Michigan Math. J. 59 (2010), 169–178

Generically Ordinary Fibrations and a Counterexample to Parshin's Conjecture

Junmyeong Jang

1. Introduction

For a proper smooth surface X of general type over the field of complex numbers $\mathbb{C}$, the Miyaoka–Yau inequality states a relation between two Chern numbers of X:

$$c_1^2(X) \leq 3c_2(X).$$

However, the Miyaoka–Yau inequality does not hold in general over a field of positive characteristic. For example, let us consider $\pi: X \to C$, a generically smooth nonisotrivial semistable fibration of a proper smooth surface to a proper smooth curve over a field of positive characteristic. If both the base genus and the fiber genus are greater than 1, then X is a minimal surface of general type. Let $\pi^{(p^n)}: X^{(p^n)} \to C$ be the base change of π by the n-iterative Frobenius morphism $F^n: C \to C$, and let $\tilde{X}^{(p^n)} \to X^{(p^n)}$ be the minimal desingulization of $X^{(p^n)}$. Then it can be easily checked that, for any $M > 0$, if n is sufficiently large then $\tilde{X}^{(p^n)}$ violates the inequality $c_1^2 \leq Mc_2$ [14, p. 195]. On the other hand, in a letter to D. Zagier, Parshin [13, p. 288] proposed that a version of the Miyaoka–Yau inequality might hold for a surface of general type whose Picard scheme is smooth. In this paper, we will construct a counterexample to this conjecture.

Theorem. *For any $M > 0$, there is a smooth proper surface of general type X over a finite field whose Picard scheme is smooth and $c_1^2(X) > Mc_2(X)$.*

The key step in the construction is the following observation.

Lemma 2.10. *If $\pi: X \to C$ is a generically ordinary semistable fibration, then*

$$\dim H^0(R^1\pi_*\mathcal{O}_X) = \dim H^0(R^1\pi_*^{(p^n)}\mathcal{O}_{X^{p^n}})$$

and

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_{X^{p^n}})$$

for any n.

From these facts and the Rieman–Roch theorem, we easily obtain the following result.

Corollary 2.11. *Under the same condition as in Lemma 2.10, all the Harder–Narasimhan slopes of $R^1\pi_*(\mathcal{O}_X)$ are nonpositive.*

Received September 23, 2008. Revision received May 1, 2009.

In Section 2, we prove Lemma 2.10 and Corollary 2.11 in addition to making other preparations. Corollary 2.11 is the semipositivity theorem [15, p. 3], which is not valid in general over a field of positive characteristic.

In Section 3 we construct a counterexample to Parshin's conjecture. Using Lemma 2.10 reduces the problem to the construction of a generically ordinary nonisotrivial smooth fibration, for which the Picard scheme of the total space is smooth. We will do this by using a reduction argument.

ACKNOWLEDGMENTS. The author is indebted to Prof. M. Kim, without whose support and instruction this work would not have been possible. It is also a pleasure to thank Prof. K. Joshi and Prof. L. Illusie for their helpful advice and encouragement.

2. Preparations

Let k be an algebraically closed field and C a projective curve over k.

DEFINITION 2.1. C is stable (resp. semistable) if:

(1) it is connected and reduced;
(2) all the singular points are normal crossing; and
(3) an irreducible component, which is isomorphic to $\mathbb{P}^1$, meets the other components in at least three (resp. two) points.

For an arbitrary base scheme, we define a (semi)stable curve as follows.

DEFINITION 2.2. A proper flat morphism of relative dimension 1 of schemes $\pi : X \to S$ is a (semi)stable curve if every geometric fiber of π is a (semi)stable curve in the sense of Definition 2.1.

In this paper we are mainly concerned with generically smooth semistable curves to proper smooth curves over a field. If $\pi : X \to C$ is such a semistable fibration over an algebraically closed field k, then X is a proper surface over k and the singular points of X are isolated. A singularity of X is a simple surface singularity of A_n type that is étale locally isomorphic to

$$k[x, y, t]/(t^{n+1} - xy).$$

If $\tilde{X} \to X$ is the minimal blow-up of these singularities, then the composition $\tilde{\pi} : \tilde{X} \to C$ is also a semistable fibration [1, p. 4]. Moreover, $\omega^1_{\tilde{X}/C}$ (the dualizing sheaf for $\tilde{\pi}$) is isomorphic to $\omega^1_{X/C}$ (the pull-back of the dualizing sheaf for π) by the blow-up and $\tilde{\pi}_* \omega^1_{\tilde{X}/C} = \pi_* \omega^1_{X/C}$ [14, p. 171]. Hence for most purposes we may assume that X is a smooth surface over k, and we do so from now on. We also assume, unless stated otherwise, that $\pi : X \to C$ is a generically smooth semistable fibration.

DEFINITION 2.3. A semistable fibration $\pi : X \to C$ is isotrivial if all the special fibers of π are isomorphic.

In particular, an isotrivial fibration $\pi : X \to C$ is a smooth fibration. If π is isotrivial, then there exists a finite étale cover $C' \to C$ such that the base change $\pi_{C'} : X \times_C C' \to C'$ is trivial. In particular, if π is isotrivial then $\deg \pi_* \omega_{X/C} = 0$.

PROPOSITION 2.4 (Szpiro). *If π is a nonisotrivial semistable fibration, then* $\deg \pi_* \omega_{X/C} > 0$. *Equivalently,* $\deg R^1\pi_*\mathcal{O}_X < 0$ [14, p. 173].

Now assume that k is a perfect field of positive characteristic p and that X is a smooth proper variety defined over k. We have the following Frobenius diagram for X/k:

$$\begin{tikzcd} X \arrow[r, "F_{X/k}"] \arrow[dr] & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ & k \arrow[r, "F_k"] & k. \end{tikzcd}$$

Here $F_{X/k}$ is the relative Frobenius morphism of X/k. When $\Omega^{\bullet}_{X/k}$ is the de Rham complex of X/k, $F_{X/k*}(\Omega^{\bullet}_{X/k})$ is an $\mathcal{O}_{X^{(p)}}$-linear complex of coherent $\mathcal{O}_{X^{(p)}}$-modules. The image of $F_{X/k*}\Omega^{i-1}_{X/k} \to F_{X/k*}\Omega^{i}_{X/k}$ is denoted by $B^i\Omega_{X/k}$ or $B^i\Omega$. Each $B^i\Omega$ is a vector bundle on $X^{(p)}$.

DEFINITION 2.5. X is ordinary (Bloch–Kato ordinary) if $H^i(B^j\Omega_{X/k}) = 0$ for all i and j.

There are many equivalent conditions to Bloch–Kato ordinarity [9, p. 209]. If X is a curve or an abelian variety, then X is ordinary if and only if it satisfies the classical definition that the order of p-torsion points of the $\mathrm{Pic}^0_{X/k}$ is maximal or that the Frobenius morphism on $H^1(\mathcal{O}_X)$ is bijective. If all the integral crystalline cohomologies of X, $H^i_{\mathrm{crys}}(X/W)$, are torsion free, then X is ordinary if and only if the Newton polygons of X are equal to the Hodge polygons of X for all degrees. Here W is the ring of Witt vectors of k.

We can extend the definition of ordinarity to any proper smooth morphism of schemes of characteristic p. Assume that $f\colon X \to S$ is a proper and smooth morphism. Let $X^{(p)} = X \times_S (S, F_S)$ and let $F_{X/S}\colon X \to X^{(p)}$ be the relative Frobenius morphism. The image $B^i_{X/S}$ of $F_{X/S*}\Omega^{i-1}_{X/S} \xrightarrow{d} F_{X/S*}\Omega^i_{X/S}$ is a vector bundle on $X^{(p)}$. We define X/S to be ordinary if $R^i f_*(B^j_{X/S}) = 0$ for all i and j.

The notion of ordinarity can also be extended to a proper generically smooth morphism to a Spec of a discrete valuation ring with normal crossing on the special fiber. We recall the definition of ordinarity for such a morphism from [7] and [8]. Let A be a local ring of a smooth curve over k. In particular, A is a discrete valuation ring of positive characteristic. Let $S = \operatorname{Spec} A$ and let $s \in S$ be the closed point.

DEFINITION 2.6. $f\colon X \to S$ is locally semistable if it is isomorphic to

$$\operatorname{Spec} A[x_1, \ldots, x_n]/(x_1 \cdots x_r - t) \to \operatorname{Spec} A$$

étale locally at a relative singular point, where t is a uniformizer of A.

The term "locally semistable" morphism is not conventional; usually such a morphism is simply called "semistable". We have introduced this definition here to avoid a conflict with our former definition of semistable curve. Note that the definition of a semistable curve is a little different from that of a locally semistable morphism. The definition of semistable curve requires that the semistable fibration be proper and relatively minimal, whereas X should be regular in Definition 2.6. Yet for a semistable curve $X \to C$, if X is regular then $X \otimes \mathcal{O}_{C,s} \to \operatorname{Spec} \mathcal{O}_{C,s}$ is locally semistable per Definition 2.6 for each $s \in C$.

Let $X \to S$ be a locally semistable morphism; let $U \subset X$ be the relative smooth locus and $u\colon U \hookrightarrow X$ the inclusion. Then $X \setminus U$ is of codimension at least 2, from which it follows that $\omega^{\cdot}_{X/S} = u_* \Omega^{\cdot}_{U/S}$ is a complex of locally free sheaves on X and that $\omega^i_{X/S} = \bigwedge^i \omega^1_{X/S}$. When X is given as $A[x_1, \dots, x_n]/(x_1 \cdots x_r - t)$ étale locally, $\omega^1_{X/S}$ is the free module of rank $n-1$ generated by

$$dx_1/x_1, \dots, dx_r/x_r, dx_{r+1}, \dots, dx_n$$

with the relation

$$\sum_{i=1}^{r} \frac{dx_i}{x_i} = 0.$$

Here $\omega^1_{X/S}$ is isomorphic to $\Omega^1_{X/S}(\log X_s/s)$, where X_s is the special fiber. Note that $\Omega^1_{X/S}$ is the subsheaf of $\omega^1_{X/S}$ generated by $dx_1, \dots, dx_n$. Also note that the highest wedge product $\omega^{n-1}_{X/S}$ is the relative dualizing sheaf of $f\colon X \to S$.

Let $X^{(p)}$ be the base change of X by the Frobenius morphism of S and let $F_{X/S}\colon X \to X^{(p)}$ be the relative Frobenius morphism. Then $F_{X/S*}\omega^{\cdot}_{X/S}$ is an $\mathcal{O}_{X^{(p)}}$-linear complex. The image and the kernel of the differentials of the complex $F_*\omega^{\cdot}_{X/S}$ are denoted by $B^i\omega_{X/S}$ and $Z^i\omega_{X/S}$ (respectively), and $\mathcal{H}^i\omega_{X/S} = Z^i\omega_{X/S}/B^i\omega_{X/S}$. Observe that $B^i\omega_{X/S}$, $Z^i\omega_{X/S}$, and $\mathcal{H}^i\omega_{X/S}$ are $\mathcal{O}_{X^{(p)}}$-coherent sheaves and are flat over S. The usual Cartier isomorphism

$$\mathrm{C}^{-1}\colon \Omega^i_{U^p/S} \to \mathcal{H}^i F_* \Omega^{\cdot}_{U/S}$$

on the smooth locus extends to an isomorphism

$$\mathrm{C}^{-1}\colon \omega^i_{X^{(p)}/S} \to \mathcal{H}^i F_* \omega^{\cdot}_{X/S},$$

[7, p. 381], where $\omega^i_{X^{(p)}/S} = F^*_C(\omega^i_{X/S})$. In particular, the Cartier isomorphism at $i = 0$ gives an exact sequence

$$0 \to \mathcal{O}_{X^{(p)}} \to F_*\mathcal{O}_X \to B^1\omega_{X/S} \to 0.$$

DEFINITION 2.7. A proper locally semistable morphism $f\colon X \to S$ is ordinary if $H^j(B^i\omega_{X/S}) = 0$ for all i, j.

Since the $B^i\omega_{X/S}$ are flat over S, it follows that f is ordinary if and only if $H^j(X_s, B^i\omega_{X/S}|_{X_s}) = 0$ for all i, j. This definition depends on the entire $f\colon X \to S$, and not only on the special fiber, because the complex $\omega^{\cdot}_{X/S}$ depends on the entire $f\colon X \to S$. But if f is smooth, then f is ordinary if and only if the special fiber is ordinary in the sense of Definition 2.5. Moreover, if the relative dimension of f is 1 and the residue field is perfect, then f is ordinary if and only if the Frobenius morphism on $H^1(\mathcal{O}_{X_s})$ is bijective. Hence the ordinarity depends only on the special fiber.

Because all these arguments are local on the base S, they remain valid when we replace the base S by a smooth curve over a perfect field of positive characteristic.

Let C be a smooth curve over a perfect field k of positive characteristic, and let $\pi : X \to C$ be a proper generically smooth semistable curve. Since each $B^i\omega_{X/C}$ is flat over C by the semicontinuity theorem, it follows that the set of points $s \in C$ satisfying the property that $X \otimes_{\mathcal{O}_C} \mathcal{O}_s$ is ordinary forms an open set in C.

DEFINITION 2.8. Let $\pi : X \to C$ be a proper semistable curve. We say that π is generically ordinary if at least one closed fiber of π is ordinary. (Hence almost all closed fibers of π are ordinary.)

Now we recall the (semi)stability and the Harder–Narasimhan slopes of a vector bundle on a smooth proper curve. Let C be a smooth proper curve defined over an algebraically closed field k. For a vector bundle V on C, the slope of V is defined as $s(V) = \deg V/\operatorname{rank} V$. We call V semistable (resp. stable) if, for any proper subbundle W of V, $s(W) \leq s(V)$ (resp. $s(W) < s(V)$).

PROPOSITION 2.9 (Harder–Narasimhan). *For any vector bundle V on C, there exists a unique filtration of V consisting of subbundles of V,*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V,$$

such that (i) V_i/V_{i-1} *is a semistable vector bundle of slope* λ_i *and* (ii) $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ [4, p. 220].

This filtration is called the Harder–Narasimhan filtration of V, and $\lambda_1, \ldots, \lambda_n$ are called the Harder–Narasimhan slopes of V. If the base field is not algebraically closed, then the Harder–Narasimhan slopes of V are defined as the Harder–Narasimhan slopes of pull-backs of the bundle along the base change to an algebraically closed field. When $\pi : X \to C$ is a semistable fibration of a proper smooth surface to a proper smooth curve over a subfield of $\mathbb{C}$, the semi-positivity theorem states that all the Harder–Narasimhan slopes of $R^1\pi_*\mathcal{O}_X$ are nonpositive [15, p. 1].

LEMMA 2.10. *If $\pi : X \to C$ is a generically ordinary semistable fibration, then*

$$\dim H^0(R^1\pi_*\mathcal{O}_X) = \dim H^0(R^1\pi_*^{(p^n)}\mathcal{O}_{X^{(p^n)}})$$

and

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_{X^{(p^n)}})$$

for any n.

Proof. Let $X^{(p)}$ be the base change of X by the absolute Frobenius morphism of C. We have the following Frobenius diagram for X/C:

$$\begin{tikzcd} X \arrow[r, "F_{X/C}"] \arrow[dr, "\pi"'] & X^{(p)} \arrow[r] \arrow[d, "\pi^{(p)}"] & X \arrow[d, "\pi"] \\ & C \arrow[r, "F_C"] & C. \end{tikzcd}$$

There is an exact sequence of coherent $\mathcal{O}_{X^{(p)}}$-modules

$$0 \to \mathcal{O}_{X^{(p)}} \to F_{X/C*}\mathcal{O}_X \to B^1\omega \to 0.$$

For this sequence, the long exact sequence via $\pi_*^{(p)}$ is

$$0 \to \mathcal{O}_C \simeq \mathcal{O}_C \to \pi_*^{(p)}(B^1\omega) \to R^1\pi_*^{(p)}(\mathcal{O}_{X^{(p)}}) \xrightarrow{F^*_{X/C}} R^1\pi_*(\mathcal{O}_X) \to R^1\pi_*^{(p)}(B^1\omega) \to 0. \quad (*)$$

Because π is generically ordinary, the restriction of $\pi_*^{(p)}B^1\omega$ to the ordinary locus in C is 0. But $B^1\omega$ is flat over $\mathcal{O}_C$, so $\pi_*^{(p)}B^1\omega = 0$. In $(*)$, then, $F^*_{X/C}$ is injective and

$$\dim H^0(R^1\pi_*^{(p)}(\mathcal{O}_{X^{(p)}})) \le \dim H^0(R^1\pi_*(\mathcal{O}_X)).$$

On the other hand, since

$$H^0(R^1\pi_*^{(p)}(\mathcal{O}_{X^{(p)}})) = H^0(F_C^*R^1\pi_*\mathcal{O}_X) = H^0(R^1\pi_*(O_X) \otimes_{\mathcal{O}_C} F_{C*}(\mathcal{O}_C))$$

and since there is an injection $R^1\pi_*\mathcal{O}_X \hookrightarrow R^1\pi_*\mathcal{O}_X \otimes F_{C*}(\mathcal{O}_C)$, it follows that

$$\dim H^0(R^1\pi_*\mathcal{O}_X) \le \dim H^0(F_C^*R^1\pi_*\mathcal{O}_X).$$

Therefore,

$$\dim H^0(R^1\pi_*\mathcal{O}_X) = \dim H^0(F_C^*R^1\pi_*\mathcal{O}_X).$$

Because

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_C) + \dim H^0(R^1\pi_*\mathcal{O}_X)$$

and

$$\dim H^1(\mathcal{O}_{X^{(p)}}) = \dim H^1(\mathcal{O}_C) + \dim H^0(R^1\pi_*^{(p)}\mathcal{O}_{X^{(p)}}),$$

we have

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_{X^{(p)}}).$$

We can apply this argument to the relative Frobenius morphism

$$F_{X^{(p^i)}/C} \colon X^{(p^i)} \to X^{(p^{i+1})}$$

for any i, since $F_{X^{(p^i)}/C} \colon X^{(p^i)} \to X^{(p^{i+1})}$ is the base change of the relative Frobenius morphism $F_{X/C} \colon X \to X^{(p)}$ by $F_C^i \colon C \to C$. Then by induction we have

$$\dim H^0(R^1\pi_*\mathcal{O}_X) = \dim H^0(F_C^{n*}R^1\pi_*\mathcal{O}_X)$$

and

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_{X^{(p^n)}})$$

for any n. □

COROLLARY 2.11. *Under the same condition as in Lemma 2.10, all the Harder–Narasimhan slopes of $R^1\pi_*(\mathcal{O}_X)$ are nonpositive.*

Proof. Assume that V is a subbundle of $R^1\pi_*\mathcal{O}_X$ with a positive degree $d > 0$ and rank r. Then $F_C^{n*}V$ is a subbundle of $F_C^{n*}R^1\pi_*\mathcal{O}_X$ with a positive degree $p^n d$ and rank r. By the Riemann–Roch theorem on C, $\dim H^0(F_C^{n*}V)$ diverges as n approaches infinity. But this contradicts the fact that $\dim H^0(F_C^{n*}R^1\pi_*\mathcal{O}_X)$ is stable (by Lemma 2.10). Hence $R^1\pi_*\mathcal{O}_X$ does not have a subbundle of positive degree, so all the Harder–Narasimhan slopes of $R^1\pi_*\mathcal{O}_X$ are nonpositive. □

REMARK 2.12. In [12], Moret-Bailly constructed a counterexample of the semi-positivity theorem consisting of a theta-divisor of a nonisotrivial supersingular abelian surface over $\mathbb{P}^1$. Note that for this example the p-rank of the generic fiber is 0. In [10] we will show that if the generic p-rank of a nonisotrivial semistable fibration is 0, then a suitable number of Frobenius pull-backs of the fibration violate the semipositivity theorem. It is an interesting phenomenon that the information of the p-rank, which is a property of a Galois action, can be expressed as a numerical property of a coherent sheaf.

In the proof of Corollary 2.11 we actually showed that, if π is generically ordinary, then $F_C^{n*}R^1\pi_*\mathcal{O}_X$ does not have a positive Harder–Narasimhan slope for any $n \in \mathbb{N}$. Hence the slope-0 part of $R^1\pi_*\mathcal{O}_X$ is strongly semistable. In fact, we can say more about the slope-0 part of $R^1\pi_*\mathcal{O}_X$. We use the following notation for convenience.

DEFINITION 2.13. For a vector bundle V on C, V_0 is the slope-0 part of V and V_- is the negative-slope part of V.

When π is generically ordinary, we have the following canonical filtration of $R^1\pi_*\mathcal{O}_X$:

$$0 \to (R^1\pi_*\mathcal{O}_X)_0 \to R^1\pi_*\mathcal{O}_X \to (R^1\pi_*\mathcal{O}_X)_- \to 0.$$

DEFINITION 2.14. A vector bundle V on C is étale trivializable if there exists a finite étale cover $f : D \to C$ such that f^*V is trivial.

PROPOSITION 2.15. *If $\pi : X \to C$ is generically ordinary, then $(R^1\pi_*\mathcal{O}_X)_0$ is étale trivializable.*

Proof. In the proof of Theorem 1, we saw that

$$F_C^* R^1\pi_*\mathcal{O}_X \hookrightarrow R^1\pi_*\mathcal{O}_X.$$

Yet in the exact sequence

$$0 \to (R^1\pi_*\mathcal{O}_X)_0 \to R^1\pi_*\mathcal{O}_X \to (R^1\pi_*\mathcal{O}_X)_- \to 0,$$

$(R^1\pi_*\mathcal{O}_X)_-$ is an iterative extension of semistable vector bundles of negative slopes. Therefore, the image of the composition

$$F^*(R^1\pi_*\mathcal{O}_X)_0 \hookrightarrow F^*R^1\pi_*\mathcal{O}_X \hookrightarrow R^1\pi_*\mathcal{O}_X$$

is contained in $(R^1\pi_*\mathcal{O}_X)_0$. Because $(R^1\pi_*\mathcal{O}_X)_0$ and $F^*(R^1\pi_*\mathcal{O}_X)_0$ are of the same rank and the same degree, they are isomorphic. Hence, by [11, Thm. 1.4, p. 75], $(R^1\pi_*\mathcal{O}_X)_0$ is étale trivializable. $\square$

REMARK 2.16. It is natural to expect that if $\pi : X \to C$ is defined over a field of characteristic 0 then $(R^1\pi_*\mathcal{O}_X)_0$ is étale trivializable. But for this problem we cannot apply the standard reduction argument directly. One reason is that we don't know whether there are infinitely many places at which the reduction is generically ordinary. This obstruction is related to Serre's ordinary reduction conjecture.

3. Counterexample to Parshin's Conjecture

In this section we will construct a counterexample to Parshin's conjecture. Let us recall the construction in [14] of a counterexample to the Miyaoka–Yau inequality over a field of positive characteristic. Let k be a perfect field of positive characteristic. Let $\pi : X \to C$ be a smooth nonisotrivial fibration of a proper smooth surface to a proper smooth curve over k of fiber genus $g \geq 2$ and of base genus $q \geq 2$. Also set $d = -\deg R^1\pi_*\mathcal{O}_X > 0$. Then

$$c_1^2(X) = 12d + 8(q-1)(g-1) \quad \text{and} \quad c_2(X) = 4(q-1)(g-1).$$

When $\pi^{(p^n)} \colon X^{(p^n)} \to C$ is the base change of π by the n-iterative Frobenius morphism of C, we have $F_C^n \colon C \to C$, $\deg R^1\pi_*^{(p^n)}\mathcal{O}_{X^{(p^n)}} = -p^n d$,

$$c_1^2(X^{(p^n)}) = 12dp^n + 8(q-1)(g-1), \quad \text{and} \quad c_2(X) = 4(q-1)(g-1).$$

For any $M > 0$, if n is sufficiently large then

$$c_1^2(X^{(p^n)}) > Mc_2(X^{(p^n)}).$$

LEMMA 3.1. *Suppose that X is a smooth proper surface over k admitting a smooth fibration $\pi : X \to C$ to a smooth proper curve C over k. If π is generically ordinary and if* Pic X *is smooth, then* Pic $X^{(p^n)}$ *is smooth for any $n \in \mathbb{N}$ when $X^{(p^n)} \to C$ is the base change of $X \to C$ by the n-iterative Frobenius morphism $F_C^n \colon C \to C$.*

Proof. Recall the Frobenius diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/C}"] \arrow[dr] & X^{(p)} \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ & C \arrow[r, "F_C"] & C. \end{tikzcd}$$

Here $\alpha \circ F_{X/C}$ is the absolute Frobenius morphism of X and $F_{X/C} \circ \alpha$ is the absolute Frobenius morphism of $X^{(p)}$. Because π is smooth, $X^{(p)}$ is smooth over k. For a smooth projective variety, the Frobenius morphism induces a bijective semilinear morphism on the rational crystalline cohomologies. Therefore,

$$\dim H^i_{\text{crys}}(X/K) = \dim H^i_{\text{crys}}(X^{(p)}/K),$$

where K is the fraction field of the ring of Witt vectors $W = W(k)$ and $H^i_{\text{crys}}(X/K) = H^i_{\text{crys}}(X/W) \otimes K$. In particular,

$$\dim H^1_{\text{crys}}(X/K) = \dim H^1_{\text{crys}}(X^{(p)}/K).$$

On the other hand, the K-dimension of the crystalline cohomology $H^i_{\text{crys}}(X/K)$ is equal to the $\mathbb{Q}_l$-dimension of the l-adic étale cohomology $H^i_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$, where l is a prime number different from the characteristic of k and $\bar{X} = X \times_k \bar{k}$. The dimension of $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ is twice the dimension of Pic X, so

$$\dim \operatorname{Pic} X = \dim \operatorname{Pic} X^{(p)}.$$

$\operatorname{Pic} X$ is smooth if and only if its dimension is equal to the k-dimension of $H^1(\mathcal{O}_X)$. Since π is generically ordinary, by Lemma 2.10 we have

$$\dim H^1(\mathcal{O}_X) = \dim H^1(\mathcal{O}_{X^{(p)}}).$$

Hence, if $\operatorname{Pic} X$ is smooth then $\operatorname{Pic} X^{(p)}$ is smooth. By the same argument, $\operatorname{Pic} X^{(p^n)}$ is smooth for any n. $\square$

THEOREM. *For any $M > 0$, there is a smooth proper surface of general type X over a finite field whose Picard scheme is smooth and $c_1^2(X) > Mc_2(X)$.*

Proof. By Lemma 3.1, it is enough to give a nonisotrivial, generically ordinary smooth fibration $X \to C$ such that $\operatorname{Pic}(X)$ is smooth. We will construct such an example by using a reduction argument.

Let F_m be the Fermat curve $x^m + y^m + z^m = 0$ over $\mathbb{C}$ with $m > 3$. Denote the genus of F_m by g, and note that $g \geq 3$. Let $\mathcal{M}_g$ be the moduli space of smooth proper curves of genus g over $\mathbb{C}$. By [5, p. 105] there exists a smooth proper curve C_0 in $\mathcal{M}_g$ passing through the point representing F_m. Then there is a finite cover $C \to C_0$ and a nonisotrivial smooth fibration $\pi : X \to C$ that induces the composition $C \to C_0 \hookrightarrow \mathcal{M}_g$. Let us choose $s \in C$ such that $X_s = X \times_C k(s) = F_m$. We can take an integral model of π with the section s over a Noetherian domain of finite type over $\mathbb{Z}$. Explicitly, we can take A, an integral domain of finite type over $\mathbb{Z}$, and a smooth fibration $\pi_A : X_A \to C_A$ over $\operatorname{Spec} A$ that satisfy the following conditions:

(1) X_A and C_A are smooth and proper over $\operatorname{Spec} A$;
(2) there is a geometric generic point of $\eta : \mathbb{C} \to \operatorname{Spec} A$ such that $\pi_A \times_A \eta$ is isomorphic to $\pi : X \to C$;
(3) there exists a section $S : \operatorname{Spec} A \to C_A$ such that $S \times_A \eta$ corresponds to s with respect to the isomorphism in (2);
(4) $S \times_{C_A} X_A$ is isomorphic to the Fermat curve over $\operatorname{Spec} A$.

Because $\operatorname{Spec} A$ is a scheme of finite type over $\mathbb{Z}$, there is a rational point of $\operatorname{Spec} A$ over a number field F. Given the coordinates of this rational point, there is a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ for which B is a localization of $\mathcal{O}_F$, the ring of integers of F. By the change of bases, we obtain a smooth fibration $\pi_B : X_B \to C_B$ over $\operatorname{Spec} B$. Then, for a place $v \in \operatorname{Spec} B$, the fiber of $\pi_v = \pi_B \times k_v$ over S_v is the Fermat curve over the residue field of v. The ordinarity of the Fermat curve over a finite field depends only on the characteristic of the field. To be precise, it is ordinary if and only if $p \equiv 1$ modulo m, where p is the characteristic of the finite field [16]. Therefore, at infinitely many places of $\operatorname{Spec} B$, the reduction of π is generically ordinary. Because $\operatorname{Pic} X_v$ is smooth for almost all $v \in \operatorname{Spec} B$, there is a place $v \in \operatorname{Spec} B$ such that π_v is generically ordinary and the Picard scheme of X_v is smooth. $\square$

REMARK 3.2. In the preceding example, while $\dim H^1(\mathcal{O}_{X^{(p^n)}})$ is stable as n is increasing, $\dim H^0(\Omega^1_{X^{(p^n)}})$ is strictly increasing [2, p. 94]. By the way, for an arbitrary proper smooth surface, the inequality

$$c_1^2 \leq 5c_2 + 6\beta_1 + 6(2h^{1,0} - \beta_1)$$

holds for $\beta_1 = \dim_{\mathbb{Q}_l} H^1_{\text{ét}}(X \times_k \bar{k}, \mathbb{Q}_l)$. In view of this inequality, it seems that the Miyaoka–Yau inequality is related to the correctness of $h^{1,0}$ and not to the correctness of $h^{0,1}$. The example we have constructed shows that the "correct" value of $h^{0,1}$ does not guarantee the Miyaoka–Yau inequality.

References

[1] M. Deschamps, *Réduction semi-stable,* Astérisque 86 (1981), 1–34.

[2] R. Fossum, *Formes différentielles non Fermées,* Astérisque 86 (1981), 90–96.

[3] T. Fujita, *On Kähler fiber spaces over curves,* J. Math. Soc. Japan 30 (1978), 779–794.

[4] G. Harder and M. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves,* Math. Ann. 212 (1975), 215–248.

[5] J. Harris, *Curves and their moduli,* Algebraic geometry (Bowdoin, 1985), Proc. Sympos. Pure Math., 46, part 1, pp. 99–143, Amer. Math. Soc., Providence, RI, 1987.

[6] L. Illusie, *Complexe de de Rham–Witt et cohomologie cristalline,* Ann. Sci. École Norm. Sup. (4) 12 (1979), 501–661.

[7] ———, *Ordinarité des intersections complètes générales,* The Grothendieck Festschrift, vol. II, Progr. Math., 87, pp. 376–405, Birkhäuser, Boston, 1990.

[8] ———, *Réduction semi-stable et décomposition de complexes de de Rham à coefficients,* Duke Math. J. 60 (1990), 139–185.

[9] L. Illusie and M. Raynaud, *Les suites spectrales associées au complexe de de Rham–Witt,* Inst. Hautes Études Sci. Publ. Math. 57 (1983), 73–212.

[10] J. Jang, *Semi-stable fibrations of generic p-rank 0,* Math. Z. (to appear).

[11] H. Langer and U. Stuhler, *Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe,* Math. Z. 156 (1977), 73–83.

[12] L. Moret-Bailly, *Familles de courbes et de variétés abéliennes sur* $\mathbb{P}^1$, Astérisque 86 (1981), 125–140.

[13] A. Parshin, *A letter to Don Zagier,* Progr. Math., 89, pp. 285–292, Birkhäuser, Boston, 1991.

[14] L. Szpiro, *Sur le théorème de rigidité de Parsin et Arakelov,* Astérisque 64 (1979), 169–202.

[15] G. Xiao, *Surfaces fibrées en courbes de genre deux,* Lecture Notes in Math., 1137, Springer-Verlag, Berlin, 1980.

[16] N. Yui, *On the Jacobian variety of the Fermat curve,* J. Algebra 65 (1980), 1–35.

School of Mathematics
Korea Institute for Advanced Study
Hoegiro 87, Dongdaemun-gu
Seoul 130-722
Korea

jjm@kias.re.kr