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Isometric Rigidity in Codimension 2

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1. Introduction

In the local theory of submanifolds, a fundamental but difficult problem is to describe the isometrically deformable isometric immersions $f : M^n \to \mathbb{R}^{n+p}$ into Euclidean space with low codimension p as compared with the dimension $n \geq 3$ of the Riemannian manifold. Moreover, one would like to understand the set of all possible isometric deformations.

Submanifolds in low codimension are generically rigid because the fundamental Gauss–Codazzi–Ricci system of equations is overdetermined. By "rigid" we mean that there are no other isometric immersions up to rigid motion of the ambient space. As a consequence, it is much easier to employ a generic assumption that implies rigidity than to describe the submanifolds that are isometrically deformable. For instance, the results in [1] and [5] establish rigidity *provided* the second fundamental is sufficiently "complicated".

The result stated by Beez [3] in 1876, but not correctly proved (by Killing [13]) until 1885, states that any deformable hypersurfaces without flat points has two nonzero principal curvatures (rank 2) at any point. For dimension 3, the deformation problem for hypersurfaces was first considered by Schur [16] as early as 1886 and then completely solved in 1905 by Bianchi [4]. The general case was solved by Sbrana [15] in 1909 and by Cartan [6] in 1916; see [10] for additional information. From their results, we have that even hypersurfaces of rank 2 are generically rigid.

Outside the hypersurfaces case, the deformation question remains essentially unanswered to this day even for low codimension $p = 2$. According to [5] or [9], any submanifold $f : M^n \to \mathbb{R}^{n+2}$ is rigid if (a) at any point the index of relative nullity satisfies $\nu_f \geq n - 5$ and (b) any shape operator has at least three nonzero principal curvatures. If only the relative nullity condition holds then we know from [12] that f is genuinely rigid. This means that, given any other isometric immersion $\tilde{f} : M^n \to \mathbb{R}^{n+2}$, there is an open dense subset of M^n such that, when restricted to any connected component, either $f|_U$ and $\hat{f}|_U$ are congruent or there exist an isometric embedding $j : U \hookrightarrow N^{n+1}$ into a Riemannian manifold N^{n+1} and either flat or isometric Sbrana–Cartan hypersurfaces $F, \hat{F} : N^{n+1} \to \mathbb{R}^{n+2}$ such that $f|_U = F \circ j$ and $\hat{f}|_U = \hat{F} \circ j$.

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One may expect the nowhere flat submanifolds $f\colon M^n \to \mathbb{R}^{n+2}$ whose second fundamental form is "as simple as possible" (i.e., with constant relative nullity index $\nu_f = n-2$) to be easily deformable. Yet we believe that, as part of a general trend, submanifolds in this class are generically rigid—as occurs in the hypersurface situation. In [8] these rank-2 submanifolds were divided into three classes: elliptic, parabolic and hyperbolic. It was shown there that the elliptic and the nonruled parabolic submanifolds are genuinely rigid; the ruled parabolic ones admit isometric immersions as hypersurfaces and have many isometric deformations. Examples of hyperbolic submanifolds that are not genuinely rigid were discussed in [8], but there are no general results for this class yet.

In this paper, we generalize [8] by showing that the nonruled parabolic submanifolds are not only genuinely rigid but are also isometrically rigid. In particular, this unexpected result provides the first known examples of locally rigid submanifolds that are of rank 2 and codimension 2. Observe that, by our result, no nonruled parabolic submanifold can be locally realized as a hypersurface of a Sbrana–Cartan hypersurface. This is certainly not the case for the elliptic submanifolds. The rest of the paper is devoted to a local parametric classification of all parabolic submanifolds.

2. Preliminaries

Let $f\colon M^n \to \mathbb{R}^{n+2}$ denote an isometric immersion of codimension 2 into Euclidean space of a Riemannian manifold of dimension $n \geq 3$. We denote its second fundamental form with values in the normal bundle by

$$\alpha_f\colon TM \times TM \to T_f^{\perp}M.$$

The shape operator $A_\xi^f\colon TM \to TM$ for any $\xi \in T_f^{\perp}M$ is defined by

$$\langle A_\xi^f X, Y\rangle = \langle \alpha_f(X,Y), \xi\rangle.$$

We assume throughout the paper that f has constant rank 2. This condition is denoted by $\operatorname{rank}_f = 2$, and it means that the relative nullity subspaces $\Delta(x) \subset T_xM$ defined by

$$\Delta(x) = \{X \in T_xM : \alpha_f(X,Y) = 0;\ Y \in T_xM\}$$

form a tangent subbundle of codimension 2. Equivalently, the index of relative nullity $\nu_f(x) = \dim \Delta(x)$ satisfies $\nu_f(x) = n-2$. It is a standard fact that the relative nullity distribution is integrable and that the $(n-2)$-dimensional leaves are totally geodesic submanifolds of the manifold and the ambient space.

The following fact was proved in [8].

PROPOSITION 1. *Let $f\colon M^n \to \mathbb{R}^{n+2}$ be an isometric immersion of* $\operatorname{rank}_f = 2$. *Assume that M^n has no open flat subset. Then, given an isometric immersion $g\colon M^n \to \mathbb{R}^{n+2}$, there exists an open dense subset of M^n such that, along each connected component V, we have*:

(i) $\text{rank}_g = 2$ *and* $\Delta_g = \Delta_f$; *or*
(ii) $\text{rank}_g = 3$ *and* $g|_V = k \circ h \colon V \to \mathbb{R}^{n+2}$ *is a composition of isometric immersions* $h \colon V \to U$ *and* $k \colon U \to \mathbb{R}^{n+2}$, *where* $U \subset \mathbb{R}^{n+1}$ *is open.*

The simplest submanifolds $f \colon M^n \to \mathbb{R}^{n+2}$ of $\text{rank}_f = 2$ are called *surfacelike*. This means that $f(M) \subset L^2 \times \mathbb{R}^{n-2}$ with $L^2 \subset \mathbb{R}^4$ or that $f(M) \subset CL^2 \times \mathbb{R}^{n-3}$, where $CL^2 \subset \mathbb{R}^5$ is a cone over a spherical surface $L^2 \subset \mathbb{S}^4$. The following characterization in terms of the splitting tensor is well known; see [10] or [12]. Recall that associated to the relative nullity foliation is the *splitting tensor* C, which is defined as follows: to each vector $T \in \Delta$ corresponds the endomorphism C_T of $\Delta^\perp$ given by

$$C_T X = -(\nabla_X T)_{\Delta^\perp}.$$

Proposition 2. *Let* $f \colon M^n \to \mathbb{R}^{n+2}$ *be an isometric immersion of* $\text{rank}_f = 2$. *Assume that* $C_T = \mu(T) I$ *for any* $T \in \Delta$. *Then each point has a neighborhood where* f *is surfacelike.*

3. Parabolic Submanifolds

A submanifold $f \colon M^n \to \mathbb{R}^{n+2}$ of $\text{rank}_f = 2$ is called *parabolic* if the following statements hold.

(i) At any $x \in M^n$,

$$T^\perp_{f(x)} M = \text{span}\{\alpha_f(X, Y);\ X, Y \in T_x M\}.$$

(ii) There is a nonsingular (asymptotic) vector field $Z \in \Delta^\perp$ such that

$$\alpha_f(Z, Z) = 0.$$

Clearly, if f is parabolic then there is a smooth orthonormal frame $\{\eta_1, \eta_2\}$ of $T_f^\perp M$ such that the shape operators have the form

$$A^f_{\eta_1}|_{\Delta^\perp} = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad A^f_{\eta_2}|_{\Delta^\perp} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \tag{1}$$

with $b, c \in C^\infty(M)$ nowhere vanishing. In particular, the asymptotic vector field Z is unique up to sign.

It is well known (see [10] or [12]) that the differential equation

$$\nabla_T A^f_\xi|_{\Delta^\perp} = A^f_\xi|_{\Delta^\perp} \circ C_T + A_{\nabla^\perp_T \xi}|_{\Delta^\perp} \tag{2}$$

holds for any $T \in \Delta$ and $\xi \in T_f^\perp M$. In particular, $A^f_\xi|_{\Delta^\perp} \circ C_T$ is symmetric. From this and (ii) it follows easily that

$$C_T = \begin{bmatrix} m & 0 \\ n & m \end{bmatrix} \tag{3}$$

for any $T \in \Delta$. From Proposition 2, we conclude that f is surfacelike if and only if $n = \langle C_T X, Z \rangle = 0$ for any $T \in \Delta$.

Recall that an isometric immersion $f: M^n \to \mathbb{R}^{n+2}$ is *ruled* when there is a foliation by open subsets of $(n-1)$-dimensional affine subspaces of $\mathbb{R}^{n+2}$. Observe that a ruled submanifold in codimension 2 need not be parabolic. In fact, generically we have $\operatorname{rank}_f = 3$.

Ruled parabolic submanifolds are never locally isometrically rigid, as shown by the converse part of the following result in [8]. In fact, the direct statement will be an important element in the proof of our main result.

PROPOSITION 3. *Let $f: M^n \to \mathbb{R}^{n+2}$ be a ruled parabolic submanifold. If M^n is simply connected then it admits an isometric immersion as a ruled hypersurface in $\mathbb{R}^{n+1}$.*

Conversely, let $g: M^n \to \mathbb{R}^{n+1}$ be a simply connected ruled hypersurface without flat points. Then the family of ruled parabolic submanifolds $f: M^n \to \mathbb{R}^{n+2}$ is parameterized by the set of ternary smooth arbitrary functions in an interval.

4. The Main Result

The following result generalizes the one in [8] and provides the first known examples of locally rigid submanifolds that are of rank 2 and codimension 2.

THEOREM 4. *Let $f: M^n \to \mathbb{R}^{n+2}$, $n \geq 3$, be a parabolic submanifold that is neither ruled nor surfacelike on any open subset of M^n. Then f is isometrically rigid.*

The following is a key ingredient of the proof of the theorem.

PROPOSITION 5. *Let $f: M^n \to \mathbb{R}^{n+2}$ be a simply connected parabolic submanifold. If f is not of surface type in any open subset of M^n and admits an isometric immersion as a hypersurface of $\mathbb{R}^{n+1}$, then f is ruled.*

Proof. Let $g: M^n \to \mathbb{R}^{n+1}$ be an isometric immersion; we denote by N its Gauss map. Given $x \in M^n$, let $\beta: T_xM \times T_xM \to \mathbb{L}^2$ be the symmetric bilinear form

$$\beta(Y,V) = \langle A^f_{\eta_1} Y, V\rangle e_1 + \langle A^g_N Y, V\rangle e_2,$$

where η_1 is as in (1) and $\{e_1, e_2\}$ is an orthonormal frame for the Lorentzian plane $\mathbb{L}^2$ such that $\|\varepsilon_1\|^2 = 1 = -\|e_2\|^2$ and $\langle \varepsilon_1, e_2\rangle = 0$. Then β is flat since, from the Gauss equations for f and g, we easily see that

$$\langle \beta(X,Y), \beta(V,W)\rangle - \langle \beta(X,W), \beta(V,Y)\rangle = 0.$$

We claim that $\Delta_g(x) = \Delta_f(x)$. Since parabolic submanifolds have no flat points, it follows that $\dim \Delta_g(x) \leq n-2$. If $\Delta_g(x) \neq \Delta_f(x)$, it is easy to see that

$$S(\beta) = \operatorname{span}\{\beta(Y,V);\ Y,V \in T_xM\}$$

satisfies $S(\beta) = \mathbb{L}^2$. From [14, Cor. 1] we obtain that

$$N(\beta) = \{Y \in T_xM : \beta(Y,V) = 0;\ V \in T_xM\}$$

satisfies $\dim N(\beta) = n - 2$; this contradicts $N(\beta) = \Delta_g \cap \Delta_f$ and so proves the claim.

Set

$$A_N^g|_{\Delta^\perp} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{c} \end{bmatrix}.$$

Using (3), we conclude from the symmetry of

$$A_N^g \circ C_T = \begin{bmatrix} \bar{a}m + \bar{b}n & \bar{b}m \\ \bar{b}m + \bar{c}n & \bar{c}m \end{bmatrix}$$

that $\bar{c}n = 0$. Then, from Proposition 3 and our assumption that f is nowhere surfacelike, it follows that $\bar{c} = 0$. In particular, given the Gauss equations for f and g, we can choose an orientation for g such that $\bar{b} = b$.

Taking the Z-component of the Codazzi equations for $A_{\eta_1}^f$ and A_N^g gives

$$2b\langle \nabla_X X, Z\rangle - a\langle \nabla_Z X, Z\rangle - Z(b) = 0 \tag{4}$$

and

$$2b\langle \nabla_X X, Z\rangle - \bar{a}\langle \nabla_Z X, Z\rangle - Z(b) = 0.$$

Thus,

$$(a - \bar{a})\langle \nabla_Z Z, X\rangle = 0.$$

Suppose that $\langle \nabla_Z Z, X\rangle = 0$ in an open subset U of M^n. From (3) we have

$$\langle \nabla_Z T, X\rangle = -\langle C_T Z, X\rangle = 0. \tag{5}$$

Taking the Z-component of the Codazzi equations for g applied to Z, T then gives

$$\langle \nabla_T Z, X\rangle = 0.$$

It follows from these equalities that the distribution $\operatorname{span}\{Z\} \oplus \Delta$ is totally geodesic on U. But then f is ruled on U as we wanted.

Assume now that $a = \bar{a}$ on an open subset. Taking the X-component of the same Codazzi equations as before gives

$$X(b) - a\langle \nabla_X Z, X\rangle - Z(a) + 2b\langle \nabla_Z X, Z\rangle + c\langle \nabla_Z^\perp \eta_1, \eta_2\rangle = 0 \tag{6}$$

and

$$X(b) - a\langle \nabla_X Z, X\rangle - Z(a) + 2b\langle \nabla_Z X, Z\rangle = 0.$$

Thus, $\langle \nabla_Z^\perp \eta_1, \eta_2\rangle = 0$. The Codazzi equation for $A_{\eta_2}^f$ applied to X, Z yields

$$c\langle \nabla_Z Z, X\rangle = b\langle \nabla_Z^\perp \eta_1, \eta_2\rangle. \tag{7}$$

Again $\langle \nabla_Z Z, X\rangle = 0$, and the proof follows. □

Proof of Theorem 4. Let $g\colon M^n \to \mathbb{R}^{n+2}$ be an isometric immersion. Since f is nowhere ruled, Proposition 5 asserts that there is no local isometric immersion of M^n as a hypersurface in $\mathbb{R}^{n+1}$. From Proposition 1 we have that $\operatorname{rank}_g = 2$ and $\Delta_g = \Delta_f$.

Take $\{\eta_1, \eta_2\}$ as in (1) and recall that (3) holds for any $T \in \Delta$. Since $A_{\bar{\eta}}^g \circ C_T$ is symmetric for any $\bar{\eta} \in T_g^\perp M$, it follows as before that Z is asymptotic for g. Therefore, we may fix an orthonormal base $\{\bar{\eta}_1, \bar{\eta}_2\}$ of $T_g^\perp M$ such that

$$A^g_{\bar\eta_1}|_{\Delta^\perp} = \begin{bmatrix} \bar a & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad A^g_{\bar\eta_2}|_{\Delta^\perp} = \begin{bmatrix} \bar c & 0 \\ 0 & 0 \end{bmatrix}.$$

Since (4) holds for both immersions, we have

$$(a-\bar a)\langle \nabla_Z Z, X\rangle = 0$$

and so conclude that $a=\bar a$. Similarly, we have from (6) that

$$c\langle \nabla^\perp_Z \eta_1, \eta_2\rangle = \bar c\langle \nabla^\perp_Z \bar\eta_1, \bar\eta_2\rangle.$$

Now computing the Z-component of the Codazzi equations for $A^f_{\eta_2}$ and $A^g_{\bar\eta_2}$ gives

$$\bar c\langle \nabla^\perp_Z \eta_1, \eta_2\rangle = c\langle \nabla^\perp_Z \bar\eta_1, \bar\eta_2\rangle.$$

It follows that

$$c=\bar c \quad \text{and} \quad \langle \nabla^\perp_Z \eta_1, \eta_2\rangle = \langle \nabla^\perp_Z \bar\eta_1, \bar\eta_2\rangle.$$

Taking the X-components now yields

$$\langle \nabla^\perp_X \eta_1, \eta_2\rangle = \langle \nabla^\perp_X \bar\eta_1, \bar\eta_2\rangle.$$

We conclude from the fundamental theorem of submanifolds (see e.g. [7]) that f and g are congruent by a rigid motion of the ambient space. □

5. Ruled Parabolic Surfaces

In this section, we give a parametric description of all ruled submanifolds in $\mathbb{R}^{n+2}$ that are parabolic.

Let $v\colon I \to \mathbb{R}^{n+2}$ be a smooth curve parameterized by arc length in an interval I in $\mathbb{R}$. Set $e_1 = dv/ds$ and let $e_2,\dots,e_{n-1}$ be orthonormal normal vector fields along $v=v(s)$ that are parallel in the normal connection of v in $\mathbb{R}^{n+2}$. Thus,

$$\frac{de_j}{ds} = b_j e_1, \quad 2\le j\le n-1, \tag{8}$$

where $b_j \in C^\infty(I)$. Set $\Delta = \operatorname{span}\{e_2,\dots,e_{n-1}\}$ and let $\Delta^\perp$ be the orthogonal complement in the normal bundle of the curve. Take a smooth unit vector field $e_0 \in \Delta^\perp$ along v such that

$$P = \operatorname{span}\left\{e_0, \left(\frac{de_1}{ds}\right)_{\Delta^\perp}\right\} \subset \Delta^\perp$$

satisfies

$$\dim P = 2 \tag{9}$$

and P is nowhere parallel in $\Delta^\perp$ along v; that is,

$$\operatorname{span}\left\{\left(\frac{de_0}{ds}\right)_{\Delta^\perp}, \left(\frac{d^2e_1}{ds^2}\right)_{\Delta^\perp}\right\} \not\subset P. \tag{10}$$

We parameterize a ruled submanifold M^n by

$$f(s,t_1,\dots,t_{n-1}) = c(s) + \sum_{j=1}^{n-1} t_j e_j(s), \tag{11}$$

where $(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$ and $c(s)$ satisfies $dc/ds = e_0$. To see that f is parabolic, first observe that

$$TM = \operatorname{span}\{f_s\} \oplus \operatorname{span}\{e_1\} \oplus \Delta,$$

where $f_s = e_0 + t_1(de_1/ds) + \sum_{j\geq 2} t_j b_j e_1$. Consider the orthogonal decomposition

$$\left(\frac{de_1}{ds}\right)_{\Delta^\perp} = a_1 e_0 + \eta. \tag{12}$$

Thus $\eta(s) \neq 0$ for all $s \in I$ by (9). Hence

$$TM = \operatorname{span}\{e_0 + t_1(a_1 e_0 + \eta)\} \oplus \operatorname{span}\{e_1\} \oplus \Delta, \tag{13}$$

and it follows easily that f is regular at any point.

Since $f_{st_j} = b_j e_1 \in TM$ for $2 \leq j \leq n-1$, we have that $\Delta \subset \Delta_f$. It follows easily from (12), (13), and $\eta(s) \neq 0$ that

$$f_{st_1} = \frac{de_1}{ds} \notin TM.$$

It is easy to see that $f_{ss} \notin \operatorname{span}\{f_{st_1}\} \oplus TM$ (i.e., $\dim N_1^f = 2$) is equivalent to

$$\left(\frac{de_0}{ds}\right)_{\Delta^\perp} + t_1\left(\frac{d^2 e_1}{ds^2}\right)_{\Delta^\perp} \notin P.$$

It follows easily that $\Delta = \Delta_f$ and that f is parabolic in at least one open dense subset of M^n.

Let $f\colon M^n \to \mathbb{R}^{n+2}$ be a ruled parabolic submanifold and let $\{e_2, \ldots, e_{n-1}\}$ be an orthonormal frame for Δ_f along a integral curve $c = c(s)$, $s \in I$, of the unit vector field X orthogonal to the rulings. Without loss of generality (see [2, Lemma 2.2]), we may assume that

$$\frac{de_j}{ds} \perp \Delta_f, \quad 2 \leq j \leq n-1.$$

Now parameterize f by (11), where $e_0 = X$ and $e_1 = Z$. That $f_{st_j} \in TM$ implies

$$\frac{de_j}{ds} \in \operatorname{span}\{e_1, f_s\}, \quad 2 \leq j \leq n-1. \tag{14}$$

Taking $t_1 = 0$, we obtain that

$$\frac{de_j}{ds} = a_j e_0 + b_j e_1, \quad 2 \leq j \leq n-1, \tag{15}$$

where $a_j, b_j \in C^\infty(I)$. Since $\dim N_1^f = 2$, we have

$$\frac{de_1}{ds} = a_1 e_0 + \left(\frac{de_1}{ds}\right)_{\Delta} + \eta.$$

where $\eta \perp \operatorname{span}\{e_0, e_1\} \oplus \Delta$ satisfies $\eta(s) \neq 0$. Hence (14) reduces to

$$a_j e_0 \in \operatorname{span}\{(1 + t_1 a_1 + \cdots + t_{n-1} a_{n-1}) e_0 + t_1 \eta\}, \quad 2 \leq j \leq n-1.$$

Therefore, $a_j = 0$. From (15) we have $de_j/ds = b_j e_1$ for $2 \leq j \leq n-1$.

We have proved the following result.

PROPOSITION 6. *Given a smooth curve* $c\colon I \subset \mathbb{R} \to \mathbb{R}^{n+2}$, *consider the orthonormal fields* $e_0 = dc/ds$ *and* $e_1(s), \dots, e_{n-1}(s)$ *satisfying* (8), (9), *and* (10) *at any point. The ruled submanifold parameterized by*

$$f(s, t_1, \dots, t_{n-1}) = c(s) + \sum_{j=1}^{n-1} t_j e_j(s) \tag{16}$$

with $(t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$ *is parabolic in an open dense subset of* M^n. *Conversely, any ruled parabolic submanifold can be parameterized as in* (16).

6. Nonruled Parabolic Surfaces

In this section, we provide a parametric description of all nonruled Euclidean parabolic submanifolds.

Let L^2 be a Riemannian manifold endowed with a global coordinate system (x, z). Let $g\colon L^2 \to \mathbb{R}^N$, $N \geq 4$, be a surface whose coordinate functions are linearly independent solutions of the parabolic equation

$$\frac{\partial^2 u}{\partial z^2} + W(u) = 0, \tag{17}$$

where $W \in TL$. In terms of the Euclidean connection, we have

$$\tilde{\nabla}_Z g_* Z + g_* W = 0,$$

where $Z = \partial/\partial z$. Thus, the coordinate field Z is asymptotic; that is, the second fundamental form of g satisfies

$$\alpha_g(Z, Z) = 0, \tag{18}$$

and also $W = -\nabla_Z Z$. In particular, the coordinate functions satisfy

$$\operatorname{Hess}_u(Z, Z) = 0. \tag{19}$$

Conversely, if $f\colon L^2 \to \mathbb{R}^N$ is a surface with a coordinate system (x, z) such that $\partial/\partial z = Z$ satisfies (18), then all coordinate functions of f satisfy (17) with $W = -\nabla_Z Z$.

Given a surface $g\colon L^2 \to \mathbb{R}^N$, $N \geq 4$, we denote

$$N_1^g(x) = \operatorname{span}\{\alpha_g(X, Y);\ X, Y \in T_x L\}.$$

We call g a *parabolic* surface if both of the following conditions hold:

(i) $\dim N_1^g(x) = 2$ at any $x \in L^2$; and
(ii) there is a nonsingular vector field $Z \in TL$ such that $\alpha_g(Z, Z) = 0$.

Let $h\colon L^2 \to \mathbb{R}^N$ be a smooth map satisfying

$$h_*(TL) \subset T_g^{\perp} L. \tag{20}$$

Set $h = U + \delta$, where $U \in TL$ and $\delta \in T_g^{\perp} L$. Given $Y \in TL$, we have

$$h_*(Y) = \nabla_Y U - A_\delta^g(Y) + \alpha_g(Y, U) + \nabla_Y^{\perp} \delta.$$

It follows that (20) is equivalent to

$$\nabla_Y U = A^g_\delta Y \quad \text{for any } \; Y \in TL. \tag{21}$$

In particular, the map $(Y, X) \mapsto \langle \nabla_Y U, X\rangle$ is symmetric; that is, the 1-form U^* is closed. Thus $U = \nabla\varphi$ for some function $\varphi \in C^\infty(L)$. We obtain from (21) that

$$\mathrm{Hess}_\varphi = A^g_\delta$$

and hence φ satisfies (19). Let Λ denote the orthogonal complement of N^g_1 in $T^\perp_g L$, so

$$T^\perp_g L = N^g_1 \oplus \Lambda.$$

We have just proved the direct statement in the following result, since the proof of the converse is immediate.

PROPOSITION 7. *Let $g\colon L^2 \to \mathbb{R}^N$ be a parabolic surface. Then any smooth map $h\colon L^2 \to \mathbb{R}^N$ satisfying* (20) *has the form*

$$h_\varphi = g_*\nabla\varphi + \gamma_1 + \gamma_0, \tag{22}$$

where the function φ satisfies (17) (*or* (19)), *the section $\gamma_1 \in N^g_1$ is unique such that $A^g_{\gamma_1} =$* Hess$_\varphi$, *and γ_0 is any section of Λ. Conversely, any function h_φ as in* (22) *satisfies* (20).

Let $f\colon M^n \to \mathbb{R}^{n+2}$ be parabolic. For simplicity, we assume that f does not split a Euclidean factor. Because we are working locally, we assume that M^n is the saturation of a transversal section L^2 to Δ. In relation to the following definition, recall that the normal bundle $T^\perp_f M$ is constant along Δ in $\mathbb{R}^{n+2}$.

DEFINITION 8. Let $f\colon M^n \to \mathbb{R}^{n+2}$ be a parabolic submanifold and let L^2 be as before. We call a *polar surface* associated to f any immersion $g\colon L^2 \to \mathbb{R}^{n+2}$ satisfying $T_{g(x)}L = T^\perp_{f(x)}M$ up to parallel identification in $\mathbb{R}^{n+2}$.

PROPOSITION 9. *Any parabolic submanifold f admits locally a polar surface. Moreover, any polar surface g associated to f is parabolic and substantial.*

Proof. Let $\eta_1, \eta_2 \in T^\perp_f M$ be orthonormal vector fields such that (1) holds. We claim that they are parallel along Δ in the normal connection. It follows from (2) that

$$\nabla_T A^f_\xi = A^f_\xi \circ C_T$$

is satisfied if $\xi \in T^\perp_f M$ is parallel along Δ. Given $x \in M^n$, let γ be a geodesic with $\gamma(0) = x$ contained in the leaf of relative nullity tangent to $\Delta(x)$. If δ_t is the parallel transport of η_x along γ, then

$$\nabla_{\gamma'} A^f_{\delta_t} = A^f_{\delta_t} \circ C_{\gamma'}.$$

Hence, $A^f_{\delta_t} = A^f_{\eta_x} e^{\int_0^t C_{\gamma'}\,d\tau}$. Thus $A^f_{\delta_t}$ has constant rank along γ. It follows that A_ξ has constant rank along the leaves of relative nullity. Now observe that η_1 is the unique (up to sign) unit vector field in $T^\perp_f M$ such that the corresponding shape operator $A^f_{\eta_1}$ has rank 1. Therefore, η_1 and hence η_2 are parallel as claimed.

Next, we claim that also the orthonormal frame $\{X, Z\}$ in $\Delta^\perp$ as in (1) is parallel along Δ. In fact, using (5) and the Codazzi equation yields

$$\nabla_T^\perp \alpha_f(Z, X) = \langle \nabla_T Z, X\rangle \alpha_f(X, X) - \langle \nabla_Z T, Z\rangle \alpha_f(Z, X).$$

Since η_1 is colinear with $\alpha_f(Z, X)$ and parallel along Δ, the claim follows easily.

Let $U, V \in TL$ be such that

$$Z = U + \delta_1 \quad \text{and} \quad X = V + \delta_2,$$

where $\delta_1, \delta_2 \in \Delta$. Now let (u, v) be a coordinate system in L^2 such that

$$\partial u =: \frac{\partial}{\partial u} = \lambda_1 U \quad \text{and} \quad \partial v =: \frac{\partial}{\partial v} = \lambda_2 V,$$

where $\lambda_1, \lambda_2 \in C^\infty(L)$. We show next that there exist linearly independent 1-forms θ_1, θ_2 such that the differential equation

$$dg = \theta_1\eta_1 + \theta_2\eta_2 \tag{23}$$

is integrable. Set

$$\theta_1 = c\lambda_2\phi\, dv \quad \text{and} \quad \theta_2 = b\lambda_1\phi\, du + \sigma\, dv, \tag{24}$$

where $\phi, \sigma \in C^\infty(L)$ and b, c denote the restriction to L^2 of the functions in M^n defined in (1). The integrability condition of (23) is

$$\begin{aligned} 0 &= d\theta_1\eta_1 + d\theta_2\eta_2 + \theta_1 \wedge d\eta_1 + \theta_2 \wedge d\eta_2 \\ &= d\theta_1\eta_1 + d\theta_2\eta_2 - \left(c\lambda_2\phi\frac{\partial\eta_1}{\partial u} - b\lambda_1\phi\frac{\partial\eta_2}{\partial v} + \sigma\frac{\partial\eta_2}{\partial u}\right) dV, \end{aligned}$$

where dV is the volume element of L^2. We have

$$\tilde\nabla_{\partial u}\eta_j = \tilde\nabla_{\lambda_1(Z-\delta_1)}\eta_j = \lambda_1\tilde\nabla_Z\eta_j \quad \text{and} \quad \tilde\nabla_{\partial v}\eta_j = \tilde\nabla_{\lambda_2(X-\delta_2)}\eta_j = \lambda_2\tilde\nabla_X\eta_j.$$

Thus, we obtain from (1) that

$$\left(c\lambda_2\phi\frac{\partial\eta_1}{\partial u} - b\lambda_1\phi\frac{\partial\eta_2}{\partial v} + \sigma\frac{\partial\eta_2}{\partial u}\right)_{TM} = 0.$$

Then

$$c\lambda_2\phi\frac{\partial\eta_1}{\partial u} - b\lambda_1\phi\frac{\partial\eta_2}{\partial v} + \sigma\frac{\partial\eta_2}{\partial u} = e\eta_1 + \ell\eta_2,$$

where $e, \ell \in C^\infty(L)$. To conclude, we observe that the integrability condition now follows from the integrability of the system

$$\begin{cases} (c\lambda_2\phi)_u = e, \\ \sigma_u - (b\lambda_1\phi)_v = \ell. \end{cases}$$

It remains to show that g is parabolic. It is clear that $N_1^g = \Delta^\perp$ and hence $\dim N_1^g = 2$. It follows from (24) that $g_*(\partial u) = b\lambda_1\phi\eta_2$. Then we obtain from (1) that $\tilde\nabla_{\partial u} g_*(\partial u) \in TL$ as we wished. □

THEOREM 10. *Let $g\colon L^2 \to \mathbb{R}^{n+2}$ be a parabolic surface and let $\Psi\colon \Lambda \to \mathbb{R}^{n+2}$ be the map defined by*

$$\Psi(\delta) = h(x) + \delta, \quad \delta \in \Lambda(x), \tag{25}$$

where $h\colon L^2 \to \mathbb{R}^{n+2}$ satisfies (20). *Then $M^n = \Psi(\Lambda)$ is, at regular points, a parabolic submanifold with polar surface g. Moreover, Ψ is nonruled if and only if g is nonruled.*

Conversely, any parabolic submanifold $f\colon M^n \to \mathbb{R}^{n+2}$ admits a parameterization (25) *locally where g is a polar surface of f.*

Proof. Because g is parabolic, there exist an orthonormal tangent frame $\{X, Z\}$ of TL and an orthonormal normal frame of $\{\eta_1, \eta_2\}$ of N_1^g such that

$$A^f_{\eta_1} = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} \quad \text{and} \quad A^f_{\eta_2} = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$$

with $b, c \in C^\infty(M)$ nowhere vanishing.

It follows from the symmetry of the tensor (see [17])

$$\alpha^2(V, Y, W) = (\nabla^\perp_V \alpha_g(Y, W))_{(N_1^g)^\perp}$$

that

$$(\nabla^\perp_Z \alpha_g(X, Z))_{(N_1^g)^\perp} = 0; \tag{26}$$

that is,

$$(\nabla^\perp_Z \eta_1)_{(N_1^g)^\perp} = 0.$$

Next we show that

$$(\tilde\nabla_Z h_*(Z))_{TL} = 0. \tag{27}$$

In fact, we have

$$\langle \tilde\nabla_Z h_*(Z), W\rangle = \langle \tilde\nabla_Z h, \alpha_g(W, Z)\rangle = \langle \tilde\nabla_W \tilde\nabla_Z h, Z\rangle = \langle \tilde\nabla_Z \tilde\nabla_W h, Z\rangle = 0$$

because Z is asymptotic.

Since h satisfies (20), it follows easily from the regularity assumption that $T_{\Psi(\delta_x)}M = T^\perp_g L(x)$. Moreover, we have that $\Delta_{\Psi(\delta_x)} = \Lambda(x)$. We now demonstrate that Ψ is parabolic. Since

$$\Psi_*(Z) = h_*(Z) + \tilde\nabla_Z \delta \in T^\perp_{g(x)} L = T_{\Psi(\delta_x)} M,$$

it follows from (26) and (27) that

$$\begin{aligned} \langle \tilde\nabla_Z \Psi_*(Z), W\rangle &= \langle h_*(Z) + \tilde\nabla_Z \delta, \alpha_g(W, Z)\rangle \\ &= \langle h_*(Z), \alpha_g(W, Z)\rangle + \langle \delta, \nabla^\perp_Z \alpha_g(W, Z)\rangle \\ &= 0. \end{aligned} \tag{28}$$

Since $\eta_1 \in T^\perp_g L = T_\Psi M$ and since $(\tilde\nabla_X \eta_1)_{TL}$ and $(\tilde\nabla_Z \eta_1)_{TL}$ are linearly independent and belong to $T^\perp_\Psi M$, it follows that $\dim N_1^\Psi = 2$. Thus Ψ is parabolic.

Let us see that Ψ is not ruled. We have from (28) that the asymptotic direction of Ψ is normal to η_1 and thus colinear with η_2. Since Ψ is ruled if and only if $\langle \tilde\nabla_Z \eta_2, \eta_1\rangle = 0$, we have from (7) that g is ruled.

For the converse, consider a polar surface $g\colon L^2 \to \mathbb{R}^{n+2}$ of f. Because f has no Euclidean factor, g is substantial. It is now easy to conclude that $\Delta_f = \Lambda$ and $TM = T^\perp_g L$ along L^2. Thus, $h = f|_L$ satisfies (20). $\square$

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