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Multi-Ideal-adic Completions of Noetherian Rings

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1. Introduction

Let R be a commutative ring with identity. A *filtration* on R is a decreasing sequence $\{I_n\}_{n=0}^{\infty}$ of ideals of R. Associated to a filtration is a well-defined completion $R^* = \varprojlim_n R/I_n$ and a canonical homomorphism $\psi\colon R \to R^*$ [13, Chap. 9]. If $\bigcap_{n=0}^{\infty} I_n = (0)$, then ψ is injective and R may be regarded as a subring of R^* [13, p. 401]. In the terminology of Northcott, a filtration $\{I_n\}_{n=0}^{\infty}$ is *multiplicative* if $I_0 = R$ and $I_n I_m \subseteq I_{n+m}$ for all $m \geq 0$ and $n \geq 0$ [13, p. 408]. A well-known example of a multiplicative filtration on R is the I-adic filtration $\{I^n\}_{n=0}^{\infty}$, where I is a fixed ideal of R.

In this paper we consider filtrations of ideals of R that are *not* multiplicative and examine the completions associated to these filtrations. We assume the ring R is Noetherian. Instead of successive powers of a fixed ideal I, we use a filtration formed from a more general descending sequence $\{I_n\}_{n=0}^{\infty}$ of ideals. We require, for each $n > 0$, that the nth ideal I_n be contained in the nth power of the Jacobson radical of R and that $I_{nk} \subseteq I_n^k$ for all $k, n \geq 0$. We call the associated completion a *multi-adic* completion and denote it by R^*. The basics of the multi-adic construction and the relationship between this completion and certain ideal-adic completions are considered in Section 2. In Section 3 we prove, for $\{I_n\}$ as just described, that R^* is Noetherian. Let $(R, \mathbf{m})$ be a Noetherian local ring. If R is excellent, Henselian, or universally catenary, we prove in Section 4 that R^* has the same property.

We were inspired to pursue this project partly because of our continuing interest in exploring completions and power series. In our previous work we constructed various examples of rings inside relatively well-understood rings such as the (x)-adic completion $k[y][[x]]$ of a polynomial ring $k[x, y]$ in two variables x and y over a field k [4; 5]. The examples we obtained demonstrate that certain properties of a ring may fail to extend to its $\mathbf{m}$-adic completion, where $\mathbf{m}$ is a maximal ideal [6].

The process of passing to completion gives an analytic flavor to algebra. Often we view completions in terms of power series or in terms of coherent sequences, as

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in [1, pp. 103–104]. Sometimes results are established by demonstrating for each n that they hold at the nth stage in the inverse limit.

Multi-adic completions are interesting from another point of view. In commutative algebra, many counterexamples can be considered as subrings of R^*/J, where R^* is a multi-adic completion of a localized polynomial ring R over a countable ground field and J is an ideal of R^*. In particular, certain counterexamples of Brodmann and Rotthaus, Heitmann, Nishimura, Ogoma, Rotthaus, and Weston can be interpreted in this way (cf. [2; 3; 7; 11; 12; 14; 15; 17; 18; 20]). For most of these examples, a particular enumeration $\{p_1, p_2, \dots\}$ of countably many nonassociate prime elements is chosen, and the ideals I_n are defined to be $I_n := (p_1 p_2 \cdots p_n)^n$. The Noetherian property in these examples is a trivial consequence of the fact that every ideal of R^* that contains one of the ideals I_n, or a power of I_n, is extended from R. In general, an advantage of R^* over the I_n-adic completion $\hat{R}_n$ is that an ideal of R^* is more likely to be extended from R than is an ideal of $\hat{R}_n$.

All rings we consider are assumed to be commutative with identity. A general reference for our notation and terminology is [9].

2. Basic Mechanics for the Multi-adic Construction

SETTING 2.1. Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$, and let $\mathbb{N}$ denote the set of positive integers. For each $n \in \mathbb{N}$, let Q_n be an ideal of R. Assume that the sequence $\{Q_n\}$ is descending (i.e., $Q_{n+1} \subseteq Q_n$) and that $Q_n \subseteq \mathcal{J}^n$ for each $n \in \mathbb{N}$. Also assume, for each pair of integers $k, n \in \mathbb{N}$, that $Q_{nk} \subseteq Q_n^k$.

Let $\mathcal{F} = \{Q_k\}_{k \geq 0}$ be the filtration

$$R = Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_k \supseteq Q_{k+1} \supseteq \cdots$$

of R, and define

$$R^* := \varprojlim_k R/Q_k \tag{2.1}$$

to be the completion of R with respect to $\mathcal{F}$.

Let $\hat{R} := \varprojlim_k R/\mathcal{J}^k$ denote the completion of R with respect to the powers of the Jacobson radical $\mathcal{J}$ of R, and for each $n \in \mathbb{N}$ let

$$\hat{R}_n := \varprojlim_k R/Q_n^k \tag{2.2}$$

denote the completion of R with respect to the powers of Q_n.

REMARK 2.2. Assume notation as in Setting 2.1. For each fixed $n \in \mathbb{N}$, we have

$$R^* = \varprojlim_k R/Q_k = \varprojlim_k R/Q_{nk},$$

where $k \in \mathbb{N}$ varies. This holds because the limit of a subsequence is the same as the limit of the original sequence.

Next we establish canonical inclusion relations among $\hat{R}$ and the completions defined in (2.1) and (2.2).

PROPOSITION 2.3. *Let the notation be as in Setting 2.1. For each $n \in \mathbb{N}$, we have canonical inclusions*

$$R \subseteq R^* \subseteq \hat{R}_n \subseteq \hat{R}_{n-1} \subseteq \cdots \subseteq \hat{R}_1 \subseteq \hat{R}.$$

Proof. The inclusion $R \subseteq R^*$ is clear because the intersection of the ideals Q_k is zero. For the inclusion $R^* \subseteq \hat{R}_n$, by Remark 2.2 we have $R^* = \varprojlim_k R/Q_{nk}$. Observe that

$$Q_{nk} \subseteq Q_n^k \subseteq Q_{n-1}^k \subseteq \cdots \subseteq \mathcal{J}^k.$$

Now Lemma 2.4 completes the proof of the proposition. □

The following lemma establishes injectivity in more generality for completions with respect to ideal filtrations (see also [13, Sec. 9.5]). Here the respective completions are defined using coherent sequences as in [1, pp. 103–104].

LEMMA 2.4. *Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$, and let $\{H_k\}_{k\in\mathbb{N}}$, $\{I_k\}_{k\in\mathbb{N}}$, and $\{L_k\}_{k\in\mathbb{N}}$ be descending sequences of ideals of R such that, for each $k \in \mathbb{N}$,*

$$L_k \subseteq I_k \subseteq H_k \subseteq \mathcal{J}^k.$$

We denote the families of natural surjections arising from these inclusions as

$$\delta_k \colon R/L_k \to R/I_k, \quad \lambda_k \colon R/I_k \to R/H_k, \quad \theta_k \colon R/H_k \to R/\mathcal{J}^k;$$

the completions with respect to these families are

$$\hat{R}_L := \varprojlim_k R/L_k, \qquad \hat{R}_I := \varprojlim_k R/I_k,$$

$$\hat{R}_H := \varprojlim_k R/H_k, \qquad \hat{R} := \varprojlim_k R/\mathcal{J}^k.$$

Then the following statements hold.

(1) *These families of surjections induce canonical injective maps Δ, Λ, and Θ among the completions, as shown in the diagram in* (2).

(2) *For each positive integer k we have the following commutative diagram, where the vertical maps are the natural surjections*:

$$\begin{tikzcd}
R/L_k \arrow[r, "\delta_k"] & R/I_k \arrow[r, "\lambda_k"] & R/H_k \arrow[r, "\theta_k"] & R/\mathcal{J}^k \\
\hat{R}_L \arrow[u] \arrow[r, "\Delta"] & \hat{R}_I \arrow[u] \arrow[r, "\Lambda"] & \hat{R}_H \arrow[u] \arrow[r, "\Theta"] & \hat{R}. \arrow[u]
\end{tikzcd}$$

(3) *The composition $\Lambda \cdot \Delta$ is the canonical map induced by the natural surjections $\lambda_k \cdot \delta_k \colon R/L_k \to R/H_k$. Similarly, the other compositions in the bottom row are the canonical maps induced by the appropriate natural surjections.*

Proof. In each case there is a unique homomorphism of the completions. For example, the family of homomorphisms $\{\delta_k\}_{k\in\mathbb{N}}$ induces a unique homomorphism

$$\hat{R}_L \xrightarrow{\ \Delta\ } \hat{R}_I. \tag{2.3}$$

To define Δ, let $x = (x_k)_{k\in\mathbb{N}} \in \hat{R}_L$, where each $x_k \in R/L_k$. Then $\delta_k(x_k) \in R/I_k$ and we define $\Delta(x) := (\delta_k(x_k))_{k\in\mathbb{N}} \in \hat{R}_I$.

To show that maps on the completions are injective, consider for example the map Δ. Suppose $x = (x_k)_{k\in\mathbb{N}} \in \varprojlim_k R/L_k$ with $\Delta(x) = 0$. Then $\delta_k(x_k) = 0$ in R/I_k; that is, $x_k \in I_k/L_k = I_k(R/L_k)$ for every $k \in \mathbb{N}$. For $v \in \mathbb{N}$, consider the commutative diagram

$$\begin{array}{ccc} R/L_k & \xrightarrow{\ \delta_k\ } & R/I_k \\ {\scriptstyle\beta_{k,kv}}\uparrow & & \uparrow{\scriptstyle\alpha_{k,kv}} \\ R/L_{kv} & \xrightarrow{\ \delta_{kv}\ } & R/I_{kv}, \end{array} \tag{2.4}$$

where $\beta_{k,kv}$ and $\alpha_{k,kv}$ are the canonical surjections associated with the inverse limits. We have $x_{kv} \in I_{kv}/L_{kv} = I_{kv}(R/L_{kv})$. Therefore,

$$x_k = \beta_{k,kv}(x_{kv}) \in I_{kv}(R/L_k) \subseteq \mathcal{J}^{kv}(R/L_k)$$

for every $v \in \mathbb{N}$. Since $\mathcal{J}(R/L_k)$ is contained in the Jacobson radical of R/L_k and since R/L_k is Noetherian, it follows that

$$\bigcap_{v\in\mathbb{N}} \mathcal{J}^{kv}(R/L_k) = (0).$$

Hence $x_k = 0$ for each $k \in \mathbb{N}$ and so Δ is injective. The remaining assertions are clear. □

Lemma 2.5. *With R^* and $\hat{R}_n$ as in Setting 2.1, we have*

$$R^* = \bigcap_{n\in\mathbb{N}} \hat{R}_n.$$

Proof. The "$\subseteq$" inclusion is shown in Proposition 2.3. For the reverse inclusion, fix positive integers n and k, and let $L_\ell = Q_{nk\ell}$, $I_\ell = Q_{nk}^\ell$, and $H_\ell = Q_n^\ell$ for each $\ell \in \mathbb{N}$. Then $L_\ell \subseteq I_\ell \subseteq H_\ell \subseteq \mathcal{J}^\ell$, as in Lemma 2.4, and

$$\hat{R}_L := \varprojlim_\ell R/Q_{nk\ell} = R^*, \quad \hat{R}_I := \varprojlim_\ell R/Q_{nk}^\ell = \hat{R}_{nk}, \quad \hat{R}_H := \varprojlim_\ell R/Q_n^\ell = \hat{R}_n.$$

(Also, as before, $\hat{R} := \varprojlim_\ell R/\mathcal{J}^\ell$.) We define φ_n, φ_{nk}, $\varphi_{nk,n}$, θ, and φ to be the canonical injective homomorphisms given by Lemma 2.4 among the rings displayed in the following diagram:

$$\begin{tikzcd} \hat{R} & \hat{R} \arrow[l, "\theta"'] \\ R^* \arrow[u, "\varphi"] \arrow[ur, "\varphi_n"] \arrow[r, "\varphi_{nk}"'] & \hat{R}_{nk}. \arrow[u, "\varphi_{nk,n}"'] \end{tikzcd} \tag{2.5}$$

By Lemma 2.4, this diagram is commutative.

Let $\hat{y} \in \bigcap_{n\in\mathbb{N}} \hat{R}_n$. We show that there is an element $\xi \in R^*$ such that $\varphi(\xi) = \hat{y}$. This is sufficient to ensure that $\hat{y} \in R^*$, since the maps θ_t are injective and diagram (2.5) is commutative.

First, we define ξ. For each $t \in \mathbb{N}$, we have

$$\hat{y} = (y_{1,t}, y_{2,t}, \dots) \in \varprojlim_{\ell} R/Q_t^{\ell} = \hat{R}_t,$$

where

$$y_{1,t} \in R/Q_t,\ \ y_{2,t} \in R/Q_t^2,\ \ y_{2,t} + Q_t/Q_t^2 = y_{1,t} \in R/Q_t, \dots$$

is a coherent sequence as in [1, pp. 103–104]. Now take $z_t \in R$ so that $z_t + Q_t = y_{1,t}$. Thus $\hat{y} - z_t \in Q_t\hat{R}_t$. For positive integers s and t with $s \ge t$, we have $Q_s \subseteq Q_t$. Therefore, $z_t - z_s \in Q_t\hat{R}_t \cap R = Q_tR$. Thus $\xi := (z_t)_{t\in\mathbb{N}} \in R^*$. We have $\hat{y} - z_t \in Q_t\hat{R}_t \subseteq \mathcal{J}^t\hat{R}$ for all $t \in \mathbb{N}$, so $\varphi(\xi) = \hat{y}$. This completes the proof of Lemma 2.5. □

The following special case of Setting 2.1 has been used by Brodmann, Rotthaus, Ogoma, Heitmann, Weston, and Nishimura in their construction of numerous examples.

SETTING 2.6. Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$. For each $i \in \mathbb{N}$, let $p_i \in \mathcal{J}$ be a nonzero divisor (i.e., a regular element) on R.

For each $n \in \mathbb{N}$, let $q_n = (p_1 \cdots p_n)^n$. Let $\mathcal{F}_0 = \{(q_k)\}_{k\ge 0}$ be the filtration

$$R \supseteq (q_1) \supseteq \cdots \supseteq (q_k) \supseteq (q_{k+1}) \supseteq \cdots$$

of R, and define $R^* := \varprojlim_k R/(q_k)$ to be the completion of R with respect to $\mathcal{F}_0$.

REMARK 2.7. In Setting 2.6, assume further that $R = K[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$, the localized polynomial ring over a countable field K, and that $\{p_1, p_2, \dots\}$ is an enumeration of all the prime elements (up to associates) in R. As in Setting 2.6, let $R^* := \varprojlim_n R/(q_n)$, where each $q_n = (p_1 \cdots p_n)^n$. The ring R^* is often useful for the construction of Noetherian local rings with a bad locus (regular, Cohen–Macaulay, and normal). In particular, the authors listed in the paragraph before Setting 2.6 make use of special subrings of this multi-adic completion R^* for their counterexamples; the first such example was constructed by Rotthaus in [16]. In this paper, we obtain a regular local Nagata ring A containing a prime element ω such that the singular locus of the quotient ring $A/(\omega)$ is not closed. This ring A is situated between the localized polynomial ring R and its $*$-completion R^*; thus, in general, R^* is bigger than R. In the Rotthaus example, the singular locus of $(A/(\omega))^*$ is defined by a height-1 prime ideal Q that intersects $A/(\omega)$ in (0). Since all ideals $Q + (p_n)$ are extended from $A/(\omega)$, the singular locus of $A/(\omega)$ is not closed.

3. Preserving Noetherian under Multi-adic Completion

THEOREM 3.1. *With notation as in Setting 2.1, the ring R^* defined in* (2.1) *is Noetherian.*

Proof. It suffices to show that each ideal I of R^* is finitely generated. Since $\hat{R}$ is Noetherian, there exist $f_1, \dots, f_s \in I$ such that $I\hat{R} = (f_1, \dots, f_s)\hat{R}$. Since $\hat{R}_n \hookrightarrow \hat{R}$ is faithfully flat, $I\hat{R}_n = I\hat{R} \cap \hat{R}_n = (f_1, \dots, f_s)\hat{R}_n$ for each $n \in \mathbb{N}$.

Let $f \in I \subseteq R^*$. Then $f \in I\hat{R}_1$ and so

$$f = \sum_{i=1}^{s} \hat{b}_{i0} f_i,$$

where $\hat{b}_{i0} \in \hat{R}_1$. Consider R as "Q_0", so $\hat{b}_{i0} \in Q_0\hat{R}_1$. Since $\hat{R}_1/Q_1\hat{R}_1 \cong R/Q_1$, for all i $(1 \le i \le s)$ we have $\hat{b}_{i0} = a_{i0} + \hat{c}_{i1}$, where $a_{i0} \in R = Q_0 R$ and $\hat{c}_{i1} \in Q_1\hat{R}_1$. Then

$$f = \sum_{i=1}^{s} a_{i0} f_i + \sum_{i=1}^{s} \hat{c}_{i1} f_i.$$

Notice that

$$\hat{d}_1 := \sum_{i=1}^{s} \hat{c}_{i1} f_i \in (Q_1 I)\hat{R}_1 \cap R^* \subseteq \hat{R}_2.$$

By the faithful flatness of the extension $\hat{R}_2 \hookrightarrow \hat{R}_1$, we see that $\hat{d}_1 \in (Q_1 I)\hat{R}_2$ and hence there exist $\hat{b}_{i1} \in Q_1\hat{R}_2$ with

$$\hat{d}_1 = \sum_{i=1}^{s} \hat{b}_{i1} f_i.$$

As before, we can use $\hat{R}_2/Q_2\hat{R}_2 \cong R/Q_2$ to write $\hat{b}_{i1} = a_{i1} + \hat{c}_{i2}$, where $a_{i1} \in R$ and $\hat{c}_{i2} \in Q_2\hat{R}_2$. This implies that $a_{i1} \in Q_1\hat{R}_2 \cap R = Q_1$. We have

$$f = \sum_{i=1}^{s} (a_{i0} + a_{i1}) f_i + \sum_{i=1}^{s} \hat{c}_{i2} f_i.$$

Now set

$$\hat{d}_2 := \sum_{i=1}^{s} \hat{c}_{i2} f_i.$$

Then $\hat{d}_2 \in (Q_2 I)\hat{R}_2 \cap R^* \subseteq \hat{R}_3$ and, since the extension $\hat{R}_3 \hookrightarrow \hat{R}_2$ is faithfully flat, $\hat{d}_2 \in (Q_2 I)\hat{R}_3$. We repeat the process. By a simple induction argument,

$$f = \sum_{i=1}^{s} (a_{i0} + a_{i1} + a_{i2} + \cdots) f_i,$$

where $a_{ij} \in Q_j$ and $a_{i0} + a_{i1} + a_{i2} + \cdots \in R^*$. Thus $f \in (f_1, \dots, f_s)R^*$. Hence I is finitely generated and R^* is Noetherian. □

COROLLARY 3.2. *With notation as in Setting 2.1, the maps $R \hookrightarrow R^*$, $R^* \hookrightarrow \hat{R}_n$, and $R^* \hookrightarrow \hat{R}$ are faithfully flat.*

We use Proposition 3.3 in the next section on preserving excellence.

PROPOSITION 3.3. *Assume notation as in Setting 2.1, and let the ring R^* be defined as in* (2.1). *If M is a finitely generated R^*-module, then*

$$M \cong \varprojlim_k (M/Q_k M);$$

in other words, M is $*$-complete.

Proof. If $F = (R^*)^n$ is a finitely generated free R^*-module, then one can see directly that

$$F \cong \varprojlim_k F/Q_k F$$

and so F is $*$-complete.

Let M be a finitely generated R^*-module. Consider an exact sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is a finitely generated free R^*-module. This induces an exact sequence

$$0 \longrightarrow \tilde{N} \longrightarrow F^* \longrightarrow M^* \longrightarrow 0,$$

where $\tilde{N}$ is the completion of N with respect to the induced filtration $\{Q_k F \cap N\}_{k \geq 0}$ (cf. [1, (10.3)]).

This gives the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "\cong"'] & M \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \tilde{N} \arrow[r] & F^* \arrow[r] & M^* \arrow[r] & 0,
\end{tikzcd}$$

where γ is the canonical map $\gamma : M \to M^*$. The diagram shows that γ is surjective. We have

$$\bigcap_{k=1}^{\infty} (Q_k M) \subseteq \bigcap_{k=1}^{\infty} J^k M = (0),$$

where the last equality is by [1, (10.19)]. Therefore, γ is also injective. $\square$

REMARK 3.4. Let the notation be as in Setting 2.1, and let B be a finite R^*-algebra. Let $\hat{B}_n \cong B \otimes_{R^*} \hat{R}_n$ denote the Q_n-adic completion of B. By Proposition 2.3 and Corollary 3.2, we have a sequence of inclusions

$$B \hookrightarrow \cdots \hookrightarrow \hat{B}_{n+1} \hookrightarrow \hat{B}_n \hookrightarrow \cdots \hookrightarrow \hat{B}_1 \hookrightarrow \hat{B},$$

where $\hat{B}$ denotes the completion of B with respect to $\mathcal{J}B$. Let $\mathcal{J}_0$ denote the Jacobson radical of B. Since every maximal ideal of B lies over a maximal ideal of R^*, we have $\mathcal{J}B \subseteq \mathcal{J}_0$.

THEOREM 3.5. *With the notation of Setting 2.1, let B be a finite R^*-algebra and let $\hat{B}_n \cong B \otimes_{R^*} \hat{R}_n$ denote the Q_n-adic completion of B. Let $\hat{I}$ be an ideal of $\hat{B}$, let $I := \hat{I} \cap B$, and let $I_n := \hat{I} \cap \hat{B}_n$ for each $n \in \mathbb{N}$. If $\hat{I} = I_n \hat{B}$ for all n, then $\hat{I} = I\hat{B}$.*

Proof. By replacing B with B/I, we may assume that $(0) = I = \hat{I} \cap B$. To prove the theorem, it suffices to show that $\hat{I} = 0$.

For each $n \in \mathbb{N}$, we define ideals $\mathfrak{c}_n$ of $\hat{B}_n$ and $\mathfrak{a}_n$ of B as

$$\mathfrak{c}_n := I_n + Q_n \hat{B}_n, \qquad \mathfrak{a}_n := \mathfrak{c}_n \cap B.$$

Since $B/Q_n B = \hat{B}_n/Q_n \hat{B}_n$, it follows that the ideals of B containing Q_n are in one-to-one inclusion-preserving correspondence with the ideals of $\hat{B}_n$ containing $Q_n \hat{B}_n$. Therefore,

$$\mathfrak{a}_n \hat{B}_n = \mathfrak{c}_n, \qquad \mathfrak{a}_{n+1} \hat{B}_n = \mathfrak{a}_{n+1} \hat{B}_{n+1} \hat{B}_n = \mathfrak{c}_{n+1} \hat{B}_n. \tag{3.1}$$

Since $\hat{B}$ is faithfully flat over $\hat{B}_n$ and since $\hat{I}$ is extended, we have

$$I_{n+1} \hat{B}_n = (I_{n+1} \hat{B}) \cap \hat{B}_n = \hat{I} \cap \hat{B}_n = I_n. \tag{3.2}$$

For all $n \in \mathbb{N}$ it thus follows by (3.1), (3.2), and $Q_{n+1} \hat{B}_n \subseteq Q_n \hat{B}_n$ that

$$\begin{aligned} \mathfrak{a}_n \hat{B}_n = \mathfrak{c}_n &= I_n + Q_n \hat{B}_n = I_{n+1} \hat{B}_n + Q_n \hat{B}_n \\ &= \mathfrak{c}_{n+1} \hat{B}_n + Q_n \hat{B}_n = \mathfrak{a}_{n+1} \hat{B}_n + Q_n \hat{B}_n. \end{aligned}$$

Because $\hat{B}_n$ is faithfully flat over B, the preceding equation implies that

$$\mathfrak{a}_{n+1} + Q_n B = (\mathfrak{a}_{n+1} \hat{B}_n + Q_n \hat{B}_n) \cap B = \mathfrak{a}_n \hat{B}_n \cap B = \mathfrak{a}_n. \tag{3.3}$$

Thus also

$$\mathfrak{a}_n \hat{B} \subseteq \mathfrak{a}_{n+1} \hat{B} + Q_n \hat{B} \subseteq I_{n+1} \hat{B} + Q_n \hat{B} = \hat{I} + Q_n \hat{B}. \tag{3.4}$$

Now $Q_n \subseteq \mathcal{J}^n \hat{B}$ and $\mathcal{J} \subseteq \mathcal{J}_0$; hence using (3.4) we obtain

$$\bigcap_{n \in \mathbb{N}} (\mathfrak{a}_n \hat{B}) \subseteq \bigcap_{n \in \mathbb{N}} (\hat{I} + Q_n \hat{B}) \subseteq \bigcap_{n \in \mathbb{N}} (\hat{I} + \mathcal{J}^n \hat{B}) = \hat{I}.$$

Since $\hat{I} \cap B = (0)$, it follows that

$$0 = \hat{I} \cap B \supseteq \left(\bigcap_{n \in \mathbb{N}} (\mathfrak{a}_n \hat{B}) \right) \cap B \supseteq \bigcap_{n \in \mathbb{N}} ((\mathfrak{a}_n \hat{B}) \cap B) = \bigcap_{n \in \mathbb{N}} \mathfrak{a}_n,$$

where the last equality is because $\hat{B}$ is faithfully flat over B. Thus $\bigcap_{n \in \mathbb{N}} \mathfrak{a}_n = (0)$.

Claim: $\hat{I} = (0)$.

Proof of Claim. Suppose $\hat{I} \neq 0$. Then there exists a $d \in \mathbb{N}$ such that $\hat{I} \not\subseteq \mathcal{J}_0^d \hat{B}$. By hypothesis, $\hat{I} = I_d \hat{B}$ and so $I_d \hat{B} \not\subseteq \mathcal{J}_0^d \hat{B}$. Since $\hat{B}$ is faithfully flat over $\hat{B}_d$, we have $I_d \not\subseteq \mathcal{J}_0^d \hat{B}_d$. By (3.1),

$$\mathfrak{a}_d \hat{B}_d = \mathfrak{c}_d = I_d + Q_d \hat{B}_d \not\subseteq \mathcal{J}_0^d \hat{B}_d,$$

so there must exist an element $y_d \in \mathfrak{a}_d$ with $y_d \notin \mathcal{J}_0^d$.

By (3.3), $\mathfrak{a}_{d+1} + Q_d B = \mathfrak{a}_d$. Hence there exist $y_{d+1} \in \mathfrak{a}_{d+1}$ and $q_d \in Q_d B$ such that $y_{d+1} + q_d = y_d$. Recursively we construct, for each $n \geq d$, sequences of elements $y_n \in \mathfrak{a}_n$ and $q_n \in Q_n B$ such that $y_{n+1} + q_n = y_n$.

The sequence $\xi = (y_n + Q_n B) \in \varprojlim_n B/Q_n B = B$ corresponds to a nonzero element $y \in B$ such that, for every $n \geq d$, we have $y = y_n + g_n$ for some element

$g_n \in Q_n B$. This shows that $y \in \mathfrak{a}_n$ for all $n \geq d$ and hence $\bigcap_{n\in\mathbb{N}} \mathfrak{a}_n \neq 0$, a contradiction. Thus $\hat{I} = (0)$. □

4. Preserving Excellence and Henselian under Multi-adic Completion

The first four results of this section concern preservation of excellence.

THEOREM 4.1. *Assume notation as in Setting 2.1, and let the ring R^* be defined as in* (2.1). *If* $(R, \mathbf{m})$ *is an excellent local ring, then R^* is excellent.*

The following result is critical to the proof of Theorem 4.1.

LEMMA 4.2 [9, Thm. 32.5, p. 259]. *Let A be a semilocal Noetherian ring. Assume that $(\hat{B})_Q$ is a regular local ring for every local domain $(B, \mathbf{n})$ that is a localization of a finite A-algebra and for every prime ideal Q of the $\mathbf{n}$-adic completion $\hat{B}$ such that $Q \cap B = (0)$. Then A is a G-ring; that is, $A \hookrightarrow \hat{A}_p$ is regular for every prime ideal p of A. Thus all of the formal fibers of all the local rings of A are geometrically regular.*

We use also Proposition 4.3 in the proof of Theorem 4.1.

PROPOSITION 4.3. *Let $(R, \mathbf{m})$ be a Noetherian local ring with geometrically regular formal fibers. Then R^* has geometrically regular formal fibers.*

Proof. Let B be a domain that is a finite R^*-algebra, and let $P \in \operatorname{Sing}(\hat{B})$ (i.e., $\hat{B}_P$ is not a regular local ring). To prove that R^* has geometrically regular formal fibers, by Lemma 4.2 it suffices to prove that $P \cap B \neq (0)$.

The Noetherian complete local ring $\hat{R}$ has the property J-2 in the sense of Matsumura: for every finite $\hat{R}$-algebra, such as $\hat{B}$, the subset $\operatorname{Reg}(\operatorname{Spec}(\hat{B}))$ of primes where the localization of $\hat{B}$ is regular is an open subset (cf. [8, pp. 246–249]). Hence there is a radical ideal $\hat{I}$ in $\hat{B}$ such that

$$\operatorname{Sing}(\hat{B}) = \mathcal{V}(\hat{I}).$$

If $\hat{I} = (0)$ (i.e., if (0) is a radical ideal), then $\hat{B}$ is a reduced ring and, for all minimal primes Q of $\hat{B}$, the localization $\hat{B}_Q$ is a field, contradicting $Q \in \operatorname{Sing}(\hat{B})$. Thus $\hat{I} \neq (0)$. For all $n \in \mathbb{N}$,

$$\hat{B}_n \cong \hat{R}_n \otimes_{R^*} B$$

is a finite $\hat{R}_n$-algebra. Because $\hat{R}_n$ has geometrically regular formal fibers [18], so has $\hat{B}_n$. This implies that $\hat{I}$ is extended from $\hat{B}_n$ for all $n \in \mathbb{N}$. By Theorem 3.5, $\hat{I}$ is extended from B and so $\hat{I} = I\hat{B}$, where $0 \neq I := \hat{I} \cap B$. Since $\hat{I} \subseteq P$, we have $(0) \neq I \subseteq P \cap B$. □

Proof of Theorem 4.1. It remains to show that R^* is universally catenary. We have injective local homomorphisms $R \hookrightarrow R^* \hookrightarrow \hat{R}$ and that R^* is Noetherian with $\hat{R^*} = \hat{R}$. Proposition 4.4 then implies that R^* is universally catenary. □

Proposition 4.4. *Let* $(A, \mathbf{m})$ *be a Noetherian local universally catenary ring, and let* $(B, \mathbf{n})$ *be a Noetherian local subring of the* $\mathbf{m}$*-adic completion* $\hat{A}$ *of* A *with* $A \subseteq B \subseteq \hat{A}$ *and* $\hat{B} = \hat{A}$*, where* $\hat{B}$ *is the* $\mathbf{n}$*-adic completion of* B*. Then* B *is universally catenary.*

Proof. By [9, Thm. 31.7], it suffices to show for $P \in \operatorname{Spec}(B)$ that $\hat{A}/P\hat{A}$ is equidimensional. We may assume that $P \cap A = (0)$ and hence that A is a domain. Let Q and W in $\operatorname{Spec}(\hat{A})$ be minimal primes over $P\hat{A}$.

Claim: $\dim(\hat{A}/Q) = \dim(\hat{A}/W)$.

Proof of Claim. Since B is Noetherian, the canonical morphisms $B_P \to \hat{A}_Q$ and $B_P \to \hat{A}_W$ are flat. By [9, Thm. 15.1],

$$\dim(\hat{A}_Q) = \dim(B_P) + \dim(\hat{A}_Q/P\hat{A}_Q),$$
$$\dim(\hat{A}_W) = \dim(B_P) + \dim(\hat{A}_W/P\hat{A}_W).$$

Since Q and W are minimal over $P\hat{A}$, it follows that

$$\dim(\hat{A}_Q) = \dim(\hat{A}_W) = \dim(B_P).$$

Let $q \subseteq Q$ and $w \subseteq W$ be minimal primes of $\hat{A}$ such that

$$\dim(\hat{A}_Q) = \dim(\hat{A}_Q/q\hat{A}_Q), \qquad \dim(\hat{A}_W) = \dim(\hat{A}_W/w\hat{A}_W).$$

Because we have reduced to the case where A is a universally catenary domain, its completion $\hat{A}$ is equidimensional and so

$$\dim(\hat{A}/q) = \dim(\hat{A}/w).$$

Since a complete local ring is catenary [9, Thm. 29.4], we have

$$\dim(\hat{A}/q) = \dim(\hat{A}_Q/q\hat{A}_Q) + \dim(\hat{A}/Q),$$
$$\dim(\hat{A}/w) = \dim(\hat{A}_W/w\hat{A}_W) + \dim(\hat{A}/W).$$

Since $\dim(\hat{A}/q) = \dim(\hat{A}/w)$ and $\dim(\hat{A}_Q) = \dim(\hat{A}_W)$, it follows that

$$\dim(\hat{A}/Q) = \dim(\hat{A}/W).$$

This completes the proof of Proposition 4.4. □

Remark 4.5. Let R be a universally catenary Noetherian local ring. Proposition 4.4 implies that *every* Noetherian local subring B of $\hat{R}$ with $R \subseteq B$ and $\hat{B} = \hat{R}$ is universally catenary. Hence, for each ideal I of R, the I-adic completion of R is universally catenary. Moreover, R^* as in Setting 2.1 is universally catenary. Proposition 4.4 also implies that the Henselization of R is universally catenary. Seydi [19] shows that the I-adic completions of universally catenary rings are universally catenary. Proposition 4.4 establishes this result for a larger class of rings.

Proposition 4.6. *With notation as in Setting 2.1, let* $(R, \mathbf{m}, k)$ *be a Noetherian local ring. If* R *is Henselian, then* R^* *is Henselian.*

Proof. Assume that R is Henselian. It is well known that every ideal-adic completion of R is Henselian (cf. [16, p. 6]); thus $\hat{R}_n$ is Henselian for all $n \in \mathbb{N}$. Let $\mathbf{n}$ denote the nilradical of $\hat{R}$. Then $\mathbf{n} \cap R^*$ is the nilradical of R^* and, to prove R^* is Henselian, it suffices to prove that $R' := R^*/(\mathbf{n} \cap R^*)$ is Henselian [10, (43.15)]. To prove R' is Henselian, by [16, Prop. 3, p. 76] it suffices to show:

> *If $f \in R'[x]$ is a monic polynomial and its image $\bar{f} \in k[x]$ has a simple root, then f has a root in R'.*

Let $f \in R'[x]$ be a monic polynomial such that $\bar{f} \in k[x]$ has a simple root. Because $\hat{R}_n/(\mathbf{n} \cap \hat{R}_n)$ is Henselian, for each $n \in \mathbb{N}$ there exist $\hat{\alpha}_n \in \hat{R}_n/(\mathbf{n} \cap \hat{R}_n)$ with $f(\hat{\alpha}_n) = 0$. Since f is monic and $\hat{R}/(\mathbf{n} \cap \hat{R})$ is reduced, f has only finitely many roots in $\hat{R}/(\mathbf{n} \cap \hat{R})$. Hence there is an α such that $\alpha = \hat{\alpha}_n$ for infinitely many $n \in \mathbb{N}$. By Lemma 2.5, $R^* = \bigcap_{n \in \mathbb{N}} \hat{R}_n$. Therefore,

$$R' = R^*/(\mathbf{n} \cap R^*) = \bigcap_{n \in \mathbb{N}} \hat{R}_n/(\mathbf{n} \cap \hat{R}_n)$$

and so there exists an $\alpha \in R'$ such that $f(\alpha) = 0$. □

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