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Almost Regular Sequences and the Monomial Conjecture

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To Mel Hochster on his 65th birthday

The monomial conjecture of Mel Hochster has been one of the most important open problems in commutative algebra for many years. The conjecture is as follows.

Conjecture. *Let R be a Noetherian local ring of dimension d, and suppose that $x_1, x_2, \dots, x_d$ is a system of parameters for R. Then for all integers $t \geq 0$ we have*

$$x_1^t x_2^t \cdots x_d^t \notin (x_1^{t+1}, \dots, x_d^{t+1}),$$

where $(x_1^{t+1}, \dots, x_d^{t+1})$ denotes the ideal generated by $x_1^{t+1}, \dots, x_d^{t+1}$.

This conjecture has assumed a central role because it is simply stated and it implies several other important conjectures, notably the canonical element conjecture, for rings of positive or mixed characteristic. In fact, when this conjecture was first announced it had numerous further consequences, some of which (such as the new intersection conjecture) were later proved by different means. We refer to Hochster [6; 8] for descriptions of these conjectures and their status at various times.

The monomial conjecture is almost trivial for rings that contain the rational numbers and is not difficult for rings of positive charateristic, but it remains an open problem for rings of mixed characteristic. The most recent advance was made by Heitmann [5], who proved it in mixed characteristic in dimension 3.

One traditional method for approaching this and other conjectures has been to construct Cohen–Macaulay modules for which a system of parameters for the ring becomes a regular sequence. It is unknown whether one can find finitely generated modules with this property, but Hochster showed many years ago that for equicharacteristic rings one can find infinitely generated modules (and even algebras) with this property (see [6]).

In the course of Heitmann's proof, he shows that a weaker condition than being a regular sequence suffices to prove these conjectures. We call a sequence of elements with this property an *almost regular sequence* and we give a precise definition in Section 1.

In this paper we first review some of the known facts about almost regular sequences and then discuss some related questions in the equicharacteristic case. Finally, we discuss a variation on this concept for rings of mixed characteristic and its relation to the monomial conjecture.

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1. Almost Regular Sequences

The inspiration for the concept of almost regular sequence that we use came from two sources. The first was Heitmann's proof of the monomial conjecture in dimension 3 mentioned previously. The second was the work of Faltings on p-adic Hodge theory in [1] and the resulting work of Gabber and Ramero [2]. This theory was developed to give a firm foundation to the results of Faltings, and these ideas have their origins in a classic work of Tate on p-divisible groups [14]. Our use of this concept is comparatively simple, but it illustrates the main questions in looking at certain homological conjectures, as we shall explain.

Let A be an integral domain, and let v be a valuation on A with values in the abelian group of rational numbers. That is, v is a function from A to $\mathbb{Q} \cup \{\infty\}$ such that

(1) $v(a) = \infty$ if and only if $a = 0$,
(2) $v(ab) = v(a)v(b)$ for all $a, b \in A$, and
(3) $v(a + b) \geq \inf\{v(a), v(b)\}$ for all $a, b \in A$.

We will also assume that $v(a) \geq 0$ for $a \in A$. Later in this work we will also consider more general functions that do not satisfy the first condition.

We note that the following definitions depend on the choice of a valuation, so the concept of being almost zero depends on this choice. However, we usually assume that we have fixed a valuation and the definitions are in terms of this valuation.

DEFINITION 1. Let A be a ring with a valuation v as in (1)–(3), and let M be an A-module. We say that M is *almost zero* with respect to v if, for every $m \in M$ and for every $\varepsilon > 0$, there is an $a \in A$ with $v(a) < \varepsilon$ and $am = 0$.

DEFINITION 2. We say that a sequence $x_1, \ldots, x_d$ is an *almost regular sequence* with respect to v if, for each $i = 1, \ldots, d$, the module

$$((x_1, \ldots, x_{i-1}) : x_i)/(x_1, \ldots, x_{i-1})$$

is almost zero. If a system of parameters is an almost regular sequence with respect to v, then we say that A is *almost Cohen–Macaulay* with respect to v.

Observe that if we require these modules to be zero rather than almost zero then we have the usual definitions of a regular sequence and a Cohen–Macaulay ring.

Although this definition was inspired in part by the work of Gabber and Ramero [2], it is not quite the same as their definition. They define a module to be almost zero if it is annihilated by a given ideal $\mathfrak{m}$ for which $\mathfrak{m} = \mathfrak{m}^2$. The corresponding definition of almost regular would be that $((x_1, \ldots, x_{i-1}) : x_i)/(x_1, \ldots, x_{i-1})$ is annihilated by $\mathfrak{m}$. In many situations their condition is stronger than ours.

We remark also that Hochster and Huneke [9] defined a closure operation using this idea, which they call "dagger closure", and showed that it agrees with tight closure in positive characteristic.

The situation we consider is when A is an integral extension of a Noetherian ring. Let R be a complete regular local ring of dimension d, and let $x_1, \ldots, x_d$ be a system of parameters for R. Let R^+ denote the integral closure of R in the algebraic closure of its fraction field; R^+ is called the *absolute integral closure* of R. The ring A will denote a ring between R and R^+, and in many cases we take A to be R^+ itself.

2. Almost Cohen–Macaulay Rings in the Equicharacteristic Case

The main question we consider is whether R^+ is almost Cohen–Macaulay with respect to some valuation v. This is easy to prove if R has positive characteristic; in fact, if S is a normal Noetherian domain of positive characteristic and if S^∞ denotes the extension of S obtained by adjoining all (p^n)th roots of elements of S, then S^∞ is almost Cohen–Macaulay with respect to any valuation. A proof of this can be found in [13], and we shall give a brief outline of it. As do virtually all results of this type, it uses some version of the following theorem (which we use again later as well).

THEOREM 1. *Let R be a complete local ring of dimension d. Then there is an ideal I of R such that the following statements hold.*

(1) *The support of I is the set of prime ideals $\mathfrak{p}$ for which $R_\mathfrak{p}$ is not Cohen–Macaulay.*
(2) *For every system of parameters $x_1, \ldots, x_d$ of R and every element a of R with $ax_i \in (x_1, \ldots, x_{i-1})$ for some i between 1 and d, we have $ca \in (x_1, \ldots, x_{i-1})$ for all $c \in I$.*
(3) *I annihilates the local cohomology $H^i_\mathfrak{m}(R)$ for $i = 0, 1, \ldots, d-1$.*

For a proof of this or a similar fact, refer to Roberts [12] or Hochster and Huneke (see discussion at beginning of Sec. 3 in [10]).

In the case we are considering, we use that if $ax_i \in (x_1, \ldots, x_{i-1})$ then applying the Frobenius map yields $a^{p^n} x_i^{p^n} \in (x_1^{p^n}, \ldots, x_{i-1}^{p^n})$, so $ca^{p^n} \in (x_1^{p^n}, \ldots, x_{i-1}^{p^n})$ for some nonzero element c in S. Taking (p^n)th roots, we have $c^{1/p^n} a \in (x_1, \ldots, x_{i-1})$. Since $v(c^{1/p^n}) = (1/p^n)v(c)$ goes to zero as n goes to infinity, this proves the result for any valuation v.

In [10] Hochster and Huneke proved the considerably deeper fact that, for an excellent local domain R of positive characteristic, the ring R^+ is Cohen–Macaulay (see also [11]). We remark that the subring S^∞ may not be Cohen–Macaulay in general.

If R is a local domain containing a field of characteristic 0, then R^+ is a big Cohen–Macaulay algebra only if the dimension of R is at most 2. In fact, if R is a normal ring of characteristic 0 that is not Cohen–Macaulay, then the field trace map shows that R is a direct summand of any finite extension of R. Consequently, a nontrivial relation on a system of parameters for R remains nontrivial in finite extensions and hence also in R^+. However, it is not known whether R^+ is almost Cohen–Macaulay with respect to some valuation v when R is a ring of characteristic 0.

In [13] we showed that, for certain graded rings in characteristic 0, the image of local cohomology group $H^2_{\mathfrak{m}}(R)$ in $H^2_{\mathfrak{m}}(R^+)$ is almost zero. In addition, we compute how this works for two examples in detail. We discuss here some properties of these examples and of the further questions that they suggest.

First of all, both examples are graded integrally closed non-Cohen–Macaulay domains of dimension 3. The valuation used is the one given by the grading. We describe the second of these examples in detail.

The simplest way to define this ring is as a Segre product. Let k be an algebraically closed field, let $A = k[X, Y, Z]/(X^3 + Y^3 + Z^3)$, and let $B = k[U, V]$, where both A and B have the usual gradings. Let R be the Segre product

$$R = A \# B = \bigoplus_n (A_n \otimes_k B_n).$$

Then R is a standard graded ring of dimension 3 generated over k by the six degree-1 elements $X \otimes U$, $Y \otimes U$, $Z \otimes U$, $X \otimes V$, $Y \otimes V$, and $Z \otimes V$. By a result of Goto and Watanabe [3, Thm. 4.1.5], the local cohomology module $H^2_{\mathfrak{m}}(R)$ is

$$H^2_{\mathfrak{m}}(A) \# B = \bigoplus_n (H^2_{\mathfrak{m}}(A))_n \otimes B_n.$$

Since B only has nonzero components in nonnegative degrees, the only component of $H^2_{\mathfrak{m}}(A)$ in nonnegative degree is in degree 0, and this component is isomorphic to k, it follows that $H^2_{\mathfrak{m}}(R) \cong k$. We do not go into the computation of the local cohomology of R, but we do use the facts that it suffices to consider the corresponding element of $H^2_{\mathfrak{m}}(A)$ of degree 0 and that this element is given by Z^2/XY, the homology of the Čech complex

$$0 \to A \to A_X \oplus A_Y \to A_{XY} \to 0.$$

This element is not zero in $H^2_{\mathfrak{m}}(A)$ because $Z^2 \notin (X, Y)$. It seems relevant to the computations in [13] (although not used explicitly) that Z^2 is integral not only over (X, Y) but also over $(X, Y)^2$, and the same holds for the other example from that paper. Furthermore, one of the few families of examples of non-Cohen–Macaulay normal domains is in the last section of Heitmann [4], and the dimension-3 examples given there also have the property that, for the system of parameters x, y, z, the given elements $a \notin (x, y)$ with $za \in (x, y)$ are integral over $(x, y)^2$. However, the following example (shown to me by A. Singh) shows that this is not necessarily true in general.

Let $A = k[X, Y, Z]/(X^2 + Y^3 + Z^7)$, where X, Y, and Z have degrees 21, 14, and 6, respectively. Then the element Z^6/XY defines a nonzero element of $H^2_{\mathfrak{m}}(A)$ of degree equal to $6 \times 6 - 21 - 14 = 1$, so by the formula of Goto and Watanabe it defines a nonzero element $H^2_{\mathfrak{m}}(R)$, where $R = A \# k[U, V]$ as in the previous example. However, Z^6 is not integral over $(X, Y)^2$: if we divide by the ideal generated by Y, the image of Z^6 is not integral over (X^2). Taking the Segre product, this produces an example in which (x, y, z) is a system of parameters, $a \notin (x, y)$ with $za \in (x, y)$, and no representative of a modulo (x, y) is integral over $(x, y)^2$.

However, for local cohomology coming from Segre products as in these examples, we can state the following theorem.

THEOREM 2. *Let R be a graded integral domain that is a finite extension of the polynomial ring $k[X,Y]$, where X and Y have positive degrees. Let w/XY be an element of $H^2_{\mathfrak{m}}(R)$ of nonnegative degree. Then there exists a nonzero constant $c \in R$ such that*

$$cw^n \in (X^n, Y^n)$$

for all $n \geq 0$.

To prove this we let X and Y have degrees i and j, respectively, and let w have degree d; then the fact that w/XY has nonnegative degree implies $d \geq i + j$. We note that w is integral over $k[X,Y]$, so there is an integer k such that every power w^n of w can be expressed as

$$w^n = w^k f_k(X,Y) + w^{k-1} f_{k-1}(X,Y) + \cdots + f_0(X,Y),$$

where each $f_m(X,Y)$ is a homogeneous polynomial. The degree of $f_m(X,Y)$ is the degree of w^n minus the degree of w^m, which is $d(n-m)$. Let c be any monomial in X and Y of degree at least dk. We claim that c satisfies the required property.

Given our preceding expression for w^n, it suffices to show that each $cf_m(X,Y)$ is in (X^n, Y^n). Now, since $f_m(X,Y)$ has degree $d(n-m)$, it follows that $cf_m(X,Y)$ has degree $d(n-m) + dk \geq dn$. Let $X^r Y^s$ be a monomial with nonzero coefficient in $cf_m(X,Y)$. Then its degree, which is $ri + sj$, satisfies

$$ri + sj \geq dn.$$

Since $d \geq i + j$, this gives

$$ri + sj \geq ni + nj,$$

so we have $r \geq n$ or $s \geq n$. Thus $cw^n \in (X^n, Y^n)$.

An interesting consequence is the existence of an ideal I of R with $I + (X,Y) = \{a \in R \mid aZ \in (X,Y)\}$ such that, for every element a of I, there is a c with $ca^n \in (X^n, Y^n)$ even though certain elements of (X,Y) itself, such as $X+Y$, do not have this property.

Theorem 2 applies only to Segre products. We observe, however, that if R is a graded domain that is the coordinate ring of a smooth projective variety (of characteristic 0) then the related fact that the local cohomology has no elements of negative degree follows from the Kodaira vanishing theorem.

3. A Variant on Almost Regular Sequences

In this section we consider another version of almost regular sequences for rings of mixed characteristic.

Let $R = \hat{\mathbb{Z}}_p[[X_2, \ldots, X_d]]$, a regular local ring of mixed characteristic p of dimension d, and let S be a ring between R and R^+. We will assume that $d \geq 3$ throughout this section.

We first introduce a function similar to a valuation but not satisfying the condition that $v(a) = \infty$ only if $a = 0$. Let v_0 be the $\mathfrak{m}$-adic valuation defined by the maximal ideal of R/pR, extended to a function on R by defining it to be infinity on pR. Let $\mathfrak{p}$ be an extension of pR to R^+; that is, $\mathfrak{p}$ is a minimal prime ideal over pR and $\mathfrak{p} \cap R = pR$. Then the valuation v_0 on R/pR extends to a valuation on $R^+/\mathfrak{p}$. We let v be this function, extended to R^+ by setting it equal to infinity on $\mathfrak{p}$.

The next proposition shows that, if we choose the correct convention in defining $0 \cdot \infty$, then this function v has the properties of a valuation *except* for the property of taking the value ∞ only at 0.

PROPOSITION 1. *Given the prime ideal $\mathfrak{p}$ and the function v as before, and making the convention that $\infty \cdot 0 = \infty$, we have*:

(1) $v(ab) = v(a)v(b)$ *for all $a, b \in R^+$; and*
(2) $v(a + b) \geq \inf\{v(a), v(b)\}$ *for all $a, b \in R^+$.*

Proof. If a and b are not in $\mathfrak{p}$, then these properties follow from the fact that v defines a valuation on $R^+/\mathfrak{p}$. If $a \in \mathfrak{p}$ and $b \notin \mathfrak{p}$, then

$$v(ab) = \infty = \infty \cdot v(b) = v(a)v(b)$$

and

$$v(a + b) = v(b) = \inf\{\infty, v(b)\} = \inf\{v(a), v(b)\}.$$

If both a and b are in $\mathfrak{p}$, then both sides of both equations are infinite. □

We will use the expression “there exists a small element c” to mean “for every $\varepsilon > 0$ there is an element c with $v(c) < \varepsilon$”, where v is defined as before. With this terminology, we say that a module M is almost zero if every element of M is annihilated by a small element.

We prove the following two theorems.

THEOREM 3. *Let S be a ring between R and R^+ as before. Suppose these two conditions hold for every system of parameters of the form $p, x_2, \ldots, x_d$ of the ring S*:

(1) *for each $i = 2, \ldots, d$ and any rational number $\alpha > 0$, if ax_i is in the ideal $(p^\alpha, x_2, \ldots, x_{i-1})$ then there exists a small element c in R^+ and a rational number $\alpha' > 0$ such that $ca \in (p^{\alpha'}, x_2, \ldots, x_{i-1})$*;
(2) *if $ap \in (x_2, \ldots, x_d)$, then there exists a small element c such that we have $ca \in (x_2, \ldots, x_d)$.*

Then the monomial conjecture holds for S.

THEOREM 4. *The first condition of Theorem 3 always holds.*

Proof of Theorem 3. We first recall that, in mixed characteristic, it suffices to prove the monomial conjecture for systems of parameters of the form $p, x_2, \ldots, x_d$ (see [7, Sec. 6]).

We next show that condition (2) implies the corresponding condition for powers of p in place of p. Suppose that condition (2) holds, and suppose we have

$$ap^m \in (x_2, \ldots, x_d)$$

for some positive integer m. Let $\varepsilon > 0$. By condition (2), since $(ap^{m-1})p \in (x_2, \ldots, x_d)$ we can find a c_1 with $v(c_1) < \varepsilon/m$ and $c_1ap^{m-1} \in (x_2, \ldots, x_d)$. We can then find c_2 with $v(c_2) < \varepsilon/m$ and $c_2c_1ap^{m-2} \in (x_2,\dots,x_d)$. Continuing, we find $c_1,\dots,c_m$ with $v(c_i) < \varepsilon/m$ and $c_m\cdots c_1a \in (x_2,\dots,x_d)$. Letting $c = c_m\cdots c_1$, we then have $v(c) < \varepsilon$ and $ca \in (x_2,\dots,x_d)$.

We now prove that conditions (1) and (2) imply the monomial conjecture. Suppose we have a counterexample to the monomial conjecture with ring S and system of parameters $p, x_2,\dots,x_d$. This means that for some t we have

$$p^t x_2^t\cdots x_d^t \in (p^{t+1}, x_2^{t+1},\dots,x_d^{t+1}).$$

Write this in the form

$$p^t x_2^t\cdots x_d^t = a_1p^{t+1} + a_2x_2^{t+1} + \cdots + a_dx_d^{t+1}.$$

Moving a_1p^{t+1} to the other side and factoring out p^t, we obtain

$$p^t(x_2^t\cdots x_d^t - a_1p) \in (x_2^{t+1},\dots,x_d^{t+1}).$$

By condition (2) extended to powers as here and then applied to the system of parameters $p, x_2^{t+1},\dots,x_d^{t+1}$, there exists a small element c such that

$$cx_2^t\cdots x_d^t \in (p, x_2^{t+1},\dots,x_d^{t+1}).$$

We now carry out one more step in detail. Write

$$cx_2^t\cdots x_d^t = b_1p + b_2x_2^{t+1} + \cdots + b_dx_d^{t+1}.$$

Moving $b_2x_2^{t+1}$ to the left-hand side of the equation and factoring out x_2^t, we obtain

$$x_2^t(cx_3^t\cdots x_d^t - b_2x_2) = b_1p + b_3x_3^{t+1} + \cdots + b_dx_d^{t+1}.$$

We now apply condition (1) to the system of parameters $p, x_2^{t+1},\dots,x_d^{t+1}$ to conclude that there is a rational number $\alpha_2 > 0$ and a small element c_2 with

$$c_2(cx_3^t\cdots x_d^t - b_2x_2) \in (p^{\alpha_2}, x_3^{t+1},\dots,x_d^{t+1}),$$

and from this we have that

$$c_2cx_3^t\cdots x_d^t \in (p^{\alpha_2}, x_2, x_3^{t+1},\dots,x_d^{t+1}).$$

Repeating this step for $x_3,\dots,x_d$, we finally show that there exist small elements $c, c_2, c_3,\dots,c_d$ and an $\alpha_d > 0$ with

$$cc_2c_3\cdots c_d \in (p^{\alpha_d}, x_2,\dots,x_d).$$

Thus we can write

$$cc_2c_3\cdots c_d = e_1p^{\alpha_d} + e_2x_2 + \cdots + e_dx_d.$$

However, we can make $v(cc_2c_3\cdots c_d)$ arbitrarily small, and by Proposition 1 we have

$$\begin{aligned} v(cc_2c_3\cdots c_d) &= v(e_1p^{\alpha_d} + e_2x_2 + \cdots + e_dx_d) \\ &\geq \inf\{v(p^{\alpha_d}), v(x_2),\dots,v(x_d)\} = \inf\{v(x_2),\dots,v(x_d)\} > 0. \end{aligned}$$

This contradiction proves the theorem. □

Before proving Theorem 4, we establish the following lemma.

LEMMA 1. *Let b be an element of R^+ such that b^p divides p. Then the Frobenius map induces an isomorphism*

$$R^+/bR^+ \to R^+/b^pR^+.$$

Proof. Since b^p divides p, it is clear that R^+/b^pR^+ has characteristic p; hence the Frobenius map defines a ring homomorphism f from R^+/b^pR^+ to itself. Since $f(b) = b^p = 0$, it follows that f induces a map (which we also denote f) from R^+/bR^+ to R^+/b^pR^+. Furthermore, since R^+ is closed under taking pth roots, f is surjective.

To prove that f is injective, let $r \in R^+$, let s be the image of r in R^+/bR^+, and assume that $f(s) = 0$ in R^+/b^pR^+. This means that $r^p = ab^p$ for some $a \in R^+$. Let c be a pth root of a. Then $(bc)^p = b^pc^p = b^pa = r^p$. Hence $r = \zeta bc$, where ζ is a pth root of 1. Thus $r \in bR^+$, so f is injective. □

Proof of Theorem 4. Let $p, x_2, \dots, x_d$ be a system of parameters for R^+, and assume that

$$ax_i \in (p^\alpha, x_2, \dots, x_{i-1})$$

for some $\alpha > 0$ and i between 2 and d. We may assume that $\alpha \le 1$. Because this relation involves a finite number of elements of R^+, it will hold in some subring S that is finite over R; we can also assume that S is integrally closed. As a result, the ideal I of Theorem 1 in S will have height 2, so there is an element c in I that is not in any prime ideal minimal over $p^\alpha S$. The element c will have the following property: For any system of parameters $y_1, \dots, y_d$ of S, if

$$by_i \in (y_1, \dots, y_{i-1})S$$

for some $b \in S$ and i between 1 and d, then

$$cb \in (y_1, \dots, y_{i-1})S.$$

We now consider the previous expression in $S/p^\alpha S$, denoting the image of an element s in $S/p^\alpha S$ by $\bar{s}$. We have

$$\bar{a}\bar{x}_i \in (\bar{x}_2, \dots, \bar{x}_{i-1}).$$

Since $S/p^\alpha S$ has characteristic p, we can apply the Frobenius map and obtain

$$\bar{a}^{p^n}\bar{x}_i^{p^n} \in (\bar{x}_2^{p^n}, \dots, \bar{x}_{i-1}^{p^n})$$

for all positive integers n. In terms of S, this translates to

$$a^{p^n}x_i^{p^n} \in (p^\alpha, x_2^{p^n}, \dots, x_{i-1}^{p^n});$$

since $p^\alpha, x_2^{p^n}, \dots, x_d^{p^n}$ is a system of parameters for S, this implies that

$$ca^{p^n} \in (p^\alpha, x_2^{p^n}, \dots, x_{i-1}^{p^n}).$$

Hence we can write

$$ca^{p^n} = a_1p^\alpha + a_2x_2^{p^n} + \cdots + a_{i-1}x_{i-1}^{p^n} \qquad (*)$$

for some elements $a_1, \dots, a_{i-1}$ in S.

Note that c is in no prime ideal minimal over $p^\alpha S$ and so is not, in particular, in $\mathfrak{p} \cap S$, where $\mathfrak{p}$ is the prime ideal in the definition of v. Thus $v(c) < \infty$.

We now apply Lemma 1, which implies that the nth power of the Frobenius map induces an isomorphism $f\colon R^+/p^{\alpha/p^n}R^+ \to R^+/p^\alpha R^+$. Lifting the elements $c, a_2, \ldots, a_{i-1}$ in equation $(*)$ to elements $d, b_2, \ldots, b_{i-1}$ yields the existence of an element $b_1 \in R^+$ with

$$da = b_1 p^{\alpha/p^n} + b_2 x_2 + \cdots + b_{i-1} x_{i-1}.$$

Since $d^{p^n} - c \in p^\alpha R^+$, we have $v(d^{p^n}) = v(c)$ and so $v(d) = v(c)/p^n$. Letting n go to infinity, we obtain elements with $v(d)$ arbitrarily small with

$$da \in (p^{\alpha'}, x_2, \ldots, x_d),$$

where $\alpha' = \alpha/p^n$. □

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