mathbb{P}(W)$ containing T.

Schemes are defined over $\mathbb{C}$ and, unless we state the contrary, the topology is the Zariski topology. Let W be a finite-dimensional complex vector space: $\mathcal{O}_{\mathbb{P}(W)}(1)$ is the line bundle on $\mathbb{P}(W)$ with fiber $L^\vee$ on the point $L \in \mathbb{P}(W)$. Given $F \in \mathrm{Sym}^d W^\vee$, let $V(F) \subset \mathbb{P}(W)$ be the subscheme defined by the vanishing of F. If $E \to X$ is a vector bundle, then we denote by $\mathbb{P}(E)$ the projective fiber bundle with fiber $\mathbb{P}(E(x))$ over x and define $\mathcal{O}_{\mathbb{P}(W)}(1)$ accordingly. For Y a subscheme of X, we let $\mathrm{Bl}_Y X \to X$ denote the blow-up of Y.

1. Symmetric Resolutions and Double Covers

In Section 1.1 we describe a method (well known to experts) for constructing double covers, and in Section 1.2 we show how this method can be used to construct double EPW-sextics. Section 1.3 contains the main ingredients needed to construct the simultaneous desingularization described in item (3) of Section 0.

1.1. Product Formula and Double Covers

Let R be an integral Noetherian ring. Let N be an R-module with a free resolution

$$0 \to U_1 \xrightarrow{\lambda} U_0 \xrightarrow{\pi} N \to 0, \quad \operatorname{rk} U_1 = \operatorname{rk} U_0 = d > 0. \tag{1.1.1}$$

Let $\{a_1, \ldots, a_d\}$ and $\{b_1, \ldots, b_d\}$ be bases of U_0 and U_1, respectively. Let M_λ be the matrix associated to λ by our choice of bases, and observe that $\det M_\lambda$ annihilates N. Given a homomorphism

$$\beta \colon N \to \operatorname{Ext}^1(N, R), \tag{1.1.2}$$

we may define a product $m_\beta \colon N \times N \to R/(\det M_\lambda)$ as follows. Applying the $\operatorname{Hom}(\cdot, R)$-functor to (1.1.1) yields the exact sequence

$$0 \to U_0^\vee \xrightarrow{\lambda^t} U_1^\vee \xrightarrow{\rho} \operatorname{Ext}^1(N, R) \to 0. \tag{1.1.3}$$

In particular, $\det M_\lambda$ kills $\operatorname{Ext}^1(N, R)$. Now apply the functor $\operatorname{Hom}(N, \cdot)$ to the exact sequence

$$0 \to R \xrightarrow{\det M_\lambda} R \to R/(\det M_\lambda) \to 0. \tag{1.1.4}$$

Since $\operatorname{Ext}^1(N, R) \to \operatorname{Ext}^1(N, R)$ amounts to multiplication by $\det M_\lambda$, we obtain the coboundary isomorphism

$$\partial \colon \operatorname{Hom}(N, R/(\det M_\lambda)) \xrightarrow{\sim} \operatorname{Ext}^1(N, R). \tag{1.1.5}$$

Let

$$\begin{aligned} N \times N &\xrightarrow{m_\beta} R/(\det M_\lambda), \\ (n, n') &\longmapsto (\partial^{-1}\beta(n))(n'). \end{aligned} \tag{1.1.6}$$

We will give an explicit formula for m_β. Let $\pi : U_0 \to N$ be as in (1.1.1). Then $\beta \circ \pi$ lifts to a homomorphism $\mu^t : U_0 \to U_1^\vee$ (the map is written as a transpose in order to conform to the notation for double EPW-sextics; see Section 1.2). It follows that there exists an $\alpha \colon U_1 \to U_0^\vee$ such that

$$\begin{tikzcd} 0 \arrow[r] & U_1 \arrow[r, "\lambda"] \arrow[d, "\alpha"] & U_0 \arrow[r, "\pi"] \arrow[d, "\mu^t"] & N \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & U_0^\vee \arrow[r, "\lambda^t"] & U_1^\vee \arrow[r, "\rho"] & \operatorname{Ext}^1(N, R) \arrow[r] & 0 \end{tikzcd} \tag{1.1.7}$$

is a commutative diagram. Let $\{a_1^\vee, \ldots, a_d^\vee\}$ and $\{b_1^\vee, \ldots, b_d^\vee\}$ be the bases of $U_0^\vee$ and $U_1^\vee$ that are dual to the chosen bases of U_0 and U_1. Let M_{μ^t} be the matrix associated to μ^t by our choice of bases.

PROPOSITION 1.1. *With notation as before, we have*

$$m_\beta(\pi(a_i), \pi(a_j)) \equiv (M_\lambda^c \cdot M_{\mu^t})_{ji} \text{ modulo } \det M_\lambda, \tag{1.1.8}$$

where M_λ^c is the matrix of cofactors of M_λ.

Proof. Equation (1.1.3) gives an isomorphism

$$\nu\colon \operatorname{Ext}^1(N,R) \xrightarrow{\sim} U_1^\vee/\lambda^t(U_0^\vee). \tag{1.1.9}$$

Let $\det(U_\bullet) := \bigwedge^d U_1^\vee \otimes \bigwedge^d U_0$. We will define an isomorphism

$$\theta\colon U_1^\vee/\lambda^t(U_0^\vee) \xrightarrow{\sim} \operatorname{Hom}(N,\det(U_\bullet)/(\det\lambda)). \tag{1.1.10}$$

First let

$$\begin{gathered} U_1^\vee = \bigwedge^{d-1} U_1 \otimes \bigwedge^d U_1^\vee \xrightarrow{\hat\theta} \bigwedge^{d-1} U_0 \otimes \bigwedge^d U_1^\vee = \operatorname{Hom}(U_0,\det(U_\bullet)),\\ \zeta\otimes\xi \mapsto \bigwedge^{d-1}(\lambda)(\zeta)\otimes\xi. \end{gathered} \tag{1.1.11}$$

We claim that

$$\operatorname{im}(\hat\theta) = \{\phi\in\operatorname{Hom}(U_0,\det(U_\bullet)) \mid \phi\circ\lambda(U_1)\subset(\det\lambda)\}. \tag{1.1.12}$$

In fact, by Cramer's formula we have

$$M_\lambda^c\cdot M_\lambda^t = M_\lambda^t\cdot M_\lambda^c = \det M_\lambda\cdot 1 \tag{1.1.13}$$

and then (1.1.12) follows. Thus $\hat\theta$ induces a surjective homomorphism

$$\tilde\theta\colon U_1^\vee \to \operatorname{Hom}(N,\det(U_\bullet)/(\det\lambda)). \tag{1.1.14}$$

One easily checks that $\lambda^t(U_0^\vee) = \ker\tilde\theta$ (use Cramer's formula again). We define θ to be the homomorphism induced by $\tilde\theta$; we have already proved that it is an isomorphism.

We claim that

$$\theta\circ\nu = \partial^{-1} \quad \text{for } \partial \text{ as in (1.1.5).} \tag{1.1.15}$$

Let K be the fraction field of R, and let $0\to R\xrightarrow{\iota} I^0\to I^1\to\cdots$ be an injective resolution of R with $I^0 = \det(U_\bullet)\otimes K$ and $\iota(1) = \det\lambda\otimes 1$. Then $\operatorname{Ext}^\bullet(N,R)$ is the cohomology of the double complex $\operatorname{Hom}(U_\bullet,I^\bullet)$ and also, of course, of the single complexes $\operatorname{Hom}(U_\bullet,R)$ and $\operatorname{Hom}(N,I^\bullet)$. One checks easily that the isomorphism ∂ of (1.1.5) is equal to the isomorphism $H^1(\operatorname{Hom}(N,I^\bullet)) \xrightarrow{\sim} H^1(\operatorname{Hom}(U_\bullet,I^\bullet))$; that is,

$$\begin{aligned} \partial\colon \operatorname{Hom}(N,\det(U_\bullet)/(\det\lambda)) &= \operatorname{Hom}(N,I^0/\iota(R))\\ &\xrightarrow{\sim} H^1(\operatorname{Hom}(U_\bullet,I^\bullet)). \end{aligned} \tag{1.1.16}$$

Let $f\in\operatorname{Hom}(N,\det(U_\bullet)/(\det\lambda))$; a representative of $\partial(f)$ in the double complex $\operatorname{Hom}(U_\bullet,I^\bullet)$ is given by $g^{0,1} := f\circ\pi\in\operatorname{Hom}(U_0,I^1)$. Let $g^{0,0}\in\operatorname{Hom}(U_0,\det(U_\bullet))$ be a lift of $g^{0,1}$ and let $g^{1,0}\in\operatorname{Hom}(U_1,\det(U_\bullet))$ be defined as $g^{1,0} := g^{0,0}\circ\lambda$. One can check that $\operatorname{im}(g^{1,0})\subset\det\lambda$ and hence that there exists a $g\in\operatorname{Hom}(U_1,R)$ such that $g^{1,0} = \iota\circ g$. By construction, g represents a class $[g]\in H^1(\operatorname{Hom}(U_\bullet,R)) = U_1^\vee/\lambda^t(U_0^\vee)$ and $[g] = \nu\circ\partial(f)$. An explicit computation shows that $[g] = \theta^{-1}(f)$, which proves (1.1.15). Now we prove (1.1.8). From (1.1.15) it follows that

$$m_\beta(\pi(a_i),\pi(a_j)) = (\partial^{-1}\beta\pi(a_i))(\pi(a_j)) = (\theta\nu\beta\pi(a_i))(\pi(a_j)). \tag{1.1.17}$$

Unwinding the definition of θ, we find that the right-hand side of this equation equals the right-hand side of (1.1.8). $\square$

Let m_β be given by (1.1.6). We may define a product on $R/(\det M_\lambda) \oplus N$ as follows. Let $(r,n),(r',n') \in R/(\det M_\lambda) \oplus N$, and set

$$(r,n) \cdot (r',n') := (rr' + m_\beta(n,n'), rn' + r'n). \tag{1.1.18}$$

This product is neither associative nor commutative in general, but we will give an example in which it is both. Suppose

$$0 \to U^\vee \xrightarrow{\gamma} U \xrightarrow{\pi} N \to 0, \quad \gamma^t = \gamma; \tag{1.1.19}$$

here U is a free R-module of rank $d > 0$ and the sequence is assumed to be exact. We get the commutative diagram (1.1.7) by letting

$$U_0 := U,\ U_1 := U^\vee, \quad \lambda = \gamma,\ \alpha = \mathrm{Id}_{U^\vee},\ \mu^t = \mathrm{Id}_U,$$

and $\beta = \beta(\gamma)\colon N \to \mathrm{Ext}^1(N,R)$ the map induced by Id_U. Abusing notation, we let $m_\gamma\colon N \times N \to R/(\det M_\gamma)$ be the map defined by $m_{\beta(\gamma)}$.

Proposition 1.2. *Suppose we have the exact sequence* (1.1.19)*. Then the product on $R/(\det M_\gamma) \oplus N$ defined by m_γ is associative and commutative.*

Proof. Let $d := \operatorname{rk} U > 0$. Let $\{a_1,\ldots,a_d\}$ be a basis of U, and let $\{a_1^\vee,\ldots,a_d^\vee\}$ be the dual basis of $U^\vee$. Let $M = M_\gamma$ (i.e., the matrix associated to γ by our choice of bases). According to (1.1.8), we have

$$m_\gamma(\pi(a_i),\pi(a_j)) \equiv M^c_{ji} \text{ modulo } \det M. \tag{1.1.20}$$

Since γ is a symmetric map, it follows that M is a symmetric matrix; hence M^c is a symmetric matrix. By (1.1.20) we know that m_γ is symmetric. It remains to prove that m_γ is associative. For $1 \le i < k \le d$ and $1 \le h \ne j \le d$, let $M^{i,k}_{h,j}$ be the $(d-2)\times(d-2)$ matrix obtained by deleting from M the rows i and k and the columns h and j. Let $X_{ijk} = (X^h_{ijk}) \in R^d$ be defined by

$$X^h_{ijk} := \begin{cases} (-1)^{i+k+j+h} \det M^{i,k}_{j,h} & \text{if } h < j, \\ 0 & \text{if } h = j, \\ (-1)^{i+k+j+h-1} \det M^{i,k}_{j,h} & \text{if } h > j. \end{cases} \tag{1.1.21}$$

A tedious but straightforward computation gives that

$$M^c_{ij} a_k - M^c_{jk} a_i = \gamma\left(\sum_{h=1}^{d} X^h_{ijk} a^\vee_h\right). \tag{1.1.22}$$

This equation proves the associativity of m_γ. □

Retaining the hypotheses of Proposition 1.2, we let

$$X_\gamma := \mathrm{Spec}(R/(\det M_\lambda) \oplus N), \qquad Y_\gamma := \mathrm{Spec}(R/(\det M_\lambda)). \tag{1.1.23}$$

Let $f_\gamma\colon X_\gamma \to Y_\gamma$ be the structure map. We may realize X_γ as a subscheme of $\mathrm{Spec}(R[\xi_1,\ldots,\xi_d])$ as follows. Because the ring $R/(\det M_\gamma) \oplus N$ is associative and commutative, there is a well-defined surjective morphism of R-algebras

$$R[\xi_1,\dots,\xi_d] \to R/(\det M_\gamma) \oplus N \tag{1.1.24}$$

mapping ξ_i to a_i. Thus we have an inclusion

$$X_\gamma \hookrightarrow \operatorname{Spec}(R[\xi_1,\dots,\xi_d]). \tag{1.1.25}$$

CLAIM 1.3. *With reference to inclusion* (1.1.25), *the ideal of X_γ is generated by the entries of the matrices*

$$M_\gamma \cdot \xi, \quad \xi \cdot \xi^t - M_\gamma^c, \tag{1.1.26}$$

where ξ is viewed as a column matrix.

Proof. By (1.1.20), the ideal of X_γ is generated by $\det M_\gamma$ and the entries of the matrices in (1.1.26). By Cramer's formula, $\det M_\gamma$ belongs to the ideal generated by the entries of the two matrices. This proves that the ideal of X_γ is as claimed. □

Now we suppose in addition that R is a finitely generated $\mathbb{C}$-algebra. Let $p \in \operatorname{Spec} R$ be a closed point; we are interested in the localization of X_γ at points in $f_\gamma^{-1}(p)$. Let $J \subset U^\vee(p)$ be a subspace complementary to $\ker \gamma(p)$. Let $\mathbf{J} \subset U^\vee$ be a free submodule whose fiber over p is equal to J. Let $\mathbf{K} \subset U^\vee$ be the submodule orthogonal to $\mathbf{J}$; that is,

$$\mathbf{K} := \{u \in U^\vee \mid \gamma(a)(u) = 0 \ \forall a \in \mathbf{J}\}. \tag{1.1.27}$$

The localization of $\mathbf{K}$ at p is free. Let $K := \mathbf{K}(p)$ be the fiber of $\mathbf{K}$ at p; clearly, $K = \ker \gamma(p)$. Localizing at p, we have

$$U_p^\vee = \mathbf{K}_p \oplus \mathbf{J}_p. \tag{1.1.28}$$

Corresponding to (1.1.28) we may write $\gamma_p = \gamma_{\mathbf{K}} \oplus_\perp \gamma_{\mathbf{J}}$, where $\gamma_{\mathbf{K}} : \mathbf{K}_p \to \mathbf{K}_p^\vee$ and $\gamma_J : \mathbf{J}_p \to \mathbf{J}_p^\vee$ are symmetric maps. Note that we have an equality of germs

$$(Y_\gamma, p) = (Y_{\gamma_{\mathbf{K}}}, p). \tag{1.1.29}$$

We claim that there is a compatible isomorphism of germs $(X_{\gamma_{\mathbf{K}}}, f_{\gamma_{\mathbf{K}}}^{-1}(p)) \cong (X_\gamma, f_\gamma^{-1}(p))$. Let $k := \dim K$ and $d := \operatorname{rk} U$. Choose bases of $\mathbf{K}_p$ and $\mathbf{J}_p$; then, by (1.1.28), we have a basis of $U_p^\vee$. The dual bases of $\mathbf{K}_p^\vee$, $\mathbf{J}_p^\vee$, and $U_p^\vee$ are compatible with respect to the decomposition that is dual to (1.1.28). Corresponding to the chosen bases we have embeddings $X_{\gamma_K} \hookrightarrow Y_{\gamma_K} \times \mathbb{C}^k$ and $X_\gamma \hookrightarrow Y_\gamma \times \mathbb{C}^d$. The decomposition dual to (1.1.28) gives an embedding $j : Y_{\gamma_K} \times \mathbb{C}^k \hookrightarrow Y_\gamma \times \mathbb{C}^d$.

CLAIM 1.4. *The composition*

$$X_{\gamma_K} \hookrightarrow (Y_{\gamma_K} \times \mathbb{C}^k) \xrightarrow{j} (Y_\gamma \times \mathbb{C}^d) \tag{1.1.30}$$

defines an isomorphism of germs in the analytic topology,

$$(X_{\gamma_{\mathbf{K}}}, f_{\gamma_{\mathbf{K}}}^{-1}(p)) \xrightarrow{\sim} (X_\gamma, f_\gamma^{-1}(p)), \tag{1.1.31}$$

that commutes with the maps $f_{\gamma_{\mathbf{K}}}$ and f_γ.

Proof. This follows by writing $\gamma_p = \gamma_{\mathbf{K}} \oplus_{\perp} \gamma_{\mathbf{J}}$ and then recalling (1.1.20). We pass to the analytic topology so that we can extract the square root of a regular nonzero function. □

PROPOSITION 1.5. *Assume that R is a finitely generated $\mathbb{C}$-algebra. Suppose that we have the exact sequence* (1.1.19). *Then the following statements hold.*

(1) $f_\gamma^{-1} Y_\gamma(1) \to Y_\gamma(1)$ *is a topological covering of degree 2.*
(2) *Let $p \in (Y_\gamma \setminus Y_\gamma(1))$ be a closed point. The fiber $f_\gamma^{-1}(p)$ consists of a single point q. Let ξ_i be the coordinates on X_γ associated to embedding* (1.1.25); *then $\xi_i(q) = 0$ for $i = 1, \ldots, d$.*

Proof. (1) Localize at $p \in Y_\gamma(1)$ and then apply Claim 1.3.

(2) Since cork $M_\gamma(p) \geq 2$, we have $M_\gamma^c(p) = 0$. Hence this part follows from Claim 1.3. □

We may associate a double cover $f_\gamma \colon X_\gamma \to Y_\gamma$ to a map β that is symmetric in the derived category.

HYPOTHESIS 1.6. *We have* (1.1.7) *with α an isomorphism and $\alpha = \mu$.*

PROPOSITION 1.7. *Assume that Hypothesis 1.6 holds. Then $R/(\det M_\lambda) \oplus N$ equipped with the product given by* (1.1.18) *is a commutative* (*and associative*) *ring.*

Proof. Let $\gamma := \lambda \circ \mu^{-1}$ and $U := U_0$. Then (1.1.19) holds, and the product defined by m_β is equal to the product defined by m_γ. From Proposition 1.2 it follows that $R/(\det M_\lambda) \oplus N$ is a commutative associative ring. □

DEFINITION 1.8. Suppose that Hypothesis 1.6 holds. Then the *symmetrization* of (1.1.7) is exact sequence (1.1.19) with γ and U as in the proof of Proposition 1.7.

1.2. Structure Sheaf of Double EPW-Sextics

Let $A \in \mathbb{LG}\left(\bigwedge^3 V\right)$ and suppose that $Y_A \neq \mathbb{P}(V)$. We will define the associated double cover $X_A \to Y_A$ by applying the results of Section 1.1. Since A is Lagrangian, the symplectic form defines a canonical isomorphism $\bigwedge^3 V/A \cong A^\vee$; thus (0.0.3) defines a map of vector bundles $\lambda_A \colon F \to A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)}$. Let $i \colon Y_A \hookrightarrow \mathbb{P}(V)$ be the inclusion map. Then, since a local generator of $\det \lambda_A$ annihilates $\operatorname{coker}(\lambda_A)$, there is a unique sheaf ζ_A on Y_A such that we have the exact sequence

$$0 \longrightarrow F \xrightarrow{\lambda_A} A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} \longrightarrow i_* \zeta_A \longrightarrow 0. \tag{1.2.1}$$

Now choose $B \in \mathbb{LG}\left(\bigwedge^3 V\right)$ transversal to A. Thus we have a direct sum decomposition $\bigwedge^3 V = A \oplus B$ and hence a projection map $\bigwedge^3 V \to A$ inducing a map $\mu_{A,B} \colon F \to A \otimes \mathcal{O}_{\mathbb{P}(V)}$. We claim that there is a commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{\lambda_A} & A^\vee \otimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & i_*\zeta_A & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mu_{A,B}} & & \Big\downarrow{\scriptstyle \mu^t_{A,B}} & & \Big\downarrow{\scriptstyle \beta_A} & & \\
0 & \longrightarrow & A \otimes \mathcal{O}_{\mathbb{P}(V)} & \xrightarrow{\lambda^t_A} & F^\vee & \longrightarrow & \mathrm{Ext}^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) & \longrightarrow & 0.
\end{array}
\tag{1.2.2}
$$

The second row is obtained by applying the $\mathrm{Hom}(\cdot, \mathcal{O}_{\mathbb{P}(V)})$-functor to (1.2.1), and the equality $\mu^t_{A,B} \circ \lambda_A = \lambda^t_A \circ \mu_{A,B}$ holds because F is a Lagrangian subbundle of $\bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)}$. Finally, β_A is defined as the unique map making the diagram commutative; it exists because the rows are exact. Observe that, as suggested by the notation, the map β_A is independent of the choice of B.

Next, by applying the $\mathrm{Hom}(i_*\zeta_A, \cdot)$-functor to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(6) \to \mathcal{O}_{Y_A}(6) \to 0, \tag{1.2.3}$$

we obtain the exact sequence

$$
\begin{aligned}
0 \to i_* \mathrm{Hom}(\zeta_A, \mathcal{O}_{Y_A}(6)) &\xrightarrow{\partial} \mathrm{Ext}^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \\
&\xrightarrow{n} \mathrm{Ext}^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}(6)),
\end{aligned}
\tag{1.2.4}
$$

where n is locally equal to multiplication by $\det \lambda_A$. Since the second row of (1.2.2) is exact, it follows that a local generator of $\det \lambda_A$ annihilates $\mathrm{Ext}^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)})$; thus $n = 0$ and hence we get a canonical isomorphism

$$\partial^{-1} \colon \mathrm{Ext}^1(i_*\zeta_A, \mathcal{O}_{\mathbb{P}(V)}) \xrightarrow{\sim} i_* \mathrm{Hom}(\zeta_A, \mathcal{O}_{Y_A}(6)). \tag{1.2.5}$$

We define $\tilde{m}_A$ by setting

$$
\begin{aligned}
\zeta_A \times \zeta_A &\xrightarrow{\tilde{m}_A} \mathcal{O}_{Y_A}(6), \\
(\sigma_1, \sigma_2) &\longmapsto (\partial^{-1} \circ \beta_A(\sigma_1))(\sigma_2).
\end{aligned}
\tag{1.2.6}
$$

Let $\xi_A := \zeta_A(-3)$. Tensorizing both sides of (1.2.6) by $\mathcal{O}_{Y_A}(-6)$ yields the multiplication map

$$\xi_A \times \xi_A \xrightarrow{m_A} \mathcal{O}_{Y_A}. \tag{1.2.7}$$

Thus we have defined a multiplication map on $\mathcal{O}_{Y_A} \oplus \xi_A$. The following result is well known to experts.

PROPOSITION 1.9. *With notation as before, let $A \in \mathbb{LG}(\bigwedge^3 V)$ and suppose that $Y_A \neq \mathbb{P}(V)$. Then:*

(1) *β_A is an isomorphism; and*

(2) *the multiplication map m_A is associative and commutative.*

Proof. Let $[v_0] \in \mathbb{P}(V)$. Choose $B \in \mathbb{LG}(\bigwedge^3 V)$ transversal to F_{v_0} (and to A, of course). Then $\mu_{A,B}$ is an isomorphism in an open neighborhood U of $[v_0]$, whence β_A is an isomorphism in a neighborhood of $[v_0]$; this proves (1). To prove (2), let

$B \in \mathbb{LG}(\bigwedge^3 V)$ and let U be as before; we may assume that U is affine. Let $N := H^0(i_*\zeta_A|_U)$ and $\beta := H^0(\beta_A|_U)$. Then $\beta \colon N \to \operatorname{Ext}^1(N, \mathbb{C}[U])$. By Proposition 1.7 and the commutativity of diagram (1.2.2), the multiplication map m_β is associative and commutative. Yet m_β is the multiplication induced by m_A on N; since $[v_0]$ is an arbitrary point of $\mathbb{P}(V)$, it follows that m_A is also associative and commutative. □

We let $X_A := \operatorname{Spec}(\mathcal{O}_{Y_A} \oplus \xi_A)$ and let $f_A \colon X_A \to Y_A$ be the structure morphism. Then X_A is the double EPW-sextic associated to A, and f_A is its structure map. The *covering involution* of X_A is the automorphism $\phi_A \colon X_A \to X_A$ corresponding to the involution of $\mathcal{O}_{Y_A} \oplus \xi_A$ with (-1)-eigensheaf equal to ξ_A.

1.3. Local Models of Double Covers

In this section we assume that R is a finitely generated $\mathbb{C}$-algebra. Let $\mathcal{W}$ be a finite-dimensional complex vector space, and suppose we have the exact sequence

$$0 \to R \otimes \mathcal{W}^\vee \xrightarrow{\gamma} R \otimes \mathcal{W} \to N \to 0, \quad \gamma = \gamma^t. \tag{1.3.1}$$

Thus we have a double cover $f_\gamma \colon X_\gamma \to Y_\gamma$. Let $p \in Y_\gamma$ be a closed point. We will examine X_γ in a neighborhood of $f_\gamma^{-1}(p)$ when the corank of $\gamma(p)$ is small. We may view γ as a regular map $\operatorname{Spec} R \to \operatorname{Sym}^2 \mathcal{W}$; it therefore makes sense to consider the differential

$$d\gamma(p) \colon T_p \operatorname{Spec} R \to \operatorname{Sym}^2 \mathcal{W}. \tag{1.3.2}$$

Let $K(p) := \ker \gamma(p) \subset \mathcal{W}^\vee$. We will consider the linear map

$$\begin{aligned} T_p \operatorname{Spec} R &\xrightarrow{\delta_\gamma(p)} \operatorname{Sym}^2 K(p)^\vee, \\ \tau &\longmapsto d\gamma(p)(\tau)|_{K(p)}. \end{aligned} \tag{1.3.3}$$

Let $d := \dim \mathcal{W}$; choosing a basis of $\mathcal{W}$, we realize X_γ as a subscheme of $\operatorname{Spec} R \times \mathbb{C}^d$ with ideal given by Claim 1.3. We will assume that $\operatorname{cork} \gamma(p) \geq 2$. Proposition 1.5 gives that $f_\gamma^{-1}(p)$ consists of a single point q; in fact, the ξ_i-coordinates of q are all zero. Throughout this section, we let

$$f_\gamma^{-1}(p) = \{q\}. \tag{1.3.4}$$

Claim 1.10. *Suppose that $d = \dim \mathcal{W} = 2$ and $\gamma(p) = 0$. Then $I(X_\gamma)$ is generated by the entries of $\xi \cdot \xi^t - M_\gamma^c$.*

Proof. This follows from Claim 1.3 and a straightforward computation. □

Example 1.11. Let $R = \mathbb{C}[x, y, z], \mathcal{W} = \mathbb{C}^2$. Suppose that the matrix associated to γ is

$$M_\gamma = \begin{pmatrix} x & y \\ y & z \end{pmatrix}. \tag{1.3.5}$$

Then $f_\gamma \colon X_\gamma \to Y_\gamma$ is identified with

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow V(xz - y^2), \\ (\xi_1, \xi_2) &\mapsto (\xi_2^2, -\xi_1\xi_2, \xi_1^2), \end{aligned} \tag{1.3.6}$$

that is, with the quotient map for the action of $\langle -1 \rangle$ on $\mathbb{C}^2$.

PROPOSITION 1.12. *Suppose that*

(a) $\operatorname{cork} \gamma(p) = 2$ *and*
(b) *the localization R_p is regular.*

Then X_γ is smooth at q if and only if $\delta_\gamma(p)$ is surjective.

Proof. Applying Claim 1.4 allows us to assume that $d = 2$. Let

$$M_\gamma = \begin{pmatrix} a & b \\ b & c \end{pmatrix}. \tag{1.3.7}$$

By Claim 1.10, the ideal of X_γ in $\operatorname{Spec} R \times \mathbb{C}^2$ is generated by the entries of $\xi \cdot \xi^t - M_\gamma^c$; that is,

$$I(X_\gamma) = (\xi_1^2 - c, \xi_1\xi_2 + b, \xi_2^2 - a). \tag{1.3.8}$$

Therefore,

$$\operatorname{cod}(T_q X_\gamma, T_q(\operatorname{Spec} R \times \mathbb{C}^2)) = \dim\langle da(p), db(p), dc(p)\rangle. \tag{1.3.9}$$

On the other hand, $\operatorname{cod}_q(X_\gamma, \operatorname{Spec} R \times \mathbb{C}^2) = 3$ and so, at q, X_γ is smooth if and only if $\delta_\gamma(p)$ is surjective. □

CLAIM 1.13. *Retain the preceding notation and hypotheses, and suppose that* $\operatorname{cork} \gamma(p) \geq 3$. *Then X_γ is singular at q.*

Proof. Let I be the ideal of X_γ in $\operatorname{Spec} R[\xi_1, \dots, \xi_d]$. By Claim 1.3, I is nontrivial; however, the differential at q of an arbitrary $g \in I$ is zero. □

Next we discuss in greater detail those X_γ whose corank at $f_\gamma^{-1}(p)$ is equal to 3. We begin by identifying the "universal" example (the universal example for corank 2 is Example 1.11). Let $\mathcal{V}$ be a 3-dimensional complex vector space. We view $\operatorname{Sym}^2 \mathcal{V}$ as an affine (6-dimensional) space and let $R := \mathbb{C}[\operatorname{Sym}^2 \mathcal{V}]$ be its ring of regular functions. We identify $R \otimes_{\mathbb{C}} \mathcal{V}$ and $R \otimes_{\mathbb{C}} \mathcal{V}^\vee$ with the space of (respectively) $\mathcal{V}$-valued and $\mathcal{V}^\vee$-valued regular maps on $\operatorname{Sym}^2 \mathcal{V}$. Let

$$R \otimes_{\mathbb{C}} \mathcal{V}^\vee \xrightarrow{\gamma} R \otimes_{\mathbb{C}} \mathcal{V} \tag{1.3.10}$$

be the map induced on the spaces of global sections by the tautological map of vector bundles, $\operatorname{Spec} R \times \mathcal{V}^\vee \longrightarrow \operatorname{Spec} R \times \mathcal{V}$. The map γ is symmetric. Let N be the cokernel of γ; then

$$0 \to R \otimes_{\mathbb{C}} \mathcal{V}^\vee \xrightarrow{\gamma} R \otimes_{\mathbb{C}} \mathcal{V} \to N \to 0 \tag{1.3.11}$$

is an exact sequence. Since γ is symmetric, it defines a double cover $f\colon X(\mathcal{V}) \to Y(\mathcal{V})$ for

$$Y(\mathcal{V}) := \{\alpha \in \operatorname{Sym}^2 \mathcal{V} \mid \operatorname{rk} \alpha < 3\} \tag{1.3.12}$$

the variety of degenerate quadratic forms.

Let

$$\phi \colon X(\mathcal{V}) \to X(\mathcal{V}) \tag{1.3.13}$$

be the covering involution of f. Then $X(\mathcal{V})$ may be described explicitly as follows. Let

$$(\mathcal{V} \otimes \mathcal{V})_1 := \{\mu \in (\mathcal{V} \otimes \mathcal{V}) \mid \operatorname{rk} \mu \le 1\}. \tag{1.3.14}$$

Thus $(\mathcal{V} \otimes \mathcal{V})_1$ is the cone over the Segre variety $\mathbb{P}(\mathcal{V}) \times \mathbb{P}(\mathcal{V})$. We have the following finite degree-2 map:

$$\begin{aligned} (\mathcal{V} \otimes \mathcal{V})_1 &\xrightarrow{\sigma} Y(\mathcal{V}) \\ \mu &\mapsto \mu + \mu^t. \end{aligned} \tag{1.3.15}$$

PROPOSITION 1.14. *There exists a commutative diagram*

$$\begin{tikzcd} (\mathcal{V} \otimes \mathcal{V})_1 \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & X(\mathcal{V}) \arrow[dl, "f"] \\ & Y(\mathcal{V}), & \end{tikzcd} \tag{1.3.16}$$

where τ is an isomorphism. Let ϕ be involution (1.3.13). *Then*

$$\phi \circ \tau(\mu) = \tau(\mu^t) \quad \forall \mu \in (\mathcal{V} \otimes \mathcal{V})_1. \tag{1.3.17}$$

Proof. To define τ, we will give a coordinate-free version of inclusion (1.1.25) for the case of $X(\mathcal{V})$. Let

$$\begin{aligned} \operatorname{Sym}^2 \mathcal{V} \times \left(\mathcal{V}^\vee \otimes \textstyle\bigwedge^3 \mathcal{V}\right) &\xrightarrow{\Psi} \left(\mathcal{V} \otimes \textstyle\bigwedge^3 \mathcal{V}\right) \\ &\qquad \times \left(\mathcal{V}^\vee \otimes \mathcal{V}^\vee \otimes \textstyle\bigwedge^3 \mathcal{V} \otimes \textstyle\bigwedge^3 \mathcal{V}\right), \\ (\alpha, \xi) &\mapsto \left(\alpha \circ \xi, \xi^t \circ \xi - \textstyle\bigwedge^2 \alpha\right). \end{aligned} \tag{1.3.18}$$

A few words of explanation are in order. In the definition of the first component of $\Psi(\alpha, \xi)$ we view ξ as belonging to $\operatorname{Hom}\left(\bigwedge^3 \mathcal{V}^\vee, \mathcal{V}^\vee\right)$; whereas, in the definition of the second component of $\Psi(\alpha, \xi)$, we view ξ as belonging to $\operatorname{Hom}\left(\mathcal{V} \otimes \bigwedge^3 \mathcal{V}^\vee, \mathbb{C}\right)$. We also make the obvious choice of isomorphism, $\mathbb{C} \cong \mathbb{C}^\vee$. Moreover,

$$\begin{aligned} \textstyle\bigwedge^2 \alpha \in \operatorname{Hom}\left(\textstyle\bigwedge^2 \mathcal{V}^\vee, \textstyle\bigwedge^2 \mathcal{V}\right) &= \operatorname{Hom}\left(\mathcal{V} \otimes \textstyle\bigwedge^3 \mathcal{V}^\vee, \mathcal{V}^\vee \otimes \textstyle\bigwedge^3 \mathcal{V}\right) \\ &= \mathcal{V}^\vee \otimes \mathcal{V}^\vee \otimes \textstyle\bigwedge^3 \mathcal{V} \otimes \mathcal{V}. \end{aligned} \tag{1.3.19}$$

Choosing a basis of $\mathcal{V}$, we obtain the embedding $X(\mathcal{V}) \subset \operatorname{Sym}^2 \mathcal{V} \times \mathbb{C}^3$; see (1.1.25). Claim 1.3 now gives equality of pairs

$$\left(\operatorname{Sym}^2 \mathcal{V} \times \left(\mathcal{V}^\vee \otimes \textstyle\bigwedge^3 \mathcal{V}\right), \Psi^{-1}(0)\right) = \left(\operatorname{Sym}^2 \mathcal{V} \times \mathbb{C}^3, X(\mathcal{V})\right), \tag{1.3.20}$$

where $\Psi^{-1}(0)$ is the scheme-theoretic fiber of Ψ. Note that we have the following isomorphism:

$$\begin{aligned}\mathcal{V}\otimes\mathcal{V} &\xrightarrow[\sim]{T} \operatorname{Sym}^2\mathcal{V}\times\left(\mathcal{V}^\vee\otimes\textstyle\bigwedge^3\mathcal{V}\right),\\ \varepsilon &\longmapsto (\varepsilon+\varepsilon^t,\varepsilon-\varepsilon^t).\end{aligned} \tag{1.3.21}$$

Let $\tau := T|_{(\mathcal{V}\otimes\mathcal{V})_1}$. We then have the embedding

$$\tau\colon (\mathcal{V}\otimes\mathcal{V})_1 \hookrightarrow \operatorname{Sym}^2\mathcal{V}\times\left(\mathcal{V}^\vee\otimes\textstyle\bigwedge^3\mathcal{V}\right). \tag{1.3.22}$$

We shall demonstrate the equality of schemes

$$\operatorname{im}(\tau)=\Psi^{-1}(0)\quad (=X(\mathcal{V})). \tag{1.3.23}$$

First, let

$$\begin{aligned}\mathcal{V}\oplus\mathcal{V} &\xrightarrow{\rho} (\mathcal{V}\otimes\mathcal{V})_1,\\ (\eta,\beta) &\mapsto \eta^t\circ\beta.\end{aligned} \tag{1.3.24}$$

Observe that ρ is the quotient map for the $\mathbb{C}^\times$-action on $\mathcal{V}\oplus\mathcal{V}$ defined by $t(\eta,\beta):=(t\eta,t^{-1}\beta)$. We have

$$\tau\circ\rho(\eta,\beta)=(\eta^t\circ\beta+\beta^t\circ\eta,\eta\wedge\beta). \tag{1.3.25}$$

Second, let's prove that

$$\Psi^{-1}(0)\supset\operatorname{im}(\tau). \tag{1.3.26}$$

Notice that $\mathrm{Gl}(\mathcal{V})$ acts on $(\mathcal{V}\otimes\mathcal{V})_1$ with a unique dense orbit—namely, $\{\eta^t\circ\beta \mid \eta\wedge\beta\neq 0\}$. An easy computation shows that $\tau(\eta^t\circ\beta)\in\Psi^{-1}(0)$ for a conveniently chosen $\eta^t\circ\beta$ in the dense orbit of $(\mathcal{V}\otimes\mathcal{V})_1$; it follows that (1.3.26) holds. On the other hand, T defines an isomorphism of pairs,

$$(\mathcal{V}\otimes\mathcal{V},(\mathcal{V}\otimes\mathcal{V})_1)\cong\left(\operatorname{Sym}^2\mathcal{V}^\vee\times\left(\mathcal{V}^\vee\otimes\textstyle\bigwedge^3\mathcal{V}\right),\operatorname{im}(\tau)\right). \tag{1.3.27}$$

Since the ideal of $(\mathcal{V}\otimes\mathcal{V})_1$ in $\mathcal{V}\otimes\mathcal{V}$ is generated by nine linearly independent quadrics, it follows that the ideal of $\operatorname{im}(\tau)$ in $\operatorname{Sym}^2\mathcal{V}^\vee\times\left(\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V}\right)$ is also generated by nine linearly independent quadrics. The ideal of $\Psi^{-1}(0)$ in $\operatorname{Sym}^2\mathcal{V}\times\left(\mathcal{V}^\vee\otimes\bigwedge^3\mathcal{V}\right)$ is likewise generated by nine linearly independent quadrics; see (1.3.18). Since $\Psi^{-1}(0)\supset\operatorname{im}(\tau)$, the ideals of $\Psi^{-1}(0)$ and of $\operatorname{im}(\tau)$ are the same and hence (1.3.23) holds. This proves that τ is an isomorphism between $(\mathcal{V}\otimes\mathcal{V})_1$ and $X(\mathcal{V})$. Diagram (1.3.16) is commutative by construction, and (1.3.17) is equivalent to

$$\phi(\tau\circ\rho(\beta,\eta))=\tau\circ\rho(\eta,\beta)). \tag{1.3.28}$$

This equality holds because $\beta\wedge\eta=-\eta\wedge\beta$. □

The following result is an immediate consequence of Proposition 1.14.

COROLLARY 1.15. $\operatorname{sing}X(\mathcal{V})=\tau(0)=f^{-1}(0)$.

2. The Divisor Δ

2.1. Parameter Counts

Let $\Delta_+\subset\mathbb{LG}\left(\bigwedge^3 V\right)$ and $\tilde\Delta_+,\tilde\Delta_+(0)\subset\mathbb{LG}\left(\bigwedge^3 V\right)\times\mathbb{P}(V)^2$ be defined as follows:

$$\Delta_+ := \left\{A \in \mathbb{LG}\left(\textstyle\bigwedge^3 V\right) \mid |Y_A[3]| > 1\right\}, \tag{2.1.1}$$

$$\tilde{\Delta}_+ := \{(A, [v_1], [v_2]) \mid [v_1] \neq [v_2],\ \dim(A \cap F_{v_i}) \geq 3\}, \tag{2.1.2}$$

$$\tilde{\Delta}_+(0) := \{(A, [v_1], [v_2]) \mid [v_1] \neq [v_2],\ \dim(A \cap F_{v_i}) = 3\}. \tag{2.1.3}$$

Note that $\tilde{\Delta}_+$ and $\tilde{\Delta}_+(0)$ are locally closed.

LEMMA 2.1. *With notation as before, we have that*

(1) $\tilde{\Delta}_+$ *is irreducible of dimension 53, and*
(2) Δ_+ *is constructible and* $\operatorname{cod}\left(\Delta_+, \mathbb{LG}\left(\bigwedge^3 V\right)\right) \geq 2$.

Proof. (1) We start by proving that $\tilde{\Delta}_+(0)$ is irreducible of dimension 53. Consider the map

$$\begin{aligned} \tilde{\Delta}_+(0) &\xrightarrow{\eta} \mathrm{Gr}\left(3, \textstyle\bigwedge^3 V\right)^2 \times \mathbb{P}(V)^2, \\ (A, [v_1], [v_2]) &\mapsto (A \cap F_{v_1}, A \cap F_{v_2}, [v_1], [v_2]). \end{aligned} \tag{2.1.4}$$

We have

$$\operatorname{im} \eta = \{(K_1, K_2, [v_1], [v_2]) \mid K_i \in \mathrm{Gr}(3, F_{v_i}),\ K_1 \perp K_2,\ [v_1] \neq [v_2]\}. \tag{2.1.5}$$

We stratify $\operatorname{im}\eta$ according to $i := \dim(K_1 \cap F_{v_2})$ and $j := \dim(K_1 \cap K_2)$; of course, $j \leq i$. Let $(\operatorname{im}\eta)_{i,j} \subset \operatorname{im}\eta$ be the stratum corresponding to i, j. A straightforward computation gives that

$$\begin{aligned} &\dim \eta^{-1}(\operatorname{im}\eta)_{i,j} \\ &\quad = 10 + 7(3-i) + j(i-j) + (3-j)(4+i) + \frac{1}{2}(j+5)(j+4) \\ &\quad = 53 - 4i - \frac{1}{2}j(j-1). \end{aligned} \tag{2.1.6}$$

Since $0 \leq i, j$, it follows that the maximum is achieved for $i = j = 0$ and that it equals 53; hence $\tilde{\Delta}_+(0)$ is irreducible of dimension 53. Yet because $\tilde{\Delta}_+(0)$ is clearly dense in $\tilde{\Delta}_+$, part (1) holds.

(2) Let $\pi_+ \colon \tilde{\Delta}_+ \to \mathbb{LG}\left(\bigwedge^3 V\right)$ be the forgetful map, $\pi_+([v_1], [v_2], A) = A$; then $\pi_+(\tilde{\Delta}_+) = \Delta_+$. From (1) we get that $\dim \Delta_+ \leq 53$; therefore, since $\dim \mathbb{LG}\left(\bigwedge^3 V\right) = 55$, part (2) follows. □

PROPOSITION 2.2. *The following statements hold.*

(1) Δ *is closed irreducible of codimension 1 in* $\mathbb{LG}\left(\bigwedge^3 V\right)$ *and is not equal to* Σ.
(2) *If* $A \in \Delta$ *is generic, then* $Y_A[3] = Y_A(3)$ *and consists of a single point.*

Proof. (1) Let

$$\begin{aligned} \tilde{\Delta} &:= \{(A, [v]) \mid \dim(F_v \cap A) \geq 3\}, \\ \tilde{\Delta}(0) &:= \{(A, [v]) \mid \dim(F_v \cap A) = 3\}. \end{aligned} \tag{2.1.7}$$

Then $\tilde{\Delta}$ is a closed subset of $\mathbb{LG}\left(\bigwedge^3 V\right) \times \mathbb{P}(V)$ and $\tilde{\Delta}(0)$ is an open subset of $\tilde{\Delta}$. Let $\pi \colon \tilde{\Delta} \to \mathbb{LG}\left(\bigwedge^3 V\right)$ be the forgetful map. Thus $\pi(\tilde{\Delta}) = \Delta$ and, since π is projective, it follows that Δ is closed. Projecting $\tilde{\Delta}(0)$ to $\mathbb{P}(V)$ yields that $\tilde{\Delta}(0)$ is smooth irreducible of dimension 54. A standard dimension count shows that $\tilde{\Delta}(0)$ is open dense in $\tilde{\Delta}$ and so $\tilde{\Delta}$ is irreducible of dimension 54. It follows that Δ is irreducible. By Lemma 2.1 we know that $\dim \tilde{\Delta}_+ \leq 53$. Therefore, the generic fiber of $\tilde{\Delta} \to \Delta$ is a single point—in particular, $\dim \Delta = 54$—and hence $\operatorname{cod}\big(\Delta, \mathbb{LG}\big(\bigwedge^3 V\big)\big) = 1$ because $\dim \mathbb{LG}\big(\bigwedge^3 V\big) = 55$. A dimension count shows that $\dim(\Delta \cap \Sigma) < 54$ and hence $\Delta \neq \Sigma$.

(2) Let $A \in \Delta$ be generic. We have already observed that there exists a unique $[v] \in \mathbb{P}(V)$ such that $([v], A) \in \tilde{\Delta}$; that is, $Y_A[3]$ consists of a single point. Since $\tilde{\Delta}(0)$ is dense in $\tilde{\Delta}$ and since $\dim \tilde{\Delta} = \dim \Delta$, it follows that $([v], A) \in \tilde{\Delta}(0)$; that is, $Y_A[3] = Y_A(3)$. $\square$

2.2. First-Order Computations

Let $(A, [v_0]) \in \tilde{\Delta}(0)$. We will study the differential of $\pi \colon \tilde{\Delta} \to \mathbb{LG}\big(\bigwedge^3 V\big)$ at $(A, [v_0])$. First we give a local description of $\tilde{\Delta}$ as degeneracy locus. Let

$$\mathbb{N}(V) := \big\{A \in \mathbb{LG}\big(\textstyle\bigwedge^3 V\big) \mid Y_A = \mathbb{P}(V)\big\}. \tag{2.2.1}$$

Notice that $\mathbb{N}(V)$ is closed. Let $\mathcal{Y}$ be the tautological family of EPW-sextics:

$$\mathcal{Y} := \big\{(A, [v]) \in \big(\mathbb{LG}\big(\textstyle\bigwedge^3 V\big) \setminus \mathbb{N}(V)\big) \times \mathbb{P}(V) \mid \dim(A \cap F_v) > 0\big\}. \tag{2.2.2}$$

Because $\mathcal{Y}$ may be described as a determinantal variety, it has a natural scheme structure. For $\mathcal{U} \subset \big(\mathbb{LG}\big(\bigwedge^3 V\big) \setminus \mathbb{N}(V)\big)$ open, we let $\mathcal{Y}_{\mathcal{U}} := \mathcal{Y} \cap (\mathcal{U} \times \mathbb{P}(V))$. Given $B \in \mathbb{LG}\big(\bigwedge^3 V\big)$, let

$$U_B := \big\{A \in \mathbb{LG}\big(\textstyle\bigwedge^3 V\big) \mid A \pitchfork B\big\} \setminus \mathbb{N}(V). \tag{2.2.3}$$

(Here $A \pitchfork B$ means that A intersects B transversely; i.e., $A \cap B = \{0\}$.) Let $i_{U_B} \colon \mathcal{Y}_{U_B} \hookrightarrow U_B \times \mathbb{P}(V)$ be the inclusion and let $\mathcal{A}$ be the tautological rank-10 vector bundle on $\mathbb{LG}\big(\bigwedge^3 V\big)$ (the fiber of $\mathcal{A}$ over A is A itself). Going through the argument that produced commutative diagram (1.2.2), we find that there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{U_B} \boxtimes F \arrow[r, "\lambda_{U_B}"] \arrow[d, "\mu_{U_B}"] & (\mathcal{A}^\vee|_{U_B}) \boxtimes \mathcal{O}_{\mathbb{P}(V)} \arrow[r] \arrow[d, "\mu^t_{U_B}"] & i_{U_B,*}\zeta_{U_B} \arrow[r] \arrow[d, "\beta_{U_B}"] & 0 \\
0 \arrow[r] & (\mathcal{A}|_{U_B}) \boxtimes \mathcal{O}_{\mathbb{P}(V)} \arrow[r, "\lambda^t_{U_B}"] & \mathcal{O}_{U_B} \boxtimes F^\vee \arrow[r] & \operatorname{Ext}^1(i_{U_B,*}\zeta_{U_B}, \mathcal{O}_{U_B \times \mathbb{P}(V)}) \arrow[r] & 0.
\end{tikzcd} \tag{2.2.4}$$

Now let $(A, [v_0]) \in \mathcal{Y}$. Choose $B \in \mathbb{LG}\big(\bigwedge^3 V\big)$ such that $B \pitchfork A$ and $B \pitchfork F_{v_0}$. Let $\mathcal{N} \subset \mathbb{P}(V)$ be an open neighborhood of $[v_0]$ such that $B \pitchfork F_w$ for all $w \in \mathcal{N}$. The restriction to U_B of $\mathcal{A}$ is trivial, as is the restriction to $\mathcal{N}$ of F. Moreover, the restriction of μ_{U_B} to $U_B \times \mathcal{N}$ is an isomorphism. Let

$$\gamma := (\lambda_{U_B}|_{U_B \times \mathcal{N}}) \circ (\mu_{U_B}|_{U_B \times \mathcal{N}})^{-1}. \tag{2.2.5}$$

We have the exact sequence

$$0 \to (\mathcal{A}|_{U_B}) \boxtimes \mathcal{O}_{\mathcal{N}} \xrightarrow{\gamma} (\mathcal{A}^\vee|_{U_B}) \boxtimes \mathcal{O}_{\mathcal{N}} \to i_{U_B,*}\zeta_{U_B}|_{U_B \times \mathcal{N}} \to 0. \tag{2.2.6}$$

The map γ is symmetric; in fact, it is the symmetrization of the restriction of (2.2.4) to $U_B \times \mathcal{N}$ (see Definition 1.8). Then $\tilde{\Delta} \cap (U_B \times \mathcal{N})$ is the symmetric degeneration locus

$$\tilde{\Delta} \cap (U_B \times \mathcal{N}) = \{(A', [v]) \in (U_B \times \mathcal{N}) \mid \operatorname{cork} \gamma(A', [v]) \geq 3\} \tag{2.2.7}$$

and so it inherits the natural structure of a closed subscheme of $\mathbb{LG}\big(\bigwedge^3 V\big) \times \mathbb{P}(V)$.

In order to study the differential of the forgetful map $\tilde{\Delta} \to \mathbb{P}(V)$, we introduce some notation. Given $v \in V$, we define a quadratic form $\phi_v^{v_0}$ on F_{v_0} as follows. Let $\alpha \in F_{v_0}$; then $\alpha = v_0 \wedge \beta$ for some $\beta \in \bigwedge^2 V$. We set

$$\phi_v^{v_0}(\alpha) := \operatorname{vol}(v_0 \wedge v \wedge \beta \wedge \beta). \tag{2.2.8}$$

This expression gives a well-defined quadratic form on F_{v_0} because β is determined up to addition by an element of F_{v_0}. Of course, $\phi_v^{v_0}$ depends only on the class of v in $V/[v_0]$.

Choose a direct sum decomposition

$$V = [v_0] \oplus V_0. \tag{2.2.9}$$

We have the isomorphism

$$\begin{aligned} \lambda_{V_0}^{v_0} \colon \textstyle\bigwedge^2 V_0 &\xrightarrow{\sim} F_{v_0}, \\ \beta &\longmapsto v_0 \wedge \beta. \end{aligned} \tag{2.2.10}$$

Under this identification, the Plücker quadratic forms on $\bigwedge^2 V_0$ correspond to the quadratic forms $\phi_v^{v_0}$ for v varying in V_0. Let $K := A \cap F_{v_0}$ and

$$\begin{aligned} V_0 &\xrightarrow{\tau_K^{v_0}} \operatorname{Sym}^2 K^\vee, & \operatorname{Sym}^2 A^\vee &\xrightarrow{\theta_K^A} \operatorname{Sym}^2 K^\vee, \\ v &\longmapsto \phi_v^{v_0}|_K; & q &\longmapsto q|_K. \end{aligned} \tag{2.2.11}$$

The isomorphism

$$\begin{aligned} V_0 &\xrightarrow{\sim} \mathbb{P}(V) \setminus \mathbb{P}(V_0), \\ v &\longmapsto [v_0 + v] \end{aligned}$$

defines an isomorphism $V_0 \cong T_{[v_0]}\mathbb{P}(V)$. Recall that the tangent space to $\mathbb{LG}\big(\bigwedge^3 V\big)$ at A is canonically identified with $\operatorname{Sym}^2 A^\vee$.

Proposition 2.3. *If we make the choice* (2.2.9), *then*

$$T_{(A,[v_0])}\tilde{\Delta} \subset T_{(A,[v_0])}\big(\mathbb{LG}\big(\textstyle\bigwedge^3 V\big) \times \mathbb{P}(V)\big) = \operatorname{Sym}^2 A^\vee \oplus V_0 \tag{2.2.12}$$

is given by

$$T_{([v_0],A)}\tilde{\Delta} = \{(q, v) \mid \theta_K^A(q) - \tau_K^{v_0}(v) = 0\}. \tag{2.2.13}$$

Proof. From the (local) degeneracy description (2.2.7) it follows that $(q, v) \in T_{([v_0],A)}\tilde{\Delta}$ if and only if

$$0 = d\gamma(A, [v_0])(q, v)|_K = d\gamma(A, [v_0])(q, 0)|_K + d\gamma(A, [v_0])(0, v)|_K.$$

It is clear that $d\gamma(A,[v_0])(q,0)|_K = \theta^A_K(q)$. On the other hand, equation (2.26) of [12] gives that

$$d\gamma(A,[v_0])(0,v)|_K = -\tau^{v_0}_K(v). \tag{2.2.14}$$

The proposition follows. □

COROLLARY 2.4. *$\tilde{\Delta}(0)$ is smooth and of codimension 6 in $\mathbb{LG}\left(\bigwedge^3 V\right) \times \mathbb{P}(V)$. Let $(A,[v_0]) \in \tilde{\Delta}(0)$ and $K := A \cap F_{v_0}$. Then the differential $d\pi(A,[v_0])$ is injective if and only if $\tau^{v_0}_K$ is injective.*

Proof. Let $(A,[v_0]) \in \tilde{\Delta}(0)$ and $K := A \cap F_{v_0}$. The map θ^A_K is surjective, and by Proposition 2.3 we have that $T_{(A,[v_0])}\tilde{\Delta}(0)$ has codimension 6 in the space $T_{(A,[v_0])}\left(\mathbb{LG}\left(\bigwedge^3 V\right) \times \mathbb{P}(V)\right)$. Yet the description of $\tilde{\Delta}(0)$ as a symmetric degeneration locus, as in (2.2.7), gives that $\tilde{\Delta}(0)$ has codimension at most 6 in $\mathbb{LG}\left(\bigwedge^3 V\right) \times \mathbb{P}(V)$. These two statements together imply that $\tilde{\Delta}(0)$ is smooth of codimension 6 in $\mathbb{LG}\left(\bigwedge^3 V\right) \times \mathbb{P}(V)$. Our claim about the injectivity of $d\pi(A,[v_0])$ follows immediately from Proposition 2.3. □

REMARK. The statement in Corollary 2.4 about the smoothness of $\tilde{\Delta}(0)$ is *not* contained in the proof of Proposition 2.2 because, in that proof, we consider $\tilde{\Delta}(0)$ with its reduced structure.

Before stating the next result, we give the following definition. For $A \in \mathbb{LG}\left(\bigwedge^3 V\right)$, let

$$\Theta_A := \left\{W \in \mathrm{Gr}(3,V) \mid \textstyle\bigwedge^3 W \subset A\right\}. \tag{2.2.15}$$

PROPOSITION 2.5. *Let $(A,[v_0]) \in \tilde{\Delta}(0)$ and let $K := A \cap F_{v_0}$. Then $\tau^{v_0}_K$ is injective if and only if:*

(1) *no $W \in \Theta_A$ contains v_0; or*

(2) *there is exactly one $W \in \Theta_A$ containing v_0 and, moreover,*

$$A \cap F_{v_0} \cap \left(\textstyle\bigwedge^2 W \wedge V\right) = \textstyle\bigwedge^3 W. \tag{2.2.16}$$

If (1) (*respectively,* (2)) *holds, then* $\operatorname{im}\tau^{v_0}_K$ *belongs to the unique open* (*respectively, closed*) $\mathrm{PGL}(K)$*-orbit of* $\mathrm{Gr}(5,\mathrm{Sym}^2 K^\vee)$.

Proof. Let $V_0 \subset V$ be a codimension-1 subspace transversal to $[v_0]$, and let

$$\rho^{v_0}_{V_0} \colon F_{v_0} \xrightarrow{\sim} \textstyle\bigwedge^2 V_0 \tag{2.2.17}$$

be the inverse of isomorphism (2.2.10). Let $\mathbf{K} := \mathbb{P}(\rho^{v_0}_{V_0}(K)) \subset \mathbb{P}\left(\bigwedge^2 V_0\right)$, in which case $\mathbf{K}$ is a projective plane. Isomorphism $\rho^{v_0}_{V_0}$ identifies the space of quadratic forms $\phi^{v_0}_v$, $v \in V_0$, with the space of Plücker quadratic forms on $\bigwedge^2 V_0$. Because the ideal of $\mathrm{Gr}(2,V_0) \subset \mathbb{P}\left(\bigwedge^2 V_0\right)$ is generated by the Plücker quadratic forms, we get that $\tau^{v_0}_K$ is identified with the natural restriction map

$$V_0 = H^0(\mathcal{I}_{\mathrm{Gr}(2,V_0)}(2)) \xrightarrow{\tau^{v_0}_K} H^0(\mathcal{O}_{\mathbf{K}}(2)) = \mathrm{Sym}^2 K^\vee. \tag{2.2.18}$$

It follows that, if the scheme-theoretic intersection $\mathbf{K}\cap \mathrm{Gr}(2, V_0)$ is neither empty nor a single reduced point, then $\tau_K^{v_0}$ is not injective.

Now suppose that $\mathbf{K}\cap \mathrm{Gr}(2, V_0)$ is either

(1′) empty (i.e., part (1) of the proposition holds) or
(2′) a single reduced point (i.e., part (2) holds).

Let

$$\mathbb{P}\left(\bigwedge^2 V_0\right) \overset{\Phi}{\dashrightarrow} |H^0(\mathcal{I}_{\mathrm{Gr}(2,V_0)}(2))|^\vee = \mathbb{P}(V_0^\vee) \tag{2.2.19}$$

be the natural map: it associates to $[\alpha]\notin \mathrm{Gr}(2, V_0)$ the projectivization of $\mathrm{supp}\,\alpha$. We have a tautological identification

$$\mathbf{K} \overset{\Phi|_{\mathbf{K}}}{\dashrightarrow} \mathbb{P}(\mathrm{im}\,\tau_K^{v_0})^\vee,$$

where $\Phi|_{\mathbf{K}}$ is the Veronese embedding $\mathbf{K}\to |\mathcal{O}_{\mathbf{K}}(2)|^\vee$ followed by the projection with center $\mathbb{P}(\mathrm{Ann}(\mathrm{im}\,\tau_K^{v_0}))$. Notice that $\tau_K^{v_0}$ is not injective if and only if $\dim \mathbb{P}(\mathrm{Ann}(\mathrm{im}\,\tau_K^{v_0}))\geq 1$. Suppose that (1′) holds. Then $\Phi|_{\mathbf{K}}$ is regular and is, in fact, an isomorphism onto its image (see [15, Lemma 2.7]. Since the chordal variety of the Veronese surface in $|\mathcal{O}_{\mathbf{K}}(2)|^\vee$ is a hypersurface, it follows that $\dim \mathbb{P}(\mathrm{Ann}(\mathrm{im}\,\tau_K^{v_0}))<1$ and hence $\tau_K^{v_0}$ is injective. We also get that $\mathrm{Ann}(\mathrm{im}\,\tau_K^{v_0})$ is a point in $|\mathcal{O}_{\mathbf{K}}(2)|^\vee$ that does not belong to the chordal variety of the Veronese surface; it therefore belongs to a unique open $\mathrm{PGL}(K)$-orbit. Now suppose that (2′) holds. Assume that $\tau_K^{v_0}$ is not injective, in which case $\dim \mathbb{P}(\mathrm{Ann}(\mathrm{im}\,\tau_K^{v_0}))\geq 1$. It follows that there exist $[x]\neq[y]\in\mathbf{K}$ in the regular locus of $\Phi|_{\mathbf{K}}$ (i.e., neither x nor y is decomposable) such that $\Phi([x])=\Phi([y])$. By the preceding description of Φ in terms of supports, we have that $\mathrm{supp}(x)=\mathrm{supp}(y)=U$ for $\dim U=4$; since $\mathrm{Gr}(2,U)$ is a hypersurface in $\mathbb{P}\left(\bigwedge^2 U\right)$, the line $\langle[x],[y]\rangle\subset\mathbb{P}\left(\bigwedge^2 V_0\right)$ intersects $\mathrm{Gr}(2,U)$ in a subscheme of length 2. Since $\langle[x],[y]\rangle\subset\mathbf{K}$ it follows that $\mathbf{K}\cap\mathrm{Gr}(2,V_0)$ contains a scheme of length 2, which contradicts (2′). This proves that if (2′) holds then $\tau_K^{v_0}$ is injective. It also follows that $\mathrm{Ann}(\tau_K^{v_0})$ belongs to the Veronese surface in $|\mathcal{O}_{\mathbf{K}}(2)|^\vee$; that is, $\mathrm{im}(\tau_K^{v_0})$ belongs to the unique closed $\mathrm{PGL}(K)$-orbit. □

3. Simultaneous Resolution

In Section 3.1 we analyze families of double EPW-sextics and their singular locus. Section 3.2 shows how to construct the simultaneous desingularization described in item (3) of Section 0 (the relation with the Hilbert square of a K3 surface will be given in Section 4).

3.1. Families of Double EPW-Sextics

Let $\mathcal{U}\subset\left(\mathbb{LG}\left(\bigwedge^3 V\right)\setminus\mathbb{N}(V)\right)$ (see (2.2.1)) be open. Suppose there exist a scheme $\mathcal{X}_{\mathcal{U}}$ and a finite $f_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}}\to\mathcal{Y}_{\mathcal{U}}$ such that, for every $A\in\mathcal{U}$, the induced map $f^{-1}Y_A\to Y_A$ is identified with $f_A\colon X_A\to Y_A$. Then we say that a *tautological family of double EPW-sextics parameterized by $\mathcal{U}$ exists*—or, more simply, that

$f_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ exists. Composing $f_{\mathcal{U}}$ with the natural map $\mathcal{Y}_{\mathcal{U}} \to \mathcal{U}$ yields a map $\rho_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{U}$ such that $\rho_{\mathcal{U}}^{-1}(A) \cong X_A$.

PROPOSITION 3.1. *Let $B \in \mathbb{LG}\big(\bigwedge^3 V\big)$. Then there exists a tautological family of double EPW-sextics parameterized by U_B, where U_B is given by* (2.2.3).

Proof. Let $\nu\colon \mathcal{Y}_{U_B} \to \mathbb{P}(V)$ be projection. Let $\xi_{U_B} := \zeta_{U_B} \otimes \nu^*\mathcal{O}_{\mathbb{P}(V)}(-3)$, where ζ_{U_B} is the sheaf on $\mathcal{Y}_{U_B}$ fitting in (2.2.4). Referring to commutative diagram (2.2.4) and proceeding as in the definition of multiplication on $\mathcal{O}_{Y_A} \oplus \xi_A$, we find that β_{U_B} defines a multiplication on $\mathcal{O}_{\mathcal{Y}_{U_B}} \oplus \xi_{U_B}$. Then, by Proposition 1.7, $\mathcal{O}_{\mathcal{Y}_{U_B}} \oplus \xi_{U_B}$ is an associative commutative ring. Let $\mathcal{X}_{U_B} := \operatorname{Spec}(\mathcal{O}_{\mathcal{Y}_{U_B}} \oplus \xi_{U_B})$ and let $f_{U_B}\colon \mathcal{X}_{U_B} \to \mathcal{Y}_{U_B}$ be the structure map. $\square$

Let $\mathcal{U} \subset \big(\mathbb{LG}\big(\bigwedge^3 V\big) \setminus \mathbb{N}(V)\big)$ be open and such that $f_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ exists. We will determine the singular locus of $\mathcal{X}_{\mathcal{U}}$. Let

$$\mathcal{Y}[d] := \big\{(A,[v]) \in \big(\mathbb{LG}\big(\textstyle\bigwedge^3 V\big) \setminus \mathbb{N}(V)\big) \times \mathbb{P}(V) \mid \dim(A \cap F_v) \ge d\big\}, \quad (3.1.1)$$

$$\mathcal{Y}(d) := \big\{(A,[v]) \in \big(\mathbb{LG}\big(\textstyle\bigwedge^3 V\big) \setminus \mathbb{N}(V)\big) \times \mathbb{P}(V) \mid \dim(A \cap F_v) = d\big\}. \quad (3.1.2)$$

Then $\mathcal{Y}[d]$ has the natural structure of a closed subscheme of $\mathbb{LG}\big(\bigwedge^3 V\big) \times \mathbb{P}(V)$ given by its local description as a symmetric determinantal variety (see [15, Sec. 2.2]). Let $\mathcal{U} \in \big(\mathbb{LG}\big(\bigwedge^3 V\big) \setminus \mathbb{N}(V)\big)$ be open. We let $\mathcal{Y}_{\mathcal{U}}[d] := \mathcal{Y}[d] \cap \mathcal{Y}_{\mathcal{U}}$ and similarly for $\mathcal{Y}_{\mathcal{U}}(d)$. Suppose that $f_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ is defined, and let

$$\mathcal{W}_{\mathcal{U}} := f_{\mathcal{U}}^{-1}\mathcal{Y}[3]. \quad (3.1.3)$$

Observe that the restriction of $f_{\mathcal{U}}$ to $\mathcal{W}_{\mathcal{U}}$ defines an isomorphism $\mathcal{W}_{\mathcal{U}} \xrightarrow{\sim} \mathcal{Y}_{\mathcal{U}}[3]$. We will prove the following result.

PROPOSITION 3.2. *Let $\mathcal{U} \subset \big(\mathbb{LG}\big(\bigwedge^3 V\big) \setminus \mathbb{N}(V)\big)$ be open, and suppose that $f_{\mathcal{U}}\colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ exists. Then* $\operatorname{sing} \mathcal{X}_{\mathcal{U}} = \mathcal{W}_{\mathcal{U}}$.

Proof. We may assume that $\mathcal{U} = U_B \times \mathcal{N}$, where $B \in \mathbb{LG}\big(\bigwedge^3 V\big)$ and $\mathcal{N} \subset \mathbb{P}(V)$ is an open subset such that $B \pitchfork F_w$ for all $w \in \mathcal{N}$. Then (see the proof of Proposition 3.1)

$$f_{U_B}^{-1}(\mathcal{U}) = X_\gamma, \quad (3.1.4)$$

where γ is given by (2.2.5). It therefore suffices to examine X_γ. Let $(A,[v]) \in \mathcal{U}$ and let

$$\delta_\gamma(A,[v])\colon T_{(A,[v])}\mathbb{LG}\big(\textstyle\bigwedge^3 V\big) \times \mathbb{P}(V) \to \operatorname{Sym}^2(A \cap F_v)^\vee \quad (3.1.5)$$

be as in (1.3.3). The restriction of $\delta_\gamma(A,[v])$ to the tangent space to $\mathbb{LG}\big(\bigwedge^3 V\big)$ at A is surjective, so

$$\delta_\gamma(A,[v]) \text{ is surjective.} \quad (3.1.6)$$

Let $q \in \mathcal{X}_\gamma$ and $f_{\mathcal{U}}(q) = (A,[v])$. Suppose that $q \notin \mathcal{W}_{\mathcal{U}}$ (i.e., that $\operatorname{cork}\gamma(p) \le 2$). If $\operatorname{cork}\gamma(p) = 1$ then $Y_{\mathcal{U}} = Y_\gamma$ is smooth because the differential $\delta_\gamma(A,[v])$

is surjective, and from Proposition 1.5 it then follows that $\mathcal{X}_{\mathcal{U}}$ is smooth at q. If $\operatorname{cork}\gamma(p) = 2$ then $\mathcal{X}_{\mathcal{U}}$ is smooth at q by Proposition 1.12—recall that the differential $\delta_\gamma(A,[v])$ is surjective. This proves that $\operatorname{sing}\mathcal{X}_{\mathcal{U}} \subset \mathcal{W}_{\mathcal{U}}$. On the other hand, $\mathcal{W}_{\mathcal{U}} \subset \operatorname{sing}\mathcal{X}_{\mathcal{U}}$ by Claim 1.13. □

We shall next prove a few results about the individual X_A.

LEMMA 3.3. *Let $A \in \left(\mathbb{LG}\left(\bigwedge^3 V\right) \setminus \mathbb{N}(V)\right)$ and let $[v] \in Y_A$. Suppose that $\dim(A \cap F_v) \le 2$ and that there is no $W \in \Theta_A$ (see* (2.2.15)*) containing v. Then X_A is smooth at $f_A^{-1}([v])$.*

Proof. Let $q \in f_A^{-1}([v])$, and suppose that $\dim(A \cap F_v) = 1$. By [15, Cor. 2.5], Y_A is smooth at $[v]$; hence, by Proposition 1.5, X_A is smooth at q. Suppose that $\dim(A \cap F_v) = 2$. Locally around q, the double cover $X_A \to Y_A$ is isomorphic to $X_{\bar\gamma} \to Y_{\bar\gamma}$, where $\bar\gamma$ is the symmetrization of the restriction of β_A to an affine neighborhood $\operatorname{Spec} R$ of $[v]$. Thus we may consider the differential $\delta_{\bar\gamma}([v])$ (see (1.3.3)). The differential is surjective by [15, Prop. 2.9], so X_A is smooth at q by Proposition 1.12. □

PROPOSITION 3.4. *Let $A \in \left(\mathbb{LG}\left(\bigwedge^3 V\right) \setminus \mathbb{N}(V)\right)$. Then X_A is smooth if and only if $A \in \mathbb{LG}\left(\bigwedge^3 V\right)^0$.*

Proof. If $A \in \mathbb{LG}\left(\bigwedge^3 V\right)^0$ then X_A is smooth by [12]. For the "only if", suppose that X_A is smooth. Then $A \notin \Delta$ by Claim 1.13. Assume that $A \in \Sigma$; we will reach a contradiction. Let $W \in \Theta_A$ and $[v] \in \mathbb{P}(W)$, and note that $\mathbb{P}(W) \subset Y_A$. Let $q \in f_A^{-1}([v])$. Since $A \notin \Delta$, it follows that $1 \le \dim(A \cap F_v) \le 2$. Suppose $\dim(A \cap F_v) = 1$. Then Y_A is singular at $[v]$ by [15, Cor. 2.5] and so X_A is singular at q by Proposition 1.5. Suppose now that $\dim(A \cap F_v) = 2$, and let $\bar\gamma$ be as in the proof of Lemma 3.3. Then $\delta_{\bar\gamma}([v])$ is not surjective by [15, Prop. 2.3]; hence X_A is singular at q by Proposition 1.12. □

3.2. The Desingularization

DEFINITION 3.5. Let $\mathbb{LG}\left(\bigwedge^3 V\right)^* \subset \mathbb{LG}\left(\bigwedge^3 V\right)$ be the set of A such that

(1) $A \notin \mathbb{N}(V)$,
(2) $Y_A[3] = Y_A(3)$, and
(3) $Y_A[3]$ is finite.

REMARK 3.6. $\mathbb{LG}\left(\bigwedge^3 V\right)^*$ is an open subset of $\mathbb{LG}\left(\bigwedge^3 V\right)$.

CLAIM 3.7. $\left(\mathbb{LG}\left(\bigwedge^3 V\right) \setminus \Sigma\right) \subset \mathbb{LG}\left(\bigwedge^3 V\right)^*$.

Proof. Definition 3.5(1) holds by [15, Claim 2.11]. To prove part (2), suppose that $Y_A[3] \ne Y_A(3)$; in other words, suppose there exists a $[v_0] \in \mathbb{P}(V)$ such that $\dim(A \cap F_{v_0}) \ge 4$. Let $V_0 \subset V$ be a codimension-1 subspace that is transversal to $[v_0]$ and let $\rho^{v_0}_{V_0}$ be as in (2.2.17). Let $\mathbf{K} := \mathbb{P}(\rho^{v_0}_{V_0}(A \cap F_{v_0}))$; then $\dim \mathbf{K} \geq 3$. Since $\mathrm{Gr}(2, V_0)$ has codimension 3 in $\mathbb{P}(\bigwedge^2 V_0)$, it follows that there exists an $[\alpha] \in \mathbf{K} \cap \mathrm{Gr}(2, V_0)$. Let $\tilde{\alpha} \in (A \cap F_{v_0})$ such that $\rho^{v_0}_{V_0}(\tilde{\alpha}) = \alpha$. Then $\tilde{\alpha}$ is nonzero and decomposable—a contradiction because $A \notin \Sigma$. To prove part (3), let $[v_0] \in Y_A[3] = Y_A(3)$. Then $(A, [v_0]) \in \tilde{\Delta}(0)$. Let $K := A \cap F_{v_0}$ and let $\tau^{v_0}_K$ be as in (2.2.11). We have

$$T_{[v_0]} Y_A[3] = T_{[v_0]} Y_A(3) = \ker \tau^{v_0}_K.$$

By Proposition 2.5, the map $\tau^{v_0}_K$ is injective. Hence $[v_0]$ is an isolated point of $Y_A[3]$. □

Let $A \in \mathbb{LG}(\bigwedge^3 V)^*$. Let $\mathcal{U} \subset \mathbb{LG}(\bigwedge^3 V)^*$ be a small open (in either the Zariski or the classical topology) subset containing A. In particular, $\rho_{\mathcal{U}} \colon \mathcal{X}_{\mathcal{U}} \to \mathcal{Y}_{\mathcal{U}}$ exists. Let $\pi_{\mathcal{U}} \colon \tilde{\mathcal{X}}_{\mathcal{U}} \to \mathcal{X}_{\mathcal{U}}$ be the blow-up of $\mathcal{W}_{\mathcal{U}}$, and let $E_{\mathcal{U}}$ be the exceptional set of $\pi_{\mathcal{U}}$.

CLAIM 3.8. *With notation as before, $\tilde{\mathcal{X}}_{\mathcal{U}}$ is smooth. If $\mathcal{U}$ is open and sufficiently small in the classical topology, then we have a locally trivial fibration*

$$E_{\mathcal{U}} \to Y_{\mathcal{U}}[3]. \tag{3.2.1}$$

Let $(A, [v]) \in Y_{\mathcal{U}}[3]$. The fiber of (3.2.1) over $(A, [v])$ is isomorphic to $\mathbb{P}(A \cap F_v)^\vee \times \mathbb{P}(A \cap F_v)^\vee$, and the restriction of $N_{E_{\mathcal{U}}/\tilde{\mathcal{X}}_{\mathcal{U}}}$ to the fiber is isomorphic to $\mathcal{O}_{\mathbb{P}(A\cap F_v)^\vee}(-1) \boxtimes \mathcal{O}_{\mathbb{P}(A\cap F_v)^\vee}(-1)$.

Proof. By Propositon 3.2 we know that $\tilde{\mathcal{X}}_{\mathcal{U}}$ is smooth outside $E_{\mathcal{U}}$. It remains to examine $\tilde{\mathcal{X}}_{\mathcal{U}}$ over $\mathcal{W}_{\mathcal{U}} \cong \mathcal{Y}_{\mathcal{U}}[3]$. We may assume that $\mathcal{U} = U_B \times \mathcal{N}$ is as in the proof of Proposition 3.2, and we will adopt the notation of that proof. Let $q \in \mathcal{X}_\gamma$ and $f_{\mathcal{U}}(q) = (A, [v]) = p$. A neighborhood of q in $X_{\mathcal{U}}$ is isomorphic to X_γ, where γ is given by (2.2.5) (see (3.1.4)). We assume that $q \in \mathcal{W}_{\mathcal{U}}$ and hence $\operatorname{cork} \gamma(p) = 3$. Let $f \colon X(\mathcal{V}) \to Y(\mathcal{V})$ be as in Section 1.3; that is, f is the universal double covering of corank 3 at the origin. We claim that there exists a map $\nu \colon X_\gamma \to X(\mathcal{V})$ such that the diagram

$$\begin{tikzcd} X_\gamma \arrow[r, "\nu"] \arrow[d, "f_\gamma"'] & X(\mathcal{V}) \arrow[d, "f"] \\ Y_\gamma \arrow[r, "\mu"] & Y(\mathcal{V}) \end{tikzcd} \tag{3.2.2}$$

commutes and such that X_γ is identified with the fibered product $Y_\gamma \times_{Y(\mathcal{V})} X(\mathcal{V})$. In fact, it suffices to apply the reduction procedure of Section 1.1 that led to Claim 1.4. Let $\mathbf{K}$ be as in Claim 1.4. By (1.1.29) we have $(Y_{\gamma_{\mathbf{K}}}, p) = (Y_\gamma, p)$, and by Claim 1.4 we have a natural isomorphism $(X_{\gamma_{\mathbf{K}}}, f^{-1}_{\gamma_{\mathbf{K}}}(p)) \xrightarrow{\sim} (X_\gamma, f^{-1}_\gamma(p))$ commuting with $f_{\gamma_{\mathbf{K}}}$ and f_γ. Let $\mathcal{U} = \operatorname{Spec} R$; we are free to replace $\mathcal{U}$ by any affine open subset containing $(A, [v])$. Thus we may assume that $\mathbf{K}$ is a trivial R-module; that is, $\mathbf{K} = \mathcal{V} \otimes R$ for $\mathcal{V}$ a complex 3-dimensional vector space. Hence we may view $\gamma_{\mathbf{K}}$

as a map $\gamma_{\mathbf{K}} \colon \operatorname{Spec} R \to \operatorname{Sym}^2 \mathcal{V}^\vee$. Notice that we have equality of schemes $Y_\gamma = \gamma_{\mathbf{K}}^{-1} Y(\mathcal{V})$ and so the restriction of $\gamma_{\mathbf{K}}$ to Y_γ defines a map $\mu \colon Y_\gamma \to Y(\mathcal{V})$. The claim then follows. By the surjectivity of $\delta_\gamma(A,[v])$ (see (3.1.6)) we get that the germ $(X_\gamma, f_\gamma^{-1}(p))$ is the product of a smooth germ (of dimension 54) and the germ $(X(\mathcal{V}), f^{-1}(0))$. Then the explicit description of $X(\mathcal{V})$ given by Proposition 1.14 immediately gives that $\tilde{\mathcal{X}}_{\mathcal{U}}$ is smooth over q and the remaining statements as well. We must assume that $\mathcal{U}$ is a small open subset in the classical topology in order to ensure that (3.2.1) is a locally trivial fibration. □

Remark 3.9. Let $A \in \mathbb{LG}\left(\bigwedge^3 V\right)^*$ and let $Y_A[3] = \{[v_1], \dots, [v_s]\}$. Let $\mathcal{U} \subset \mathbb{LG}\left(\bigwedge^3 V\right)^*$ be a small open (in the classical topology) subset containing A. For each $1 \le i \le s$, choose a projection

$$E_{\mathcal{U}}([v_i]) \longrightarrow \mathbb{P}(A \cap F_v)^\vee. \tag{3.2.3}$$

There exists a unique $\mathbb{P}^2$-fibration

$$\varepsilon \colon E_{\mathcal{U}} \longrightarrow \star, \tag{3.2.4}$$

where $\star$ is itself a fibration over $Y_{\mathcal{U}}[3]$ with fiber $\mathbb{P}(A \cap F_v)^\vee$ over $(A,[v])$. We say that (3.2.3) is a *choice of* $\mathbb{P}^2$*-fibration* ε *for* X_A.

Let $A \in \mathbb{LG}\left(\bigwedge^3 V\right)^*$ and choose a $\mathbb{P}^2$-fibration ε for X_A. Let $\mathcal{U} \subset \mathbb{LG}\left(\bigwedge^3 V\right)^*$ be a small open (in the classical topology) subset containing A. By Claim 3.8, the normal bundle of $E_{\mathcal{U}}$ along the fibers of (3.2.4) is $\mathcal{O}_{\mathbb{P}^2}(-1)$. Hence there exists a contraction $c_{\mathcal{U},\varepsilon} \colon \tilde{\mathcal{X}}_{\mathcal{U}} \to \mathcal{X}_{\mathcal{U}}^\varepsilon$ in the category of complex manifolds fitting into the commutative diagram

$$\begin{tikzcd} \tilde{\mathcal{X}}_{\mathcal{U}} \arrow[rr, "c_{\mathcal{U},\varepsilon}"] \arrow[dr, "\pi_{\mathcal{U}}^\varepsilon"'] & & \mathcal{X}_{\mathcal{U}}^\varepsilon \arrow[dl, "g_{\mathcal{U}}^\varepsilon"] \\ & \mathcal{X}_{\mathcal{U}}. & \end{tikzcd} \tag{3.2.5}$$

Let $f_{\mathcal{U}}^\varepsilon = f_{\mathcal{U}} \circ g_{\mathcal{U}}^\varepsilon \colon \mathcal{X}_{\mathcal{U}}^\varepsilon \to \mathcal{Y}_{\mathcal{U}}$, and let $\rho_{\mathcal{U}}^\varepsilon \colon \mathcal{X}_{\mathcal{U}}^\varepsilon \to \mathcal{U}$ be the map $f_{\mathcal{U}}^\varepsilon$ followed by $\mathcal{Y}_{\mathcal{U}} \to \mathcal{U}$. Let

$$X_A^\varepsilon := (\rho_{\mathcal{U}}^\varepsilon)^{-1}(A), \quad g_A^\varepsilon := g_{\mathcal{U}}^\varepsilon|_{X_A^\varepsilon}, \quad f_A^\varepsilon := f_{\mathcal{U}}^\varepsilon|_{X_A^\varepsilon},$$
$$\mathcal{O}_{X_A^\varepsilon}(1) := (f_A^\varepsilon)^* \mathcal{O}_{Y_A}(1), \quad H_A^\varepsilon \in |\mathcal{O}_{X_A^\varepsilon}(1)|.$$

Our notation makes no reference to $\mathcal{U}$ because the isomorphism class of the polarized couple $(X_A^\varepsilon, \mathcal{O}_{X_A^\varepsilon}(1))$ does not depend on the open set $\mathcal{U}$ containing A. Observe that if $A \in \Delta$ then $\mathcal{O}_{X_A^\varepsilon}(1)$ is not ample; in fact, it is trivial on s copies of $\mathbb{P}^2$ for $s = |Y_A[3]|$. Of course,

$$(X_A^\varepsilon, \mathcal{O}_{X_A^\varepsilon}(1)) \cong (X_A, \mathcal{O}_{X_A}(1)) \quad \text{if } A \in \left(\mathbb{LG}\left(\bigwedge^3 V\right) \setminus \Delta\right). \tag{3.2.6}$$

Proposition 3.10. *Let* $A \in \mathbb{LG}\left(\bigwedge^3 V\right)^*$*, and let* ε *be a choice of* $\mathbb{P}^2$*-fibration for* X_A.

(1) *X_A^ε is smooth away from* $(f_A^\varepsilon)^{-1}\big(\bigcup_{W\in\Theta_A}\mathbb{P}(W)\big)$.
(2) *If* $[v_i]\in Y_A[3]$, *then* $(f_A^\varepsilon)^{-1}[v_i]\cong\mathbb{P}(A\cap F_{v_i})^\vee$.
(3) *If ε' is another choice of $\mathbb{P}^2$-fibration for X_A, then there exists a commutative diagram*

$$\begin{tikzcd} X_A^\varepsilon \arrow[rr, dashed] \arrow[dr, "f_A^\varepsilon"'] & & X_A^{\varepsilon'} \arrow[dl, "f_A^{\varepsilon'}"] \\ & Y_A & \end{tikzcd} \tag{3.2.7}$$

in which the birational map is the flop of a collection of $(f_A^\varepsilon)^{-1}[v_i]$'s. Conversely, every flop of a collection of $(f_A^\varepsilon)^{-1}[v_i]$'s is isomorphic to one $X_A^{\varepsilon'}$.

Proof. To prove part (1), note that X_A^ε is smooth away from $(f_A^\varepsilon)^{-1}\big(Y_A[3]\cup\bigcup_{W\in\Theta_A}\mathbb{P}(W)\big)$ by Lemma 3.3. It remains to prove that X_A^ε is smooth at every point of $(f_A^\varepsilon)^{-1}\{[v_1],\dots,[v_s]\}$, where

$$\{[v_1],\dots,[v_s]\}=Y_A[3]\setminus\bigcup_{W\in\Theta_A}\mathbb{P}(W). \tag{3.2.8}$$

Let $\mathcal{U}\subset\mathbb{LG}\big(\bigwedge^3 V\big)^*$ be a small open (in the classical topology) subset containing A. Let $\tilde\rho_{\mathcal{U}}:=\rho_{\mathcal{U}}\circ\pi_{\mathcal{U}}$; thus $\tilde\rho_{\mathcal{U}}\colon\tilde X_{\mathcal{U}}\to\mathcal{U}$. For $1\le i\le s$, the fiber over $(A,[v_i])$ of fibration (3.2.1) is canonically isomorphic to $\mathbb{P}(A\cap F_{v_i})^\vee\times\mathbb{P}(A\cap F_{v_i})^\vee$. Let $\hat X_A\subset\tilde X_{\mathcal{U}}$ be the strict transform of X_A. Abusing notation, we write

$$\tilde\rho_{\mathcal{U}}^{-1}(A)=\hat X_A\cup\bigcup_{i=1}^{s}\mathbb{P}(A\cap F_{v_i})^\vee\times\mathbb{P}(A\cap F_{v_i})^\vee. \tag{3.2.9}$$

(Of course, $\mathbb{P}(A\cap F_{v_i})^\vee\times\mathbb{P}(A\cap F_{v_i})^\vee$ denotes the fiber over $(A,[v_i])$ of fibration (3.2.1). The components $\mathbb{P}(A\cap F_{v_i})^\vee\times\mathbb{P}(A\cap F_{v_i})^\vee$ are pairwise disjoint. We claim that, for $i=1,\dots,s$, the intersection

$$E_{A,i}:=\hat X_A\cap(\mathbb{P}(A\cap F_{v_i})^\vee\times\mathbb{P}(A\cap F_{v_i})^\vee) \tag{3.2.10}$$

is a smooth symmetric divisor in the linear system $|\mathcal{O}_{\mathbb{P}(A\cap F_{v_i})^\vee}(1)\boxtimes\mathcal{O}_{\mathbb{P}(A\cap F_{v_i})^\vee}(1)|$. In order to prove this, refer to (1.3.15) and recall that $\mathcal{V}$ is a 3-dimensional complex vector space. Pull-back by σ defines an isomorphism

$$\mathrm{Sym}^2\mathcal{V}^\vee\xrightarrow{\sigma^*}(\mathcal{V}^\vee\otimes\mathcal{V}^\vee)^{\mathbb{Z}/(2)}=:\mathrm{Sym}_2\mathcal{V}^\vee, \tag{3.2.11}$$

which is $\mathrm{Gl}(\mathcal{V})$-equivariant. Isomorphism σ^* induces a $\mathrm{PGL}(\mathcal{V})$-equivariant isomorphism of projective spaces $\mathbf{p}\colon\mathbb{P}(\mathrm{Sym}^2\mathcal{V}^\vee)\xrightarrow{\sim}\mathbb{P}(\mathrm{Sym}_2\mathcal{V}^\vee)$. Clearly, $\mathbf{p}$ maps a point in the unique open $\mathrm{PGL}(\mathcal{V})$-orbit of $\mathbb{P}(\mathrm{Sym}^2\mathcal{V}^\vee)$ to a point in the unique open $\mathrm{PGL}(\mathcal{V})$-orbit of $\mathbb{P}(\mathrm{Sym}_2\mathcal{V}^\vee)$. Now let $\mathcal{V}=(A\cap F_{v_i})^\vee$. Let $K_i:=(A\cap F_{v_i})$ and let $\tau_{K_i}^{v_i}$ be as in (2.2.11). By Proposition 2.5, $\mathrm{im}(\tau_{K_i}^{v_i})$ belongs to the unique open $\mathrm{PGL}(K_i)$-orbit of $\mathbb{P}(\mathrm{Sym}^2(A\cap F_{v_i}))$. The commutative diagram (1.3.16) then gives that $E_{A,i}$ is a symmetric smooth divisor in $|\mathcal{O}_{\mathbb{P}(A\cap F_{v_i})^\vee}(1)\boxtimes\mathcal{O}_{\mathbb{P}(A\cap F_{v_i})^\vee}(1)|$.

Thus we have described $\tilde{\rho}_{\mathcal{U}}^{-1}(A)$. Since $X_{\mathcal{U}}^{\varepsilon}$ is obtained from $\tilde{X}_{\mathcal{U}}$ by contracting $E_{\mathcal{U}}$ along the $\mathbb{P}^2$-fibration ε, it follows that X_A^{ε} is smooth at every point of $(f_A^{\varepsilon})^{-1}\{[v_1],\dots,[v_s]\}$. This proves (1). And because X_A^{ε} is obtained from $\hat{X}_A$ by contracting each of the divisors $E_{A,i}$ along the fibration $\mathbb{P}^1 \to E_{A,i} \to \mathbb{P}(A \cap F_{v_i})^{\vee}$ determined by ε (and similarly for ε'), we also get parts (2) and (3). □

Corollary 3.11. *Let $A \in \left(\mathbb{LG}\left(\bigwedge^3 V\right) \setminus \Sigma\right)$. Then $g_A^{\varepsilon}\colon X_A^{\varepsilon} \to X_A$ is a desingularization for every choice of $\mathbb{P}^2$-fibration ε for X_A.*

Proof. By Claim 3.7 we know that $A \in \mathbb{LG}\left(\bigwedge^3 V\right)^*$, so Proposition 3.10 applies to X_A^{ε}. Since $A \notin \Sigma$, it follows that X_A^{ε} is smooth by Proposition 3.10(1).

Corollary 3.1.2. *Let $A, A' \in \left(\mathbb{LG}\left(\bigwedge^3 V\right) \setminus \Sigma\right)$, and let $\varepsilon, \varepsilon'$ be choices of $\mathbb{P}^2$-fibration for X_A. Then the quasi-polarized 4-folds $(X_A^{\varepsilon}, H_A^{\varepsilon})$ and $(X_{A'}^{\varepsilon}, H_{A'}^{\varepsilon})$ are deformation equivalent.*

4. Double EPW-Sextics Parameterized by Δ

Let $A \in \Delta$ and $[v_0] \in Y_A(3)$. In Section 4.1 we will associate to $(A, [v_0])$ (under some hypotheses that are certainly satisfied if $A \notin \Sigma$) a K3 surface $S_A(v_0)$ of genus 6, which means that it comes equipped with a big and nef divisor class $D_A(v_0)$ of square 10. We will also prove a converse: given a generic such pseudo-polarized K3 surface S, there exist $A \in \Delta$ and $[v_0] \in Y_A(3)$ such that the pseudo-polarized surfaces S and $S_A(v_0)$ are isomorphic. In Section 4.2 we assume that $A \in (\Delta \setminus \Sigma)$; with this hypothesis, $D_A(v_0)$ is very ample. We will prove that there exists a bimeromorphic map $\psi\colon S_A^{[2]}(v_0) \dashrightarrow X_A^{\varepsilon}$, where ε is an arbitrary choice of $\mathbb{P}^2$-fibration for X_A. That such a map exists for generic $A \in \Delta$ could be proved by invoking the results of [14]. Here we present a direct proof that appeals neither to [14] nor to [12]. Furthermore, we will prove that if $S_A(v_0)$ contains no lines (this will be the case for generic A) then there exists a choice of ε for which ψ is regular—in particular, X_A^{ε} is projective for such ε. We conclude Section 4 by using these results to show that a smooth double cover of an EPW-sextic is a deformation of the Hilbert square of a K3 (and that the family of double EPW-sextics is a locally versal family of projective hyper-Kähler manifolds); the proof is more direct than the corresponding one in [12].

4.1. EPW-Sextics and K3 Surfaces

Assumption 4.1. $A \in \mathbb{LG}\left(\bigwedge^3 V\right)$, $[v_0] \in Y_A(3)$, and the following statements hold.

(a) There exists a codimension-1 subspace $V_0 \subset V$ such that $\bigwedge^3 V_0 \pitchfork A$; that is, $\bigwedge^3 V_0 \cap A = \{0\}$.
(b) There exists at most one $W \in \Theta_A$ containing v_0.
(c) If $W \in \Theta_A$ contains v_0 then $A \cap \left(\bigwedge^2 W \wedge V\right) = \bigwedge^3 W$.

REMARK 4.2. Let $A \in (\Delta \setminus \Sigma)$. Let $[v_0] \in Y_A(3)$ ($= Y_A[3]$ by Claim 3.7). Then Assumption 4.1 holds. In fact, parts (b) and (c) hold trivially and part (a) holds by [15, Claim 2.11, eq. (2.81)].

Let $(A, [v_0])$ be as in Assumption 4.1. We define a surface $S_A(v_0)$ of genus 6. The condition that $\bigwedge^3 V_0$ be transverse to A is open: hence we have the direct sum decomposition

$$V = [v_0] \oplus V_0. \tag{4.1.1}$$

We will denote by $\mathcal{D}$ be the direct sum decomposition of V appearing in (4.1.1). Let

$$K_A^{\mathcal{D}} := \rho_{V_0}^{v_0}(A \cap F_{v_0}), \tag{4.1.2}$$

where $\rho_{V_0}^{v_0}$ is given by (2.2.17). Choose a volume form on V_0. Wedge product followed by the volume form defines an isomorphism $\bigwedge^3 V_0 \cong \bigwedge^2 V_0^\vee$, so it makes sense to let

$$F_A^{\mathcal{D}} := \mathbb{P}(\operatorname{Ann} K_A^{\mathcal{D}}) \cap \operatorname{Gr}(3, V_0). \tag{4.1.3}$$

By Proposition 5.2 and Proposition 5.3 (see the Appendix) we know that $F_A^{\mathcal{D}}$ is a Fano 3-fold with at most one singular point. Next we will define a quadratic form on $\operatorname{Ann} K_A^{\mathcal{D}}$. By Assumption 4.1(a), the subspace A is the graph of a map $\tilde{q}_A^{\mathcal{D}} \colon \bigwedge^2 V_0 \to \bigwedge^3 V_0$; explicitly,

$$\tilde{q}_A^{\mathcal{D}}(\alpha) = \beta \iff (v_0 \wedge \alpha + \beta) \in A. \tag{4.1.4}$$

The map $\tilde{q}_A^{\mathcal{D}}$ is symmetric because A, $\bigwedge^2 V_0$, and $\bigwedge^3 V_0$ are Lagrangian subspaces of $\bigwedge^3 V$. It is clear that $\ker \tilde{q}_A^{\mathcal{D}} = K_A^{\mathcal{D}}$, so $\tilde{q}_A^{\mathcal{D}}$ induces the isomorphism

$$\tilde{r}_A^{\mathcal{D}} \colon \bigwedge^2 V_0 / K_A^{\mathcal{D}} \xrightarrow{\sim} \operatorname{Ann} K_A^{\mathcal{D}} \subset \bigwedge^3 V_0. \tag{4.1.5}$$

The inverse $(\tilde{r}_A^{\mathcal{D}})^{-1}$ defines a nondegenerate quadratic form $(r_A^{\mathcal{D}})^\vee$ on $\operatorname{Ann} K_A^{\mathcal{D}}$.

For future reference, we unwind the definitions of $(\tilde{r}_A^{\mathcal{D}})^{-1}$ and $(r_A^{\mathcal{D}})^\vee$. Let $\beta \in \operatorname{Ann} K_A^{\mathcal{D}}$; that is,

$$v_0 \wedge \alpha + \beta \in A, \quad \alpha \in \textstyle\bigwedge^2 V_0. \tag{4.1.6}$$

Then

$$(\tilde{r}_A^{\mathcal{D}})^{-1}(\beta) \equiv \alpha \pmod{K_A^{\mathcal{D}}}, \qquad (r_A^{\mathcal{D}})^\vee(\beta) = \operatorname{vol}(v_0 \wedge \alpha \wedge \beta). \tag{4.1.7}$$

Let $V((r_A^{\mathcal{D}})^\vee) \subset \mathbb{P}(\operatorname{Ann} K_A^{\mathcal{D}})$ be the 0-scheme of $(r_A^{\mathcal{D}})^\vee$: a smooth 5-dimensional quadric. Let

$$S_A^{\mathcal{D}} := V((r_A^{\mathcal{D}})^\vee) \cap F_A^{\mathcal{D}}. \tag{4.1.8}$$

We need to show that $S_A^{\mathcal{D}}$ does not depend on the choice of the subspace $V_0 \subset V$ that is complementary to $[v_0]$—in other words, that it depends only on A and $[v_0]$. Toward this end we remark that $F_A^{\mathcal{D}}$ is independent of V_0; in fact, $\bigwedge^3 V_0$ is transversal to F_{v_0}. Then, since both $\bigwedge^3 V_0$ and F_{v_0} are Lagrangians, the volume vol induces an isomorphism

$$g_{V_0} \colon \textstyle\bigwedge^3 V_0 \xrightarrow{\sim} F_{v_0}^\vee. \tag{4.1.9}$$

Thus g_{V_0} defines the inclusion

$$F_A^{\mathcal{D}} \hookrightarrow \mathbb{P}(\mathrm{Ann}\, K_A). \tag{4.1.10}$$

REMARK 4.3. The image of map (4.1.10) does not depend on V_0. It depends exclusively on A and $[v_0] \in Y_A(3)$, and we will denote it by $Z_A(v_0)$.

Similarly, g_{V_0} defines the inclusion

$$\mathbf{g}_{V_0}\colon S_A^{\mathcal{D}} \hookrightarrow \mathbb{P}(\mathrm{Ann}\, K_A). \tag{4.1.11}$$

LEMMA 4.4. *Retain our previous notation and assumptions. Then $\mathbf{g}_{V_0}(S_A^{\mathcal{D}})$ is independent of V_0; in other words, it depends exclusively on A and $[v_0] \in Y_A(3)$.*

Proof. Let $V_0' \subset V$ be a codimension-1 subspace that is complementary to $[v_0]$ and transverse to A. Let $\mathcal{D}'$ denote the corresponding direct sum decomposition of V. We must show that

$$\mathbf{g}_{V_0}(S_A^{\mathcal{D}}) = \mathbf{g}_{V_0'}(S_A^{\mathcal{D}'}). \tag{4.1.12}$$

The subspace V_0' is the graph of a linear function,

$$\begin{aligned} V_0 &\to [v_0], \\ v &\mapsto f(v)v_0; \end{aligned} \tag{4.1.13}$$

we thus have the isomorphism

$$\begin{aligned} V_0 &\xrightarrow{\psi} V_0', \\ v &\longmapsto v + f(v)v_0. \end{aligned} \tag{4.1.14}$$

Observe that

$$\textstyle\bigwedge^3 \psi(\beta) = \beta + v_0 \wedge (f \lrcorner \beta), \tag{4.1.15}$$

where $\lrcorner$ denotes contraction. In particular, $g_{V_0'} \circ \bigwedge^3 \psi = g_{V_0}$. Note also that $\phi := \bigwedge^3 \psi|_{\mathrm{Ann}\, K_A^{\mathcal{D}}}$ is an isomorphism between $\mathrm{Ann}\, K_A^{\mathcal{D}} \subset \bigwedge^3 V_0$ and $\mathrm{Ann}\, K_{A'}^{\mathcal{D}'} \subset \bigwedge^3 V_0'$. Hence it suffices to prove that

$$\phi(S_A^{\mathcal{D}}) = S_A^{\mathcal{D}'}. \tag{4.1.16}$$

We claim that

$$\phi^*(r_A^{\mathcal{D}'})^\vee - (r_A^{\mathcal{D}})^\vee \in H^0(\mathcal{I}_{F_A^{\mathcal{D}}}(2)). \tag{4.1.17}$$

If we let $\beta \in \mathrm{Ann}\, K_A^{\mathcal{D}} \subset \bigwedge^3 V_0$ then (4.1.6) holds. Then it follows from (4.1.15) that

$$v_0 \wedge (\alpha - (f \lrcorner \beta)) + \phi(\beta) = v_0 \wedge \alpha + \beta \in A. \tag{4.1.18}$$

By (4.1.15) we have

$$\begin{aligned} \phi^*(r_A^{\mathcal{D}'})^\vee(\beta) &= \mathrm{vol}(v_0 \wedge (\alpha - (f \lrcorner \beta)) \wedge \phi(\beta)) \\ &= \mathrm{vol}(v_0 \wedge \alpha \wedge \phi(\beta)) - \mathrm{vol}(v_0 \wedge (f \lrcorner \beta) \wedge \phi(\beta)) \\ &= \mathrm{vol}(v_0 \wedge \alpha \wedge \beta) - \mathrm{vol}(v_0 \wedge (f \lrcorner \beta) \wedge \beta) \\ &= (r_A^{\mathcal{D}})^\vee(\beta) - \mathrm{vol}(v_0 \wedge (f \lrcorner \beta) \wedge \beta). \end{aligned} \tag{4.1.19}$$

The second term on the last line in (4.1.19) is the restriction to $\mathbb{P}(\mathrm{Ann}\, K_A^{\mathcal{D}})$ of a Plücker quadratic form, so that term vanishes on $F_A^{\mathcal{D}}$. This proves (4.1.17) and hence (4.1.16) holds. □

Lemma 4.4 leads to the following definition.

DEFINITION 4.5. Let $A \in \mathbb{LG}(\bigwedge^3 V)$. Suppose that $[v_0] \in Y_A(3)$ and that Assumption 4.1 holds. Let $\mathcal{D}$ be the direct sum decomposition (4.1.1). Then we set

$$S_A(v_0) := \mathbf{g}_{V_0}(S_A^{\mathcal{D}}). \tag{4.1.20}$$

We single out special points of $S_A(v_0)$ as follows. Suppose that $W \in \Theta_A$ (see (2.2.15) for the definition of Θ_A) and assume that $v_0 \notin W$. Let γ be a generator of $\bigwedge^3 W$ (i.e., γ is decomposable with $\mathrm{supp}(\gamma) = W$). By hypothesis, $\bigwedge^3 V_0 \cap A = \{0\}$ and hence $W \not\subset V_0$; therefore,

$$\gamma = (v_0 + u_1) \wedge u_2 \wedge u_3, \quad u_i \in V_0. \tag{4.1.21}$$

Since $v_0 \notin W$, it follows that $u_1 \wedge u_2 \wedge u_3 \neq 0$ and so $[u_1 \wedge u_2 \wedge u_3] \in F_A^{\mathcal{D}}$. Moreover, $[u_1 \wedge u_2 \wedge u_3] \in V((r_A^{\mathcal{D}})^\vee)$ by (4.1.7) and so $[u_1 \wedge u_2 \wedge u_3] \in S_A^{\mathcal{D}}$. We let

$$\begin{aligned} \Theta_A \setminus \{W \mid v_0 \in W\} &\xrightarrow{\theta_A^{\mathcal{D}}} S_A^{\mathcal{D}} \\ W &\longmapsto [u_1 \wedge u_2 \wedge u_3]. \end{aligned} \tag{4.1.22}$$

The map

$$\theta_A(v_0) := \mathbf{g}_{V_0} \circ \theta_A^{\mathcal{D}} \colon (\Theta_A \setminus \{W \mid v_0 \in W\}) \to S_A(v_0) \tag{4.1.23}$$

is independent of $\mathcal{D}$; in other words, it depends only on A and $[v_0]$. Note that $\theta_A(v_0)$ is injective.

PROPOSITION 4.6. *Let $A \in \mathbb{LG}(\bigwedge^3 V)$. Suppose that $[v_0] \in Y_A(3)$ and that Assumption 4.1 holds. Let $\mathcal{D}$ be the direct sum decomposition* (4.1.1). *Then the set of points at which the intersection $V((r_A^{\mathcal{D}})^\vee) \cap F_A^{\mathcal{D}}$ is not transverse is equal to*

$$\mathrm{im}\, \theta_A^{\mathcal{D}} \coprod (S_A^{\mathcal{D}} \cap \mathrm{sing}\, F_A^{\mathcal{D}}). \tag{4.1.24}$$

Proof. Let $[\beta] \in S_A^{\mathcal{D}}$; in particular, β is nonzero decomposable. Let $U := \mathrm{supp}\, \beta$. Since $[\beta] \in F_A^{\mathcal{D}}$, we have that (4.1.6) holds. Let $\alpha \in \bigwedge^2 V_0$ be as in (4.1.6). We claim that

$$V((r_A^{\mathcal{D}})^\vee) \pitchfork F_A^{\mathcal{D}} \text{ at } [\beta] \quad \text{unless } \langle \alpha, K_A^{\mathcal{D}} \rangle \cap \textstyle\bigwedge^2 U \neq \emptyset. \tag{4.1.25}$$

In fact, the projective tangent space to $\mathrm{Gr}(3, V_0)$ at $[\beta]$ is given by

$$\mathbf{T}_{[\beta]} \mathrm{Gr}(3, V_0) = \mathbb{P}\big(\mathrm{Ann}\big(\textstyle\bigwedge^2 U\big)\big). \tag{4.1.26}$$

On the other hand, (4.1/7) gives that

$$\mathbf{T}_{[\beta]} V((r_A^{\mathcal{D}})^\vee) = \mathbb{P}(\mathrm{Ann}\, \alpha) \cap \mathbb{P}(\mathrm{Ann}\, K_A^{\mathcal{D}}). \tag{4.1.27}$$

Statement (4.1.25) now follows immediately from (4.1.26) and (4.1.27).

Next we prove that

$$\langle \alpha, K_A^{\mathcal{D}} \rangle \cap \bigwedge^2 U \neq \emptyset \iff [\beta] \in \operatorname{sing} F_A^{\mathcal{D}} \ \text{ or } \ [\beta] \in \operatorname{im} \theta_A^{\mathcal{D}}. \tag{4.1.28}$$

Suppose that $[\beta] \in \operatorname{sing} F_A^{\mathcal{D}}$; then Proposition 5.3(1) gives that $K_A^{\mathcal{D}} \cap \bigwedge^2 U \neq \emptyset$. Next suppose that $[\beta] \in \operatorname{im} \theta_A^{\mathcal{D}}$; then $\alpha \in \bigwedge^2 U$ by (4.1.21). This proves the "if" implication of (4.1.28). Let us prove the "only if" implication. First assume that $K_A^{\mathcal{D}} \cap \bigwedge^2 U \neq \{0\}$, and let $0 \neq \kappa_0 \in K_A^{\mathcal{D}} \cap \bigwedge^2 U$. Then κ_0 is decomposable because $\dim U = 3$, whence $[\kappa_0]$ is the unique point belonging to $\mathbb{P}(K_A^{\mathcal{D}}) \cap \operatorname{Gr}(2, V_0)$. By equation (5.8) in the Appendix, $[\beta]$ is the unique singular point of $F_A^{\mathcal{D}}$. Next assume that $K_A^{\mathcal{D}} \cap \bigwedge^2 U = \{0\}$. Then there exists a $\kappa \in K_A^{\mathcal{D}}$ such that $(\alpha + \kappa) \in \bigwedge^2 U$. Since $\kappa \in K_A^{\mathcal{D}}$, we have $(v_0 \wedge (\alpha + \kappa) + \beta) \in A$. The tensor $(v_0 \wedge (\alpha + \kappa) + \beta) \in A$ is decomposable; we use W to denote its support. Then $v_0 \notin W$ because $\beta \neq 0$ and hence $[\beta] = \theta_A^{\mathcal{D}}(W)$. This finishes the proof of (4.1.28) and hence of the proposition. □

Corollary 4.7. *Let $A \in \mathbb{LG}\big(\bigwedge^3 V\big)$. Suppose that $[v_0] \in Y_A(3)$ and that Assumption 4.1 holds. Asssume that Θ_A is finite. Then $S_A(v_0)$ is a reduced and irreducible surface with*

$$\operatorname{sing} S_A(v_0) = \operatorname{im} \theta_A(v_0) \coprod (S_A(v_0) \cap \operatorname{sing} Z_A(v_0)). \tag{4.1.29}$$

(*See Remark 4.3 for the definition of $Z_A(v_0)$.*)

Proof. By Proposition 4.6 we know that $S_A^{\mathcal{D}}$ is a smooth surface beyond the right-hand side of (4.1.29). By hypothesis, Θ_A is finite and hence the right-hand side of (4.1.29) is finite. On the other hand, by Proposition 5.3 we know that $Z_A(v_0)$ is a 3-fold with at most one singular point (which must be an ordinary quadratic singularity) and that $S_A^{\mathcal{D}}$ is the complete intersection of $Z_A(v_0)$ and a quadric hypersurface. It follows that $S_A^{\mathcal{D}}$ is reduced and irreducible with singular set, as claimed. □

Corollary 4.8. *With hypotheses as in Corollary 4.7, suppose that $S_A(v_0)$ has du Val singularities. Let $\hat{S}_A(v_0) \to S_A(v_0)$ be the minimal desingularization. Then $\hat{S}_A(v_0)$ is a K3 surface.*

Proof. Let $\mathcal{O}_{Z_A(v_0)}(1)$ be the pull-back by map (4.1.10) of the hyperplane line bundle on $\mathbb{P}(\operatorname{Ann}(F_{v_0} \cap A))$. Then $S_A(v_0) \in |\mathcal{O}_{Z_A(v_0)}(2)|$. By Proposition 5.2 and Proposition 5.3, there exist smooth divisors in $|\mathcal{O}_{Z_A(v_0)}(2)|$ and they are K3 surfaces; from the simultaneous resolution of du Val singularities it follows that $\hat{S}_A(v_0)$ is a K3 surface. □

Corollary 4.9. *Let $A \in (\Delta \setminus \Sigma)$. Let $[v_0] \in Y_A(3)$, which means* (*by Remark 4.2*) *that Assumption 4.1 holds. Then $S_A(v_0)$ is a* (*smooth*) *K3 surface.*

Proof. This is an immediate consequence of Corollary 4.8. □

Under the hypotheses of Corollary 4.8, let $\mathcal{O}_{S_A(v_0)}(1)$ be the restriction to $S_A(v_0)$ of $\mathcal{O}_{Z_A(v_0)}(1)$. Let $\mathcal{O}_{\hat{S}_A(v_0)}(1)$ be the pull-back of $\mathcal{O}_{S_A(v_0)}(1)$ to $\hat{S}_A(v_0)$. We set

$$D_A(v_0) \in |\mathcal{O}_{S_A(v_0)}(1)|, \qquad \hat{D}_A(v_0) \in |\mathcal{O}_{\hat{S}_A(v_0)}(1)|. \tag{4.1.30}$$

REMARK 4.10. With hypotheses as in Corollary 4.8, we have that $(\hat{S}_A(v_0), \hat{D}_A(v_0))$ is a quasi-polarized K3 surface of genus 6. Furthermore, the composition

$$\hat{S}_A(v_0) \to S_A(v_0) \to \mathbb{P}(\mathrm{Ann}(F_{v_0} \cap A)) \tag{4.1.31}$$

is identified (up to projectivities) with the map associated to the complete linear system $|\hat{D}_A(v_0)|$.

Remark 4.10 has a converse. In order to formulate it, we identify $F_{v_0} \cong \bigwedge^2(V/[v_0])$ (this identification is well-defined up to homothety).

ASSUMPTION 4.11. $K \in \mathrm{Gr}(3, F_{v_0})$ and

(1) $\mathbb{P}(K) \cap \mathrm{Gr}(2, V/[v_0]) = \emptyset$, or
(2) the scheme-theoretic intersection $\mathbb{P}(K) \cap \mathrm{Gr}(2, V/[v_0])$ is a single reduced point.

Let

$$W_K := \mathbb{P}(\mathrm{Ann}\, K) \cap \mathrm{Gr}(3, V/[v_0]). \tag{4.1.32}$$

(This makes sense because we have an isomorphism $\bigwedge^2(V/[v_0]) \xrightarrow{\sim} \bigwedge^3(V/[v_0])^\vee$ that is well-defined up to homothety.) Let

$$S := W_K \cap Q, \quad Q \subset \mathbb{P}(\mathrm{Ann}\, K) \text{ a quadric.} \tag{4.1.33}$$

If Q is generic then S is a linearly normal K3 surface of genus 6 (see Corollary 4.8). In fact, the family of such K3 surfaces is locally versal. Suppose more generally that Assumption 4.11 holds, that S is given by (4.1.33), and that S has du Val singularities. Let $\hat{S} \to S$ be the minimal desingularization, in which case $\hat{S}$ is a K3 surface. Let $D \in |\mathcal{O}_S(1)|$ and let $\hat{D}$ be the pull-back of D to $\hat{S}$. Consider the family $\mathcal{S} \to B$ of deformations of (S, D) obtained by deforming slightly K and Q; by Brieskorn and Tjurina there is a suitable base change $\hat{B} \to B$ such that the pull-back of $\mathcal{S}$ to $\hat{B}$ admits a simultaneous resolution of singularities $\hat{\mathcal{S}} \to \hat{B}$ with fiber $\hat{S}$ over the point corresponding to S. Of course, there is a divisor class $\hat{\mathcal{D}}$ on $\hat{\mathcal{S}}$ whose restriction to $\hat{S}$ is $\hat{D}$; hence $\hat{\mathcal{S}} \to \hat{B}$ is a family of quasi-polarized K3 surfaces. The following result is well known, so we omit the (standard) proof.

PROPOSITION 4.12. *The family $\hat{\mathcal{S}} \to \hat{B}$ is a versal family of quasi-polarized K3 surfaces.*

LEMMA 4.13. *Suppose that Assumption 4.11 holds. Let S be as in* (4.1.33), *and assume that Q is transversal to W_K outside a finite set*; *hence S is a surface with finite singular set. Then there exists a smooth quadric $Q' \subset \mathbb{P}(\mathrm{Ann}\, K)$ such that $S = W_K \cap Q'$.*

Proof. Since W_K is cut out by quadrics, Bertini's theorem gives that the generic quadric in $\mathbb{P}(\mathrm{Ann}\, K)$ containing S is smooth outside $\mathrm{sing}\, S$; let $Q_0 = V(P_0)$ be such a quadric. Let $p \in \mathrm{sing}\, S$. The generic quadric $Q' = V(P') \in |\mathcal{I}_{W_K}(2)|$ is smooth at p and so $V(P_0 + P')$ is smooth at p. Since sing S is finite, it follows that the generic quadric Q containing S is smooth at all points of sing S. Therefore, the generic quadric Q containing S is smooth. $\square$

Our next result gives the inverse of the process that yields $S_A(v_0)$ from $(A, [v_0]) \in \tilde{\Delta}(0)$ (with the extra hypotheses in Assumption 4.1).

PROPOSITION 4.14. *Suppose that Assumption 4.11 holds. Let S be as in* (4.1.33), *and assume that Q is smooth and transversal to W_K outside a finite set. Then there exist $A \in \Delta$, $[v_0] \in \mathbb{P}(V)$, and a codimension-1 subspace $V_0 \subset V$ transversal to $[v_0]$ such that*:

(1) $\bigwedge^3 V_0 \cap A = \{0\}$;
(2) *Assumptions 4.1(c) and 4.1(d) hold*; *and*
(3) *the natural isomorphism $\mathbb{P}\big(\bigwedge^3(V/[v_0])\big) \xrightarrow{\sim} \mathbb{P}\big(\bigwedge^3 V_0\big)$ maps S to $S_A^{\mathcal{D}}$, where $\mathcal{D}$ is the direct sum decomposition of V appearing in* (4.1.1).

If we (a) *replace the quadric Q with a smooth quadric $Q' \subset \mathbb{P}(\operatorname{Ann} K)$ such that $S = W_K \cap Q'$ and* (b) *let $A' \in \Delta$ be the corresponding point, then there exists a projectivity of $\mathbb{P}(V)$ that fixes $[v_0]$ and takes A to A'.*

Proof. Let $Q = V(P)$. The dual of $\operatorname{Ann} K$ is $\bigwedge^2(V/[v_0])/K$, so the polarization of P defines the nondegenerate symmetric map

$$\operatorname{Ann} K \xrightarrow{\sim} \textstyle\bigwedge^2(V/[v_0])/K. \tag{4.1.34}$$

The inverse of this map is the nondegenerate symmetric map

$$\textstyle\bigwedge^2(V/[v_0])/K \xrightarrow{\sim} \operatorname{Ann} K. \tag{4.1.35}$$

Composing on the right with $\bigwedge^2 V_0 \xrightarrow{\sim} \bigwedge^2(V/[v_0])$ and the quotient map $\bigwedge^2(V/[v_0]) \to \bigwedge^2(V/[v_0])/K$ while composing on the left with $\operatorname{Ann} K \hookrightarrow \bigwedge^3(V/[v_0])$ and $\bigwedge^3(V/[v_0]) \xrightarrow{\sim} \bigwedge^3 V_0$, we obtain the symmetric map

$$\textstyle\bigwedge^2 V_0 \to \bigwedge^3 V_0 \tag{4.1.36}$$

with 3-dimensional kernel corresponding to K. The graph of this map is a Lagrangian $A \in \mathbb{LG}\big(\bigwedge^3 V\big)$. One can then easily check that parts (1), (2), and (3) of the proposition hold. Proceeding as in the proof of Lemma 4.4, we can show that the projective equivalence of A does not depend on Q. $\square$

4.2. X_A^{ε} for $A \in (\Delta \setminus \Sigma)$

Let S be a K3 surface, and let $\Delta_S^{[2]} \subset S^{[2]}$ be the irreducible codimension-1 subset parameterizing nonreduced subschemes. Then there exists a square root of the line bundle $\mathcal{O}_{S^{[2]}}(\Delta_S^{[2]})$; we use ξ to denote its first Chern class. There is a natural morphism of integral Hodge structures $\mu\colon H^2(S) \to H^2(S^{[2]})$ such that $H^2(S^{[2]}; \mathbb{Z}) = \mu(H^2(S; \mathbb{Z})) \oplus \mathbb{Z}\xi$; see [1]. Let $(\cdot,\cdot)$ be the Beauville–Bogomolov bilinear symmetric form on $H^2(S^{[2]})$. It is known [1] that

$$(\mu(\eta),\mu(\eta)) = \int_S c_1(\eta)^2, \quad \mu(H^2(S;\mathbb{Z})) \perp \mathbb{Z}\xi, \quad (\xi,\xi) = -2. \tag{4.2.1}$$

Because S and $S^{[2]}$ are regular varieties, we may identify their respective Picard groups with $H^{1,1}_{\mathbb{Z}}(S)$ and $H^{1,1}_{\mathbb{Z}}(S^{[2]})$. Let $C \in \mathrm{Pic}(S)$. Abusing notation, we will denote by $\mu(C)$ the class in $\mathrm{Pic}(S^{[2]})$ corresponding to $\mu(\mathcal{O}_S(C)) \in H^{1,1}_{\mathbb{Z}}(S)$; if C is an integral curve then it is represented by the set of subschemes whose support intersects C. The following theorem is the main result of Section 4.2.

THEOREM 4.15. *Let $A \in (\Delta \setminus \Sigma)$ and $[v_0] \in Y_A[3]$ $(= Y_A(3)$ by Claim 3.7$)$, so $S_A(v_0)$ is a K3 surface by Corollary 4.9. Then the following statements hold.*

(1) *If $S_A(v_0)$ does not contain lines* (*which is true for generic A by Proposition 4.12*), *then there exist a choice ε of $\mathbb{P}^2$-fibration for X_A and an isomorphism*

$$\psi\colon S_A(v_0)^{[2]} \xrightarrow{\sim} X_A^{\varepsilon} \tag{4.2.2}$$

such that

$$\psi^* H_A^{\varepsilon} \sim \mu(D_A(v_0)) - \Delta^{[2]}_{S_A(v_0)}. \tag{4.2.3}$$

(2) *For arbitrary A and ε, there exists a bimeromorphic map*

$$\psi\colon S_A(v_0)^{[2]} \dashrightarrow X_A^{\varepsilon} \tag{4.2.4}$$

such that (4.2.3) *holds.*

REMARK 4.16. Suppose that $S_A(v_0)$ contains a line L. The restriction of the right-hand side of (4.2.3) to $L^{(2)}$ (embedded in $S_A(v_0)^{[2]}$) is $\mathcal{O}_{L^{(2)}}(-1)$. Since H_A^{ε} is numerically effective, in this case map (4.2.4) cannot be regular.

The proof of Theorem 4.15 will be given after a series of auxiliary results. Let $S \subset \mathbb{P}^6$ be a linearly normal K3 surface of genus 6 such that $\mathcal{I}_{S/\mathbb{P}^6}(2)$ is globally generated; then S is projectively normal and hence Riemann–Roch gives that $\dim|\mathcal{I}_S(2)| = 5$. One defines a rational map $S^{[2]} \dashrightarrow |\mathcal{I}_S(2)|^{\vee}$ as follows. Given $[Z] \in S^{[2]}$, we let $\langle Z\rangle \subset \mathbb{P}^5$ be the line spanned by Z. Let

$$\begin{aligned} \Bigl(S^{[2]} \setminus \bigcup_{L\subset S\,\text{line}} L^{(2)}\Bigr) &\xrightarrow{g} |\mathcal{I}_S(2)|^{\vee} \cong \mathbb{P}^5, \\ [Z] &\mapsto \{Q \in |\mathcal{I}_S(2)| \mid \text{s.t. } Q \supset \langle Z\rangle\}. \end{aligned} \tag{4.2.5}$$

For D a hyperplane divisor on S, one can show (see [11, Claim 5.16]) that

$$g^*\mathcal{O}_{\mathbb{P}^5}(1) \cong \mu(D) - \Delta_S^{[2]}. \tag{4.2.6}$$

(Notice that the set of lines on S is finite and hence $\bigcup_{L\subset S\,\text{line}} L^{(2)}$ has codimension 2 in $S^{[2]}$.) In fact, g can be identified with the map associated to the complete linear system $|(\mu(D) - \Delta_S^{[2]})|$. We will analyze g under the assumption that S is generic (in a precise sense).

ASSUMPTION 4.17. Assumption 4.11(1) holds; also,

$$S := W_K \cap Q \tag{4.2.7}$$

for $Q \subset \mathbb{P}(\mathrm{Ann}\, K)$ a quadric intersecting W_K transversely.

Let $S \subset \mathbb{P}(\operatorname{Ann} K)$ be as in Assumption 4.17. Then S is a linearly normal K3 surface of genus 6 and $\mathcal{I}_S(2)$ is globally generated. Thus the map g of (4.2.5) is defined. Let $F(W_K)$ be the variety that parameterizes lines in W_K. Since the set of lines in S is finite (for generic S, that set is empty by Proposition 4.12), we have the map

$$\begin{aligned} (F(W_K) \setminus \{L \mid L \subset S\}) &\to S^{[2]}, \\ L &\mapsto L \cap Q. \end{aligned} \tag{4.2.8}$$

Definition 4.18. Let $P_S^0 \subset S^{[2]}$ be the image of map (4.2.8), and let P_S be its closure in $S^{[2]}$.

We recall that $F(W_K) \cong \mathbb{P}^2$ by Iskovskih [10]; see Proposition 5.2.

Claim 4.19. *Let $S \subset \mathbb{P}(\operatorname{Ann} K)$ be as in Assumption 4.17, and suppose that S contains no lines. Let $C_1, C_2, \dots, C_s$ be the* (*smooth*) *conics contained in S* (*of course, the generic S contains no conics*)*. Then $P_S, C_1^{(2)}, \dots, C_s^{[2]}$ are pairwise disjoint subsets of $S^{[2]}$. Moreover, there exists a biregular morphism*

$$c\colon S^{[2]} \to N(S) \tag{4.2.9}$$

that contracts each of $P_S, C_1^{(2)}, \dots, C_s^{[2]}$. Hence $N(S)$ is a compact complex normal space with

$$\operatorname{sing} N(S) = \{c(P_S), \dots, c(C^{(2)}), \dots \mid C \subset S \text{ is a conic}\} \tag{4.2.10}$$

and c is an isomorphism of the complement of $P_S \cup C_1^{(2)} \cup \dots \cup C_s^{[2]}$ onto the smooth locus of $N(S)$. The map g (*which is regular on all of $S^{[2]}$ because S contains no lines*) *descends to a regular map*

$$\bar{g}\colon N(S) \to |\mathcal{I}_S(2)|^\vee, \quad \bar{g} \circ c = g. \tag{4.2.11}$$

Proof. P_S is isomorphic to $\mathbb{P}^2$ by Proposition 5.2, and each $C_i^{(2)}$ is isomorphic to $\mathbb{P}^2$ because C_i is a conic. Thus each of P_S and the C_i can be contracted individually. Let's show that $P_S, C_1^{(2)}, \dots, C_s^{[2]}$ are pairwise disjoint. Suppose that $[Z] \in P_S \cap C_i^{(2)}$, and let Λ be the plane containing C_i. Then $\Lambda \cap W_K$ contains the line $\langle Z \rangle$ and the smooth conic C_i. Since W_K is cut out by quadrics, it follows that $\Lambda \subset W_K$—which is absurd because W_K contains no planes. This proves that $P_S \cap C_i^{(2)} = \emptyset$. Yet by Corollary 5.5 there does not exist a $[Z] \in C_i^{(2)} \cap C_j^{(2)}$. We have proved that $P_S, C_1^{(2)}, \dots, C_s^{[2]}$ are pairwise disjoint, so the contraction (4.2.9) exists. It remains to prove that g is constant on each of $P_S, C_1^{(2)}, \dots, C_s^{[2]}$. In fact, if $[Z] \in P_S$ then $g([Z]) = |\mathcal{I}_{W_K}(2)|$, and if $[Z] \in C_i^{(2)}$ then

$$g([Z]) = \{Q \in |\mathcal{I}_S(2)| \mid Q \supset \langle C_i \rangle\}.$$

□

Now we return to the "general" case and suppose that Assumption 4.17 holds (although S may very well contain lines). Let

$$S_\star^{[2]} := S^{[2]} \setminus P_S \setminus \bigcup_{R \subset S \text{ line or conic}} \operatorname{Hilb}^2 R. \tag{4.2.12}$$

(If $R \subset S$ is a conic that is not smooth, then we delete all $[Z] \in S^{[2]}$ such that Z is contained in the scheme R.) The following result is essentially [14, Lemma 3.7].

PROPOSITION 4.20. *Suppose that Assumption 4.17 holds.*

(1) *The fibers of $g|_{S_\star^{[2]}}$ are finite of cardinality at most 2, and the generic fiber has cardinality 2.*

(2) *There exist an open dense subset $\mathcal{A} \subset S_\star^{[2]}$ and an anti-symplectic (and hence nontrivial) involution $\phi\colon \mathcal{A} \to \mathcal{A}$ such that*

$$(g|_{\mathcal{A}}) \circ \phi = g|_{\mathcal{A}}; \tag{4.2.13}$$

the induced map

$$\mathcal{A}/\langle\phi\rangle \to g(\mathcal{A}) \tag{4.2.14}$$

is a bijection.

(3) *If, in addition, S does not contain lines, then* (a) *ϕ descends to a regular involution $\bar{\phi}\colon N(S) \to N(S)$ such that $\bar{g} \circ \bar{\phi} = \bar{g}$ and* (b) *the induced map*

$$j\colon N(S)/\langle\bar{\phi}\rangle \to g(S^{[2]}) \tag{4.2.15}$$

is a bijection. Furthermore,

$$\operatorname{cod}(\operatorname{Fix}(\bar{\phi}), N(S)) \geq 2 \tag{4.2.16}$$

for $\operatorname{Fix}(\bar{\phi})$ *the fixed locus of $\bar{\phi}$.*

Let A and $[v_0]$ be as in the statement of Theorem 4.15; we shall perform the key computation needed to prove that theorem. Let $V_0 \subset V$ be a codimension-1 subspace transversal to $[v_0]$ and such that $\bigwedge^3 V_0 \cap A = \{0\}$. Let $\mathcal{D}$ be the decomposition $V = [v_0] \oplus V_0$, and let $S_A^{\mathcal{D}}$ be given by (4.1.8); thus $S_A^{\mathcal{D}}$ sits in $\mathbb{P}(\operatorname{Ann} K_A^{\mathcal{D}}) \cap \operatorname{Gr}(3, V_0)$ and is isomorphic to $S_A(v_0)$. Let $f \in V_0^\vee$. We let q_f be the quadratic form on $\bigwedge^3 V_0$ defined by setting

$$q_f(\omega) := \operatorname{vol}_0((f \lrcorner \omega) \wedge \omega), \tag{4.2.17}$$

where vol_0 is a volume form on V_0. Then q_f is a Plücker quadric; in fact, we have an isomorphism

$$\begin{aligned} V_0^\vee &\xrightarrow{\sim} H^0(\mathcal{I}_{\operatorname{Gr}(3,V_0)}(2)), \\ f &\longmapsto q_f. \end{aligned} \tag{4.2.18}$$

Let $V^\vee = [v_0^\vee] \oplus V_0^\vee$ be the dual decomposition of $\mathcal{D}$; thus $v_0^\vee \in \operatorname{Ann} V_0$ and $v_0^\vee(v_0) = 1$. We then have the isomorphism

$$\begin{aligned} [v_0^\vee] \oplus V_0^\vee &\xrightarrow{\sim} H^0(\mathcal{I}_{S_A^{\mathcal{D}}}(2)), \\ x v_0^\vee + f &\longmapsto x (r_A^{\mathcal{D}})^\vee + q_f. \end{aligned} \tag{4.2.19}$$

Let

$$\iota\colon |\mathcal{I}_{S_A^{\mathcal{D}}}(2)|^\vee \xrightarrow{\sim} \mathbb{P}(V) \tag{4.2.20}$$

be the projectivization of the transpose of (4.2.19).

PROPOSITION 4.21. *Let A and $[v_0]$ be as in the statement of Theorem 4.15. Let g be map* (4.2.5) *for $S_A^{\mathcal{D}}$ (this makes sense by Corollary 4.9). Then $\iota(\operatorname{im} g) \subset Y_A$.*

Proof. Let

$$[Z] \in ((S_A^{\mathcal{D}})_\star^{[2]} \setminus \Delta_{S_A^{\mathcal{D}}}^{[2]} \setminus P_{S_A^{\mathcal{D}}}). \tag{4.2.21}$$

We will show that

$$\iota(g([Z]) \in Y_A; \tag{4.2.22}$$

this will suffice to prove the lemma because the right-hand side of (4.2.21) is dense in $(S_A^{\mathcal{D}})_\star^{[2]}$ and Y_A is closed.

By hypothesis, Z is reduced; hence $Z = \{[\beta], [\beta']\}$, where $\beta, \beta' \in \bigwedge^3 V_0$ are decomposable. The line $\langle [\beta], \beta'] \rangle$ spanned by $[\beta]$ and $[\beta']$ is not contained in $F_A^{\mathcal{D}}$ because $[Z] \notin P_{S_A^{\mathcal{D}}}$. Thus $\langle [\beta], \beta'] \rangle$ is not contained in $\operatorname{Gr}(3, V_0)$, from which it follows that the vector subspaces of V_0 supporting the decomposable vectors β and β' intersect in a 1-dimensional subspace. Hence there exists a basis $\{v_1, \ldots, v_5\}$ of V_0 such that

$$\beta = v_1 \wedge v_2 \wedge v_3, \qquad \beta' = v_1 \wedge v_4 \wedge v_5. \tag{4.2.23}$$

We may also assume that $\operatorname{vol}_0(v_1 \wedge v_2 \wedge v_3 \wedge v_4 \wedge v_5) = 1$. By (4.1.6) and (4.1.7), there exist $\alpha, \alpha' \in \bigwedge^2 V_0$ such that

$$v_0 \wedge \alpha + \beta \in A, \ v_0 \wedge \alpha' + \beta' \in A, \qquad \alpha \wedge \beta = \alpha' \wedge \beta' = 0. \tag{4.2.24}$$

Because A is Lagrangian, we obtain

$$\operatorname{vol}_0(\alpha \wedge \beta') = \operatorname{vol}_0(\alpha' \wedge \beta) =: c. \tag{4.2.25}$$

Let $t_0, \ldots, t_5 \in \mathbb{C}$. Then a straightforward computation gives that

$$\left(t_0 (r_A^{\mathcal{D}})^\vee + \sum_{i=1}^{5} t_i q_{v_i^\vee} \right)(\beta + \beta') = 2ct_0 + 2t_1. \tag{4.2.26}$$

Therefore,

$$\iota(g([Z])) = [cv_0 + v_1]. \tag{4.2.27}$$

It remains to prove that

$$[cv_0 + v_1] \in Y_A. \tag{4.2.28}$$

Let $K_A^{\mathcal{D}}$ be as in (4.1.2); we claim that it suffices to prove the existence of $(x, x') \in (\mathbb{C}^2 \setminus \{(0,0)\})$ and $\kappa \in K_A^{\mathcal{D}}$ such that

$$(cv_0 + v_1) \wedge (x(v_0 \wedge \alpha + \beta) + x'(v_0 \wedge \alpha' + \beta') + v_0 \wedge \kappa) = 0. \tag{4.2.29}$$

So assume that (4.2.29) holds. Then

$$0 \neq (x(v_0 \wedge \alpha + \beta) + x'(v_0 \wedge \alpha' + \beta') + v_0 \wedge \kappa) \in A \cap F_{cv_0 + v_1} \tag{4.2.30}$$

(the inequality holds because β and β' are linearly independent). A straightforward computation now gives that (4.2.29) is equivalent to

$$x(c\beta - v_1 \wedge \alpha) + x'(c\beta' - v_1 \wedge \alpha') = v_1 \wedge \kappa. \tag{4.2.31}$$

As is easily checked, we have

$$(c\beta - v_1 \wedge \alpha), (c\beta' - v_1 \wedge \alpha') \in \Big([v_1] \wedge \big(\textstyle\bigwedge^2 \langle v_2, v_3, v_4, v_5\rangle\big)\Big) \cap \{v_2 \wedge v_3, v_4 \wedge v_5\}^{\perp}, \quad (4.2.32)$$

where perpendicularity is with respect to wedge product followed by vol_0. Multiplication by v_1 gives an injection of $K_A^{\mathcal{D}}$ into the right-hand side of (4.2.32); in fact, no nonzero element of $K_A^{\mathcal{D}}$ is decomposable because $A \notin \Sigma$. Since the right-hand side of (4.2.32) has dimension 4 and since $\dim K_A^{\mathcal{D}} = 3$, it follows that there exists $(x, x') \in (\mathbb{C}^2 \setminus \{(0,0)\})$ such that (4.2.31) holds. □

LEMMA 4.22. *Let $A \in \big(\mathbb{LG}\big(\bigwedge^3 V\big) \setminus \Sigma\big)$. Then $Y_A(1)$ is not empty, the topological double cover $f_A^{-1} Y_A(1) \to Y_A(1)$ is not trivial, and Y_A is integral.*

Proof. By Claim 3.7 we know that $Y_A[3]$ is finite. However, $(Y_A[2] \setminus Y_A[3])$ is a smooth surface by [12, Prop. 2.8]. Since $\operatorname{sing} Y_A \subset Y_A[2]$, it follows that Y_A is integral and that $Y_A(1)$ is connected. Let $[v_0] \in (Y_A[2] \setminus Y_A[3])$. By Proposition 1.5 we know that $f_A^{-1}([v_0])$ is a singleton $\{q\}$; moreover, X_A is smooth at q by Lemma 3.3. Hence there exists an open neighborhood U of $[v_0]$ in Y_A such that $f_A^{-1}U$ is smooth. Furthermore, $(f_A^{-1}Y_A[2]) \cap f_A^{-1}U$ is nowhere dense in $f_A^{-1}U$. Since $f_A^{-1}U$ is smooth, the complement $f_A^{-1}(Y_A(1) \cap U)$ is connected; since $Y_A(1)$ is connected, it follows that $f_A^{-1}Y_A(1)$ is connected. □

PROPOSITION 4.23. *With hypotheses and notation as in Proposition 4.21, we have $\iota(\overline{\operatorname{im} g}) = Y_A$.*

Proof. By Proposition 4.20(1), the map g has finite generic fiber and hence $\dim \overline{\operatorname{im} g} = 4$. By Proposition 4.21, $\iota(\overline{\operatorname{im} g})$ is an irreducible component of Y_A. But since Y_A is irreducible (by Lemma 4.22), it follows that $\iota(\overline{\operatorname{im} g}) = Y_A$. □

REMARK 4.24. With notation as in Proposition 4.21, we have

$$\iota \circ g(P^0_{S_A^{\mathcal{D}}}) = \iota(H^0(\mathcal{I}_{F_A^{\mathcal{D}}}(2))) = [v_0]. \quad (4.2.33)$$

Proof of Theorem 4.15. For part (1), let A and $[v_0]$ be as in the statement of Theorem 4.15. Let $V_0 \subset V$ be a codimension-1 subspace transversal to $[v_0]$ and such that $\bigwedge^3 V_0 \cap A = \{0\}$. Let $\mathcal{D}$ be the decomposition $V = [v_0] \oplus V_0$. In order to simplify notation, we set $S = S_A^{\mathcal{D}}$; thus $S \cong S_A(v_0)$ and, by hypothesis, S does not contain lines. Let j be the map of (4.2.15). Then, by Proposition 4.21, the composition $\iota \circ j$ is a map

$$\iota \circ j \colon N(S)/\langle \bar{\phi} \rangle \to Y_A. \quad (4.2.34)$$

We claim that $\iota \circ j$ is an isomorphism. In fact, it has finite fibers and is birational (by Proposition 4.20). Since $\dim \operatorname{sing} Y_A = 2$ (because $A \notin \Sigma$), the hypersurface Y_A is normal and thus $\iota \circ j$ is an isomorphism. Let $\pi : N(S) \to N(S)/\langle \bar{\phi} \rangle$ be the quotient map. By (4.2.16), the singular locus of $N(S)/\langle \bar{\phi} \rangle$ is the image of $\operatorname{Fix}(\bar{\phi})$ (and so is isomorphic to $\operatorname{Fix}(\bar{\phi})$); since (4.2.34) is an isomorphism, we have that

$$\begin{aligned} N(S) \setminus \mathrm{Fix}(\bar{\phi}) &\to Y_A^{sm}, \\ x &\mapsto \iota \circ j \circ \pi(x) \end{aligned} \tag{4.2.35}$$

is a topological covering of degree 2. We claim that

$$\pi_1(Y_A^{sm}) \cong \mathbb{Z}/(2). \tag{4.2.36}$$

In fact, $(N(S)\setminus\mathrm{Fix}(\bar{\phi})) \cong (S^{[2]}\setminus(P_S \cup \mathrm{Fix}(\phi|_{S^{[2]}\setminus P_S}))$. Since $(P_S \cup \mathrm{Fix}(\phi|_{S^{[2]}\setminus P_S}))$ is of codimension 2 in the simply connected manifold $S^{[2]}$, it follows that $(N(S)\setminus \mathrm{Fix}(\bar{\phi}))$ is simply connected. Thus (4.2.35) is the universal covering of Y_A^{sm} and we obtain (4.2.36). On the other hand, $Y_A^{sm} \subset Y_A(1)$ by [15, Cor. 2.5] and so, by Lemma 4.22, $f_A^{-1}Y_A^{sm} \to Y_A^{sm}$ is the universal covering of Y_A^{sm} as well. Hence both X_A and $N(S)$ are normal completions of the universal cover of Y_A^{sm} such that the extended maps to Y_A are finite; it follows that they are isomorphic (over Y_A). The singular locus of $N(S)$ is given by (4.2.10). Since sing $X_A = Y_A[3]$, by Remark 4.24 we can order the set of (smooth) conics on S (say, $C_1,\ldots,C_s$) and the set of points in $Y_A[3]$ different from $[v_0]$ (say, $[v_1],\ldots,[v_s]$) such that

$$\bar{\psi}(c(P_S)) = [v_0], \qquad \bar{\psi}(c(C_i^{(2)})) = [v_i], \quad 1 \le i \le s \tag{4.2.37}$$

(recall Remark 4.24). Let ε_0 be a choice of $\mathbb{P}^2$-fibration for X_A. Then $\bar{\psi}$ defines a birational map $\psi_0 \colon S^{[2]} \dashrightarrow X_A^{\varepsilon_0}$ such that

$$\psi_0^* H_A^{\varepsilon_0} \cong \mu(D) - \Delta_S^{[2]}, \tag{4.2.38}$$

where D is the hyperplane class of S (thus (S,D) is isomorphic to $(S_A(v_0), D_A(v_0))$). The birational map ψ_0 is an isomorphism away from

$$P_S \cup C_1^{(2)} \cup \cdots \cup C_s^{(2)}. \tag{4.2.39}$$

It follows that ψ_0 is the flop of a collection of irreducible components of (4.2.39). By Proposition 3.10 we get that there exists a choice of $\mathbb{P}^2$-fibration for X_A, call it ε, such that the corresponding birational map $\psi \colon S^{[2]} \dashrightarrow X_A^{\varepsilon}$ is biregular. Equation (4.2.3) then follows from (4.2.38). This completes the proof of Theorem 4.15(1). Part (2) of the theorem follows from part (1) and a specialization argument; we leave the details to the reader. □

We conclude this section by re-proving a previous result. Let $h_A := c_1(\mathcal{O}_{X_A}(H_A))$.

THEOREM 4.25 [12]. *Let $A \in \mathbb{LG}(\bigwedge^3 V)^0$. Then X_A is a deformation of* $(\mathrm{K3})^{[2]}$ *and $(h_A,h_A)_{X_A} = 2$. Any small deformation of (X_A,H_A) (i.e. a small deformation of X_A keeping h_A of type $(1,1)$) is isomorphic to (X_B,H_B) for some $B \in \mathbb{LG}(\bigwedge^3 V)^0$.*

Proof. Let $A_0 \in (\Delta\setminus\Sigma)$ and $[v_0] \in Y_{A_0}[3]$. Suppose that $S_{A_0}(v_0)$ does not contain lines. By Theorem 4.15, there exists a choice ε of $\mathbb{P}^2$-fibration for X_{A_0} yielding the isomorphism

$$\psi \colon S^{[2]} \xrightarrow{\sim} X_{A_0}^{\varepsilon}, \quad \psi^* H_{A_0}^{\varepsilon} \sim \mu(D_A(v_0)) - \Delta_{S_{A_0}(v_0)}^{[2]}. \tag{4.2.40}$$

On the other hand, (X_A, H_A) is a deformation of $(X^{\varepsilon}_{A_0}, H^{\varepsilon}_{A_0})$ by Corollary 3.12; this proves that (X_A, H_A) is a deformation of $(S^{[2]}, (\mu(D_A(v_0)) - \Delta^{[2]}_{S_{A_0}(v_0)}))$. By (4.2.1) we have that $(h_A, h_A)_{X_A} = 2$. Finally, we prove that an arbitrary small deformation of (X_A, H_A) is isomorphic to $(X_{A'}, H_{A'})$ for some $A' \in \mathbb{LG}\big(\bigwedge^3 V\big)^0$. The deformation space of (X_A, H_A) has dimension given by

$$\dim \operatorname{Def}(X_A, H_A) = h^{1,1}(X_A) - 1 = 20. \tag{4.2.41}$$

Yet $\mathbb{LG}\big(\bigwedge^3 V\big)^0$ is contained in the locus of points in $\mathbb{LG}$ that are stable for the natural (linearized) $\mathrm{PGL}(V)$-action (this is proved in [12]). Thus, by varying $A \in \mathbb{LG}\big(\bigwedge^3 V\big)$ we get

$$\dim \mathbb{LG}\big(\textstyle\bigwedge^3 V\big) - \dim \mathrm{SL}(V) = 55 - 35 = 20 \tag{4.2.42}$$

moduli of double EPW-sextics. Because (4.2.41) and (4.2.42) are equal, we may conclude that an arbitrary small deformation of (X_A, H_A) is isomorphic to (X_B, H_B) for some $B \in \mathbb{LG}\big(\bigwedge^3 V\big)^0$. $\square$

5. Appendix: Three-Dimensional Sections of $\mathrm{Gr}(3, \mathbb{C}^5)$

Throughout this section, V_0 is a complex vector space of dimension 5. Choose a volume form vol_0 on V_0; it defines an isomorphism

$$\begin{aligned} \textstyle\bigwedge^2 V_0 &\xrightarrow{\sim} \textstyle\bigwedge^3 V_0^{\vee}, \\ \alpha &\longmapsto \omega \mapsto \mathrm{vol}_0(\alpha \wedge \omega). \end{aligned} \tag{5.1}$$

Let $K \subset \bigwedge^2 V_0$ be a 3-dimensional subspace such that either

$$\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0) = \emptyset \tag{5.2}$$

or else

$$\mathbb{P}(K) \cap \mathrm{Gr}(2, V_0) = \{[\kappa_0]\} = \mathbb{P}(K) \cap T_{[\kappa_0]} \mathrm{Gr}(2, V_0). \tag{5.3}$$

In other words, either $\mathbb{P}(K)$ does not intersect $\mathrm{Gr}(2, V_0)$ or else the scheme-theoretic intersection is a single reduced point. We shall describe

$$W_K := \mathbb{P}(\operatorname{Ann} K) \cap \mathrm{Gr}(3, V_0). \tag{5.4}$$

First recall that the dual of $\mathrm{Gr}(3, V_0)$ is $\mathrm{Gr}(2, V_0)$. More precisely, let $[\alpha] \in \mathbb{P}\big(\bigwedge^2 V_0\big)$; then

$$\operatorname{sing}(\mathbb{P}(\operatorname{Ann} \alpha) \cap \mathrm{Gr}(3, V_0)) = \{U \in \mathrm{Gr}(3, V_0) \mid U \supset \operatorname{supp} \alpha\}. \tag{5.5}$$

In particular, $\mathbb{P}(\operatorname{Ann} \alpha)$ is tangent to $\mathrm{Gr}(3, V_0)$ if and only if $[\alpha] \in \mathrm{Gr}(2, V_0)$ (in which case it is tangent along a $\mathbb{P}^2$). Second, we record the following observation (the proof is an easy exercise).

LEMMA 5.1. *Let $U \subset V_0$ be a codimension-1 subspace, and let $\alpha \in \bigwedge^2 V_0$. Then*

$$\alpha \wedge \big(\textstyle\bigwedge^3 U\big) = 0 \tag{5.6}$$

if and only if $\operatorname{supp} \alpha \subset U$.

We recall the following result of Iskovskih.

PROPOSITION 5.2 [10]. *With notation as before, let $K \subset \bigwedge^2 V_0$ be a 3-dimensional subspace such that* (5.2) *holds. Then*

(1) *W_K is a smooth Fano 3-fold of degree 5 with $\omega_{W_K} \cong \mathcal{O}_{W_K}(-2)$,*
(2) *the Fano variety $F(W_K)$ parameterizing lines on W_K (reduced structure) is isomorphic to $\mathbb{P}^2$, and*
(3) *the projective equivalence class of W_K does not depend on K.*

PROPOSITION 5.3. *Let $K \subset \bigwedge^2 V_0$ be a subvector space of dimension 3 such that* (5.3) *holds. Then W_K is a singular Fano 3-fold of degree 5 with $\omega_{W_K} \cong \mathcal{O}_{W_K}(-2)$ and with one singular point that is ordinary quadratic and belongs to*

$$\{U \in \mathrm{Gr}(3, V_0) \mid U \supset \mathrm{supp}\,\kappa_0\}. \tag{5.7}$$

Proof. If $\kappa \in (K \setminus [\kappa_0])$, then κ is not decomposable and hence $\mathbb{P}(\mathrm{Ann}\,\kappa)$ is transverse to $\mathrm{Gr}(3, V_0)$; hence, by (5.5),

$$\mathrm{sing}\, W_K = \{U \in \mathrm{Gr}(3, V_0) \mid U \supset \mathrm{supp}\,\kappa_0\} \cap \mathbb{P}(\mathrm{Ann}\, K). \tag{5.8}$$

We claim that this intersection consists of one point. First observe that we have a natural identification

$$\{U \in \mathrm{Gr}(3, V_0) \mid U \supset \mathrm{supp}\,\kappa_0\} \cong \mathbb{P}(V_0/\mathrm{supp}\,\kappa_0) \tag{5.9}$$

and a linear map

$$\begin{aligned} K &\xrightarrow{\nu} (V_0/\mathrm{supp}\,\kappa_0)^\vee, \\ \kappa &\mapsto (\bar{v} \mapsto \mathrm{vol}_0(v \wedge \kappa_0 \wedge \kappa)); \end{aligned} \tag{5.10}$$

here $v \in V_0$ and $\bar{v}$ is its class in $V_0/\mathrm{supp}\,\kappa_0$. Given (5.8) and (5.9), we have

$$\mathrm{sing}\, W_K = \mathbb{P}(\mathrm{Ann}\,\mathrm{im}\,\nu). \tag{5.11}$$

Second, it is clear that $\kappa_0 \in \ker \nu$ and so, in order to prove that $\mathrm{sing}\, W_K$ is a singleton, it suffices to prove that $\ker \nu = [\kappa_0]$. If $\kappa \in (K \setminus [\kappa_0])$ then $\kappa_0 \wedge \kappa \neq 0$; in fact, this follows from (5.3) together with the equality

$$\mathbb{P}\Big\{\kappa \in \textstyle\bigwedge^2 V_0 \mid \kappa_0 \wedge \kappa = 0\Big\} = T_{[\kappa_0]}\,\mathrm{Gr}(2, V_0). \tag{5.12}$$

Since $\kappa_0 \wedge \kappa \neq 0$, we have $\nu(\kappa) \neq 0$; this proves that $\mathrm{sing}\, W_K$ consists of a single point. The formula for the dualizing sheaf of W_K follows at once from adjunction.

It remains only to prove that the singular point of W_K is an ordinary quadratic point. Let $\widetilde{W}_K \subset \mathbb{P}(\mathrm{supp}\,\kappa_0) \times \mathbb{P}(V_0/\mathrm{supp}\,\kappa_0) \times W_K$ be the closed subset defined by

$$\widetilde{W}_K := \{([v], U, W) \mid v \in W \subset U\}. \tag{5.13}$$

The projection $\widetilde{W}_K \to \mathbb{P}(V_0/\mathrm{supp}\,\kappa_0)$ is a $\mathbb{P}^1$-fibration and hence $\widetilde{W}_K$ is smooth. One can show that the projection $\pi : \widetilde{W}_K \to W_K$ is the blow-up of $\mathrm{sing}\, W_K$. Moreover, $\pi^{-1}(\mathrm{sing}\, W_K) \cong \mathbb{P}^1 \times \mathbb{P}^1$ and so it follows that the singularity of W_K is ordinary quadratic. □

Our last result is about the base locus of 3-dimensional linear systems of quadrics containing W_K for $K \subset \bigwedge^2 V_0$ a 3-dimensional subspace such that (5.2) holds. We begin by addressing the analogous question for the Grassmannian $\mathrm{Gr}(3, \bigwedge^3 V_0)$. Consider the rational map

$$\mathbb{P}\big(\textstyle\bigwedge^3 V_0\big) \overset{\Phi}{\dashrightarrow} |\mathcal{I}_{\mathrm{Gr}(3,V_0)}(2)|^\vee \cong \mathbb{P}(V_0), \tag{5.14}$$

where the last isomorphism is given by (4.2.18). Let $Z \subset \mathbb{P}\big(\bigwedge^3 V_0\big) \times \mathbb{P}(V_0)$ be the incidence subvariety defined by

$$Z := \{([\omega], [v]) \mid v \wedge \omega = 0\}. \tag{5.15}$$

Then we have a commutative triangle

$$\begin{tikzcd} & Z \arrow[dl, "\Psi"'] \arrow[dr, "\tilde{\Phi}"] & \\ \mathbb{P}\big(\bigwedge^3 V_0\big) \arrow[rr, dashed, "\Phi"] & & \mathbb{P}(V_0), \end{tikzcd} \tag{5.16}$$

where Ψ and $\tilde{\Phi}$ are the restrictions to Z of the two projections of $\mathbb{P}\big(\bigwedge^3 V_0\big) \times \mathbb{P}(V_0)$. Note that Ψ is the blow-up of $\mathrm{Gr}(3, V_0)$. In particular, if $\omega \in \bigwedge^3 V_0$ is not decomposable then there exists a unique $[v] \in \mathbb{P}(V_0)$ such that $v \wedge \omega = 0$ and $\Phi([\omega]) = [v]$. Let $[v] \in \mathbb{P}(V_0)$; by (4.2.18), we may view $\mathrm{Ann}(v) \subset V_0^\vee$ as a hyperplane in $|\mathcal{I}_{\mathrm{Gr}(3,V_0)}(2)|$. Then, by the commutativity of (5.16), we have

$$\bigcap_{f \in \mathrm{Ann}(v)} V(q_f) = \mathrm{Gr}(3, V_0) \cup \big\{[\omega] \in \mathbb{P}\big(\textstyle\bigwedge^3 V_0\big) \mid v \wedge \omega = 0\big\}. \tag{5.17}$$

PROPOSITION 5.4. *Let $K \subset \bigwedge^2 V_0$ be a 3-dimensional subspace such that* (5.2) *holds. Let $L \subset |\mathcal{I}_{W_K}(2)|$ be a hyperplane* (*here $\mathcal{I}_{W_K}$ is the ideal sheaf of W_K in* $\mathbb{P}(\mathrm{Ann}\, K)$). *Then*

$$\bigcap_{t \in L} Q_t = W_K \cup R_L, \tag{5.18}$$

where R_L is a plane. Furthermore, $W_K \cap R_L$ is a conic.

Proof. Restriction to $\mathbb{P}(\mathrm{Ann}\, K)$ defines an isomorphism

$$|\mathcal{I}_{\mathrm{Gr}(3,V_0)}(2)| \xrightarrow{\sim} |\mathcal{I}_{W_K}(2)|. \tag{5.19}$$

By (4.2.18), we may identify L with $\mathbb{P}(\mathrm{Ann}(v))$ for a well-defined $[v] \in \mathbb{P}(V_0)$ and may also identify each quadric Q_t for $t \in L$ with $\mathbb{P}(\mathrm{Ann}\, K) \cap V(q_f)$ for a suitable $[f] \in \mathbb{P}(\mathrm{Ann}(v))$. By (5.17),

$$\bigcap_{f \in \mathrm{Ann}(v)} (\mathbb{P}(\mathrm{Ann}\, K) \cap V(q_f)) = W_K \cup R_L; \tag{5.20}$$

here

$$R_L := \mathbb{P}(\mathrm{Ann}\, K) \cap \big\{[\omega] \in \mathbb{P}\big(\textstyle\bigwedge^3 V_0\big) \mid v \wedge \omega = 0\big\}. \tag{5.21}$$

Thus R_L is a linear space of dimension at least 2. Now observe that we have the isomorphism

$$\begin{aligned} \textstyle\bigwedge^2(V_0/[v]) &\xrightarrow{\sim} \{[\omega] \in \mathbb{P}(\textstyle\bigwedge^3 V_0) \mid v \wedge \omega = 0\}, \\ \bar{\alpha} &\longmapsto v \wedge \alpha, \end{aligned} \tag{5.22}$$

where $\alpha \in \bigwedge^2 V_0$ is an element mapped to $\bar{\alpha}$ by the quotient map $\bigwedge^2 V_0 \to \bigwedge^2(V_0/[v])$. Because $\dim(V_0/[v]) = 4$, the Grassmannian $\mathrm{Gr}(2, V_0/[v])$ is a quadric hypersurface in $\mathbb{P}(\bigwedge^2(V_0/[v]))$; it follows that either $R_L \subset W_K$ or $R_L \cap W_K$ is a quadric hypersurface in R_L. According to Lefschetz, $\mathrm{Pic}(W_K)$ is generated by the hyperplane class; it follows that W_K contains no planes and no quadric surfaces. Hence necessarily $\dim R_L = 2$; moreover, $R_L \not\subset W_K$ and the intersection $R_L \cap W_K$ is a conic. □

COROLLARY 5.5. *Let $K \subset \bigwedge^2 V_0$ be a 3-dimensional subspace such that* (5.2) *holds, and let $\mathcal{C}(W_K)$ be the variety parameterizing conics on W_K* (*reduced structure*)*. Then we have the isomorphism*

$$\begin{aligned} |\mathcal{I}_{W_K}(2)|^\vee &\xrightarrow{\sim} \mathcal{C}(W_K), \\ L &\longmapsto R_L \cap W_K, \end{aligned} \tag{5.23}$$

where R_L is as in Proposition 5.4. Furthermore, given $Z \in W_K^{[2]}$, there exists a unique conic containing Z—namely, $R_L \cap W_K$ for $L \in |\mathcal{I}_{W_K}(2)|^\vee$ the hyperplane of quadrics containing $\langle Z \rangle$.

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