Michigan Math. J. 57 (2008)

Structure Theorems for Certain Gorenstein Ideals

Juan Elias & Giuseppe Valla

Dedicated to Melvin Hochster, on the occasion of his sixty-fifth birthday

1. Introduction

Let I be an ideal in the regular local ring $(R, \mathfrak{n})$ such that $I \subseteq \mathfrak{n}^2$, and let

$$A := R/I, \quad \mathfrak{m} := \mathfrak{n}/I, \quad \mathbf{k} := R/\mathfrak{n} = A/\mathfrak{m}.$$

Let $d = \dim(A)$ be the dimension, e the multiplicity, and $h = v(\mathfrak{m}) - d$ the embedding codimension of A. We assume that $\mathbf{k}$ is a field of characteristic 0 (see the comment after Proposition 2.3).

A classical problem in the theory of local rings is the determination of the minimal number of generators $v(I) := \dim_k(I/\mathfrak{n}I)$ of the ideal I under certain restrictions on the numerical characters of A. For example, by a classical theorem of Abhyankar we know that $e \geq h + 1$, and if the equality $e = h + 1$ holds then we say that A has *minimal multiplicity* and we know that $v(I) = \binom{h+1}{2}$.

In a sequence of papers, Rosales and García-Sánchez proved the following results for A the one-dimensional local domain corresponding to a monomial curve in the affine space (see [4; 5; 6]). By difficult computations related to the numerical semigroup of the curve, they were able to prove the following: if $h + 2 \leq e \leq h + 3$, then

$$\binom{h+2}{2} - e \leq v(I) \leq \binom{h+1}{2}; \tag{1}$$

if $h + 2 \leq e \leq h + 4$ and A is Gorenstein, then

$$v(I) = \binom{h+1}{2} - 1. \tag{2}$$

We remark that the monomial curve $\{t^8 : t^{10} : t^{12} : t^{15}\}$ shows that (2) does not hold if $e = h + 5$ (see [6]).

On the other hand, the monomial curve $\{t^7 : t^8 : t^{10} : t^{19}\}$ shows that the upper bound in (1) does not hold if $e = h + 4$. In the same paper it is asked whether

$$\binom{h+2}{2} - e = \binom{h+1}{2} - 3 \leq v(I) \leq \binom{h+1}{2} + 1 \tag{3}$$

holds for $e = h + 4$.

Received June 20, 2007. Revision received September 13, 2007.

J.E. was partially supported by MTM2007-67493. G.V. was partially supported by the Consiglio Nazionale delle Ricerche (CNR).

A first motivation for our paper was to understand these results and to extend them to the general case of a local Cohen–Macaulay ring of any dimension.

A sharp upper bound for the minimal number of generators of a perfect ideal I in a regular local ring R has been given in [2] in terms of the multiplicity e and of the codimension h of R/I. The bound is

$$v(I) \leq \binom{h+t-1}{t} - r + r^{\langle t \rangle},$$

where the meaning of r, t, and $r^{\langle t \rangle}$ will be explained in the Section 2. In the same section we will also prove that

$$\binom{h+2}{2} - e \leq v(I)$$

holds for every perfect codimension-h ideal I in a regular local ring R; see Proposition 2.2. We will also see how these bounds extend (1) to a considerable extent and positively answer question (3) in a general setting.

For (2), the problem is much harder. We have a Gorenstein local ring $(A = R/I, \mathfrak{m} = \mathfrak{n}/I)$ of codimension h and multiplicity $h+2 \leq e \leq h+4$, and we want to determine the minimal number of generators of I. It is easy to see that we may assume $A = R/I$ is Artinian; then, since A is Gorenstein, the possible Hilbert functions of R/I are

$$(1, h, 1), (1, h, 1, 1), (1, h, 2, 1), (1, h, 1, 1, 1).$$

Hence, in any case, $v(\mathfrak{m}^2) \leq 2$.

Following Sally [8] we say that an Artinian local ring $(A, \mathfrak{m})$, not necessarily Gorenstein, is *stretched* if $v(\mathfrak{m}^2) = 1$. We call *almost stretched* an Artinian local ring such that $v(\mathfrak{m}^2) = 2$.

With this notation, we can strongly extend (2) proving that, if R/I is Gorenstein, stretched, or almost stretched of multiplicity e and codimension h, then $v(I) = \binom{h+1}{2} - 1$.

By the classical theorem of Macaulay on the shape of the Hilbert function of a standard graded algebra, the Hilbert function of A is given by

0	1	2	...	s	$s+1$
1	h	1	...	1	0

(with $s \geq 2$) if A is stretched or by

0	1	2	...	t	$t+1$	...	s	$s+1$
1	h	2	...	2	1	...	1	0

(with $s \geq t \geq 2$) if A is almost stretched.

The particular shape of the Hilbert function can be used to prove that

$$\binom{h+1}{2} - 1 \leq v(I) \leq \binom{h+1}{2} \quad \text{if } A \text{ is stretched,} \quad \text{and}$$

$$\binom{h+1}{2} - 2 \leq v(I) \leq \binom{h+1}{2} \quad \text{if } A \text{ is almost stretched.}$$

The case of a stretched Artinian Gorenstein local ring was studied by Sally [8], who was able to prove a structure theorem for the corresponding ideals (see also [7]). We extend this result to the case of stretched Artinian local rings of any Cohen–Macaulay type. But the unexpected and deeper result that we prove in this paper is a structure theorem for *any almost stretched Gorenstein local ring.*

These results are proved as Theorem 4.1, respectively. As a consequence, we get even more of what we wanted: if A is stretched, then $v(I) = \binom{h+1}{2} - 1$ if $\tau(A) < h$ and $v(I) = \binom{h+1}{2}$ otherwise; if A is almost stretched and Gorenstein, then $v(I) = \binom{h+1}{2} - 1$.

Another motivation for our paper came from a recent work by Casnati and Notari [1]. Let $\mathcal{H}ilb_{p(t)}(\mathbb{P}^n_k)$ denote the Hilbert scheme parameterizing closed subschemes in $\mathbb{P}^n_k$ with given Hilbert polynomial $p(t) \in \mathbb{Q}[t]$.

The case $\deg(p(t)) = 0$ is often problematic. Since it is known that any zero-dimensional Gorenstein scheme of degree d can be embedded as an arithmetically Gorenstein nondegenerate subscheme in $\mathbb{P}^{d-2}_k$, it is natural to study the open locus

$$\mathcal{H}ilb^{aG}_d(\mathbb{P}^{d-2}_k) \subseteq \mathcal{H}ilb_d(\mathbb{P}^{d-2}_k).$$

The scheme $\mathcal{H}ilb^{aG}_d(\mathbb{P}^{d-2}_k)$ has a natural stratification that reduces the problem to understanding the intrinsic structure of Artinian Gorenstein **k**-algebras of degree d. Because such an algebra is the direct sum of local, Artinian, Gorenstein **k**-algebras of degree at most d, it is natural to begin with the inspection of these elementary bricks.

If $d = 6$ then the bricks are all given by stretched local rings—excepting the Hilbert function $(1, 2, 2, 1)$, which is almost stretched and was studied deeply by Casnati and Notari.

Extending these results to the case $d \geq 7$ would begin with studying the intrinsic structure of Artinian Gorenstein local algebras with multiplicity 7. Since the Hilbert function $(1, 2, 3, 1)$ is not allowed, an Artinian Gorenstein ring $(A, \mathfrak{m})$ with multiplicity 7 is stretched or almost stretched. (See [3] for more results on the classification of Artinian algebras.) Hence, the structure theorems we prove here will shed light on these questions, too.

It would clearly be best to have a classification up to isomorphisms of Artinian Gorenstein k-algebras of a given Hilbert function, at least in the almost stretched case. We approach this difficult problem in the last part of the paper, where we give a classification of Artinian complete intersection local k-algebras with Hilbert function $(1, 2, 2, 2, 1, 1, 1)$. This example is significant because the parameter space has a one-dimensional component.

2. Upper and Lower Bounds for $v(I)$

Let $(R, \mathfrak{n})$ be a regular local ring and I an ideal in R. We assume that $(A = R/I, \mathfrak{m} = \mathfrak{n}/I)$ has dimension d, embedding codimension h, and multiplicity e. We denote by H_A the Hilbert function of A:

$$H_A(n) := \dim_{\mathbf{k}}\left(\frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}}\right)$$

for $n \geq 0$. The socle degree of an Artinian ring A is the last integer $s = s(A)$ such that $H_A(s) \neq 0$; the Cohen–Macaulay type of A is

$$\tau(A) := \dim_{\mathbf{k}}(0 : \mathfrak{m}).$$

A sharp upper bound for $v(I)$ can be given by using the notion of lex-segment ideal as in [2]. We recall that the associated graded ring of A can be presented as $\mathrm{gr}_{\mathfrak{m}}(A) = \mathrm{gr}_{\mathfrak{n}}(R)/I^*$, where I^* is the ideal generated by the $\mathfrak{n}$-initial forms of I in the polynomial ring $S = \mathrm{gr}_{\mathfrak{n}}(R)$. This implies that the Hilbert function of $A = R/I$ is the same as the Hilbert function of the standard graded algebra S/I^*.

A set of elements in I whose $\mathfrak{n}$-initial forms generate I^* is called a *standard basis* of I. It is easy to see that a standard basis is a basis, so we have $v(I) \leq v(I^*)$.

On the other hand, by a classical result of Macaulay, any homogeneous ideal P in the polynomial ring $S = k[X_1, \dots, X_n]$ has the following property: The number of minimal generators of P is less than or equal to the number of minimal generators of the unique lex-segment ideal P_{lex}, which has the same Hilbert function of P.

Hence, given the ideal I in the regular local ring $(R, \mathfrak{n})$ and the corresponding lex-segment ideal $I_{\mathrm{lex}} := (I^*)_{\mathrm{lex}}$ in $S := \mathrm{gr}_{\mathfrak{n}}(R)$, we have

$$v(I) \leq v(I^*) \leq v(I_{\mathrm{lex}}). \tag{4}$$

More difficult is obtaining a bound involving only the multiplicity and the codimension. In particular, one must compare the number of generators of all the lex-segment ideals having the given multiplicity and codimension. This has been done in [2], where the following bound was proved.

If n and i are positive integers then n can be uniquely written as

$$n = \binom{n(i)}{i} + \binom{n(i-1)}{i-1} + \dots + \binom{n(j)}{j},$$

where $n(i) > n(i-1) > \dots > n(j) \geq j \geq 1$. This is called the *$i$-binomial expansion* of n. We let

$$n^{\langle i \rangle} := \binom{n(i)+1}{i+1} + \binom{n(i-1)+1}{i} + \dots + \binom{n(j)+1}{j+1}.$$

Given two positive integers e, h with $e \geq h+1$, we define t as the unique integer such that

$$\binom{h+t-1}{t-1} \leq e < \binom{h+t}{t}$$

and

$$r := e - \binom{h+t-1}{t-1}.$$

The main result in [2] shows that, for every perfect codimension-h ideal I in the regular local ring R with $I \subseteq \mathfrak{n}^2$ and $e(R/I) = e$,

$$v(I) \leq \binom{h+t-1}{t} - r + r^{\langle t \rangle}. \tag{5}$$

For example, if $h \geq 3$ and $e = h + 2$, then $t = 2$, $r = 1$ and we have $v(I) \leq \binom{h+1}{2}$. The same bound holds also for $e = h + 3$; see (1).

If, instead, $e = h + 4$, then $t = 2$, $r = 3$, and

$$v(I) \leq \binom{h+1}{2} - 3 + 3^{\langle 2 \rangle} = \binom{h+1}{2} - 3 + 4 = \binom{h+1}{2} + 1;$$

see (3). The same bound holds also for $e = h + 5$.

A lower bound for $v(I)$ is obtained from the following easy lemma.

LEMMA 2.1. *Let $A = R/I$ be a local Artinian ring with multiplicity e and embedding codimension h. We assume that $I \subseteq \mathfrak{n}^2$. Then*

$$\binom{h+2}{2} - e \leq \binom{h+1}{2} - v(\mathfrak{m}^2) \leq v(I).$$

Proof. It is clear that the kernel of the epimorphism

$$\mathfrak{n}^2/\mathfrak{n}^3 \to \mathfrak{m}^2/\mathfrak{m}^3 = (\mathfrak{n}^2 + I)/(\mathfrak{n}^3 + I) \to 0$$

is $(\mathfrak{n}^3 + I)/\mathfrak{n}^3 \cong I/(\mathfrak{n}^3 \cap I)$. Since $I\mathfrak{n} \subseteq \mathfrak{n}^3 \cap I$, we have

$$v(\mathfrak{n}^2) - v(\mathfrak{m}^2) = \binom{h+1}{2} - v(\mathfrak{m}^2) \leq v(I).$$

Now observe that we have $e = \sum_{i=0}^{s} v(\mathfrak{m}^i)$, where s is the socle degree of A, so that $e \geq 1 + h + v(\mathfrak{m}^2)$ and

$$\binom{h+2}{2} - e \leq \binom{h+2}{2} - (1 + h + v(\mathfrak{m}^2)) = \binom{h+1}{2} - v(\mathfrak{m}^2). \qquad \square$$

As a consequence of this lemma we derive a lower bound for the number of generators of perfect ideals in a regular local ring, a bound that seems to be useful at least for low multiplicity.

PROPOSITION 2.2. *Let $A = R/I$ be a local Cohen–Macaulay ring with dimension d, multiplicity e, and embedding codimension h. Assume that $I \subseteq \mathfrak{n}^2$. Then*

$$\binom{h+2}{2} - e \leq v(I) \leq \binom{h+t-1}{t} - r + r^{\langle t \rangle}.$$

Proof. Let $J = (x_1, \ldots, x_d)$ be a maximal $\mathfrak{n}$-superficial sequence for A. Because A is Cohen–Macaulay, $x_1, \ldots, x_d$ is a regular sequence modulo I and so $I \cap J = IJ$. Let

$$\bar{I} = (I + J)/J, \qquad \bar{R} = R/J,$$

$$\bar{A} = A/(x_1, \ldots, x_d)A = \bar{R}/\bar{I}, \qquad \bar{\mathfrak{m}} = \mathfrak{m}/J.$$

Then

$$v(\bar{I}) = \dim_k(I + J/\mathfrak{n}I + J) = \dim_k(I/\mathfrak{n}I + I \cap J) = \dim_k(I/\mathfrak{n}I) = v(I).$$

We know also that the multiplicity of A is the same as the multiplicity of the Artinian local ring $A/(x_1, \ldots, x_d)A$. Finally, I and $\bar{I}$ share the same embedding codimension because $h = v(\mathfrak{m}) - d = v(\bar{\mathfrak{m}})$. The lower bound now follows from Lemma 2.1, and the upper bound is given by (5). $\square$

In Section 3 we shall establish structure theorems for stretched local rings and for almost stretched Gorenstein local rings. One of the main ingredients is the following result, which will be used several times later and is reminiscent of the *lean basis* notion introduced by Sally [8].

In the proof of the following proposition, we need to know that *if the characteristic of* $\mathbf{k}$ *is* 0, *then a Borel fixed monomial ideal* K *is strongly stable*. This means that K satisfies the following requirement: For any term $M \in K$ and any indeterminate X_j dividing M, we have $X_i(M/X_j) \in K$ for all $1 \le i < j$.

PROPOSITION 2.3. *Let* $(A, \mathfrak{m})$ *be an Artinian local ring of embedding dimension* h *and socle degree* s *such that the characteristic of the residue field* $\mathbf{k}$ *is* 0 *and* $v(\mathfrak{m}^2) \le 2$. *Then we can find a minimal basis* $x_1, \dots, x_h$ *of* $\mathfrak{m}$ *such that*

$$\mathfrak{m}^j = (x_h^j) \quad \text{for } j = 2, \dots, s$$

if A is stretched and such that

$$\mathfrak{m}^j = \begin{cases} (x_h^j, x_h^{j-1}x_{h-1}) & \text{for } j = 2, \dots, t, \\ (x_h^j) & \text{for } j = t+1, \dots, s \end{cases}$$

if A is almost stretched.

Proof. We prove the proposition for A almost stretched because the other case is easier. Let $\mathfrak{m} = (a_1, \dots, a_h)$; we know that the Hilbert function of A is the same as the Hilbert function of $\mathrm{gr}_{\mathfrak{m}}(A) = k[\xi_1, \dots, \xi_h] = S/J$, where $\xi_i := \overline{a_i} \in \mathfrak{m}/\mathfrak{m}^2$, $S = k[X_1, \dots, X_h]$, and J is an homogeneous ideal of S. Moreover, the generic initial ideal $\mathrm{gin}(J)$ of J is a Borel fixed monomial ideal, which is then strongly stable.

We claim that, after a suitable change of coordinates in S that corresponds to a change of generators for the maximal ideal $\mathfrak{m}$ of A, we may assume that a basis for S_j modulo $\mathrm{gin}(J)_j$ is given by $X_h^j, X_h^{j-1}X_{h-1}$ for $j = 2, \dots, t$ and by X_h^j for $j = t+1, \dots, s$.

In order to prove this claim, we need only remark that if a monomial ideal K is strongly stable and $K_j \ne S_j$ then $X_h^j \notin K_j$, and if $\dim_k(S_j/K_j) \ge 2$ then $X_h^{j-1}X_{h-1} \notin K_j$. Since $\mathrm{gin}(J)$ is an initial ideal, the same monomials form a basis also for S modulo J. The conclusion follows because, for every $j \ge 0$,

$$S_j/(J)_j = (\mathfrak{m}^j/\mathfrak{m}^{j+1}). \qquad \square$$

Hereafter, we will assume by this proposition that *the residue field* $\mathbf{k}$ *has characteristic* 0.

REMARK 2.4. Note that the argument used in the proof of Proposition 2.3 does not hold for codimension > 2. Take, for example, the ideals (X_1^2, X_1X_2, X_1X_3) and (X_1^2, X_1X_2, X_2^2), which are strongly stable of codimension 3 in $k[X_1, X_2, X_3]$.

3. Stretched Local Rings

We recall that Sally [8] studied several properties of stretched local rings and proved a structure theorem for stretched Artinian local rings in the Gorenstein case. Here we extend that result to any Cohen–Macaulay type.

THEOREM 3.1. *Let I be an ideal in the regular local ring $(R, \mathfrak{n})$ such that $I \subseteq \mathfrak{n}^2$ and $A := R/I$ is Artinian. Let $\mathfrak{m} := \mathfrak{n}/I$ and $h := v(\mathfrak{m})$, and let τ be the Cohen–Macaulay type of A.*

(i) *If A is stretched of socle degree s and if $\tau < h$, then we can find a basis $\{x_1, \dots, x_h\}$ of $\mathfrak{n}$ such that I is minimally generated by the elements $\{x_i x_j\}_{1 \le i < j \le h}$, $\{x_j^2\}_{2 \le j \le \tau}$, and $\{x_i^2 - u_i x_1^s\}_{\tau+1 \le i \le h}$; here the u_i are units in R.*

(ii) *If A is stretched of socle degree s and if $\tau = h$, then we can find a basis $\{x_1, x_2, \dots, x_h\}$ of $\mathfrak{n}$ such that I is minimally generated by the elements $\{x_1 x_j\}_{2 \le j \le h}$, $\{x_i x_j\}_{2 \le i \le j \le h}$, and x_1^{s+1}.*

Proof. By Proposition 2.3, we can find an element $y_1 \in \mathfrak{m}$, $y_1 \notin \mathfrak{m}^2$ such that $y_1^s \neq 0$ and $\mathfrak{m}^j = (y_1^j)$ for $2 \le j \le s$. This implies that $y_1^j \notin \mathfrak{m}^{j+1}$ for every $1 \le j \le s$.

LEMMA 3.2. *We have*

$$(0 : \mathfrak{m}) \cap \mathfrak{m}^2 = \mathfrak{m}^s.$$

Proof. If $s = 2$ then there is nothing to prove, so we let $s \ge 3$. If $a \in 0 : \mathfrak{m}$ and $a \in \mathfrak{m}^2$, then $a = y_1^2 u$ and we have $0 = y_1 a = y_1^3 u$. Since $s \ge 3$ it follows that $u \in \mathfrak{m}$, for otherwise $y_1^3 = 0$. Hence $a \in \mathfrak{m}^3$. Continuing in this fashion yields $a \in \mathfrak{m}^s$, as desired. □

Since $y_1^s \in 0 : \mathfrak{m}$ and $y_1^s \neq 0$, we can find elements $y_2, \dots, y_\tau \in \mathfrak{m}$ such that $\{y_1^s, y_2, \dots, y_\tau\}$ is a basis of the $\mathbf{k}$-vector space $0 : \mathfrak{m}$.

LEMMA 3.3. *The elements $y_1, y_2, \dots, y_\tau$ are part of a minimal basis of $\mathfrak{m}$.*

Proof. If $\sum_{i=1}^{\tau} \lambda_i y_i \in \mathfrak{m}^2$ then $\lambda_1 \in \mathfrak{m}$; otherwise, $y_1 \in 0 : \mathfrak{m} + \mathfrak{m}^2$ and $y_1^2 \in \mathfrak{m}^3$, a contradiction. Hence

$$\sum_{i=2}^{\tau} \lambda_i y_i \in (0 : \mathfrak{m}) \cap \mathfrak{m}^2 = \mathfrak{m}^s$$

and, for some $t \in R$, $\sum_{i=2}^{\tau} \lambda_i y_i + t y_1^s = 0$. This implies that $\lambda_i \in \mathfrak{m}$ for every i, because $\{y_2, \dots, y_\tau, y_1^s\}$ is a basis of the $\mathbf{k} = A/\mathfrak{m}$ vector space $0 : \mathfrak{m}$. □

Of course we can complete the set $\{y_1, y_2, \dots, y_\tau\}$ to a minimal basis of $\mathfrak{m}$, say $\mathfrak{m} = (y_1, y_2, \dots, y_\tau, z_{\tau+1}, \dots, z_h)$. Now, if $j \ge \tau+1$ then $y_1 z_j \in \mathfrak{m}^2$; hence $y_1 z_j = y_1^2 t$ and $z_j - y_1 t \in 0 : y_1$. After replacing z_j with $z_j - y_1 t$ in the minimal generators of $\mathfrak{m}$, we may assume that

$$\mathfrak{m} = (y_1, y_2, \dots, y_\tau, y_{\tau+1}, \dots, y_h)$$

with

$$y_2, \dots, y_\tau \in 0 : \mathfrak{m}, \qquad y_{\tau+1}, \dots, y_h \in 0 : y_1. \tag{6}$$

Case (i): $\tau < h$. If we choose i and j so that $\tau + 1 \le i \le j \le h$, then

$$y_i y_j \mathfrak{m} \subseteq y_i \mathfrak{m}^2 = y_i(y_1^2) = 0.$$

Hence $y_i y_j \in (0 : \mathfrak{m}) \cap \mathfrak{m}^2 = \mathfrak{m}^s = (y_1^s)$ and we can write $y_i y_j = u_{ij} y_1^s$, where $u_{ij} \in \mathfrak{m}$ if and only if $y_i y_j = 0$.

Let $J := (y_{\tau+1}, \dots, y_h)$. We can then define an inner product in the $\mathbf{k}$-vector space $V := J/J\mathfrak{m}$ by letting

$$\langle \overline{y_i}, \overline{y_j} \rangle := \overline{u_{ij}} \in A/\mathfrak{m} = \mathbf{k}.$$

This is well-defined. Namely, let $y_i = p_i + z_i$ with $p_i \in J$ and $z_i \in J\mathfrak{m}$; since $J \subseteq 0 : y_1$, we have

$$y_i y_j - p_i p_j = (p_i + z_i)(p_j + z_j) - p_i p_j \in J\mathfrak{m}^2 = y_1^2 J = 0.$$

Since the characteristic of $\mathbf{k}$ is not 2, the inner product can be diagonalized. Therefore, the generators of $\mathfrak{m}$ can be chosen to satisfy

$$y_i y_j = 0 \tag{7}$$

for every $\tau + 1 \le i < j \le h$. This implies that for every $\tau + 1 \le i \le h$ we must have $y_i^2 \ne 0$, because if $y_i^2 = 0$ then we would get $y_i \in 0 : \mathfrak{m}$, a contradiction. Hence, for every $\tau + 1 \le i \le h$,

$$y_i^2 = u_i y_1^s \tag{8}$$

with $u_i \notin \mathfrak{m}$.

As a consequence we can prove the first part of the theorem. Let $x_i \in \mathfrak{n}$ such that $\overline{x_i} = y_i$. From (6), (7), and (8) it is clear that all the elements

$$\{x_i x_j\}_{1 \le i < j \le h}, \quad \{x_j^2\}_{2 \le j \le \tau}, \quad \{x_i^2 - u_i x_1^s\}_{\tau+1 \le i \le h}$$

are in I. Let J be the ideal they generate; then $J \subseteq I$, so that $H_{R/I}(n) \le H_{R/J}(n)$ for every $n \ge 0$. We claim that equality holds here for every $n \ge 0$. In particular,

$$x_1^{s+1} = (u_h)^{-1} x_1 x_h^2 \in J$$

so that $I^* \supseteq J^* \supseteq K$, where K is the ideal in $S = \mathbf{k}[X_1, \dots, X_h]$ generated by X_1^{s+1} and all degree-2 monomials except X_1^2. Since the Hilbert function of S/K is the same as the Hilbert function of R/I, the claim follows. Hence R/J and R/I have the same finite length and so the canonical surjection $R/J \to R/I$ is a bijection and $I = J$.

Finally, the given elements are a minimal basis of I because the generators of $\mathfrak{n}$ are analytically independent.

Case (ii): $\tau(A) = h$. If the Cohen–Macaulay type of A is h, the maximum allowed, then by (6) we have $\mathfrak{m} = (y_1, y_2, \dots, y_h)$, where $(y_2, \dots, y_h) \subseteq 0 : \mathfrak{m}$. This implies that $y_1 y_i = 0$ for every $i = 2, \dots, h$ and $y_i y_j = 0$ for every $2 \le i \le j \le h$. We also have $y_1^{s+1} = 0$. The conclusion follows as in case (i) but is now even easier because the generators of J are monomials. □

REMARK 3.4. It is clear that, for a stretched local ring $A = R/I$ of maximal type, the minimal set of generators of I found in Theorem 3.1 are a standard basis for I. Namely, we have that I^* is the ideal generated by X_1^{s+1} and the degree-2 monomials in S except for X_1^2. This is not true when $\tau(A) < h$. In this case, the initial forms of the generators of I in $S = \mathrm{gr}_{\mathfrak{n}}(R) = \mathbf{k}[X_1, X_2, \ldots, X_h]$ are the degree-2 monomials in S except for X_1^2. The ideal I^* is, as before, the ideal generated by X_1^{s+1} and the degree-2 monomials in S except for X_1^2.

REMARK 3.5. Given two integers $1 \le \tau \le h$ and a regular local ring $(R, \mathfrak{n})$ with maximal ideal $\mathfrak{n}$ minimally generated by $(x_1, x_2, \ldots, x_h)$, the ideals I generated as in Theorem 3.1 have the property that $A := R/I$ is a stretched local ring of type τ.

We have proved that if R/I is a stretched Artinian local ring of embedding dimension h, Cohen–Macaulay type $\tau < h$, and socle degree s, then there exists a minimal system of generators $x_1, \ldots, x_h$ of $\mathfrak{n}$ such that

$$I = (\{x_i x_j\}_{1 \le i < j \le h}, \{x_j^2\}_{2 \le j \le \tau}, \{x_i^2 - u_i x_1^s\}_{\tau+1 \le i \le h}),$$

where the u_i are units in R. For every $\underline{u} = (u_j)_{j=\tau+1,\ldots,h}$, we let $I(\underline{u})$ be such an ideal.

We shall often use the following easy and well-known lemma, a consequence of Hensel's lemma.

LEMMA 3.6. *Let $(A, \mathfrak{m})$ be an Artinian local ring with residue field $\mathbf{k}$, and let a be an element in A such that $\bar{a} \in \mathbf{k}^*$. If $\bar{b}^n = \bar{a}$ for some $\bar{b} \in \mathbf{k}$, then $c^n = a$ for some $c \in A$, $c \notin \mathfrak{m}$.*

PROPOSITION 3.7. *Let $I(\underline{u})$ be as before and assume that the residue field $\mathbf{k} = R/\mathfrak{n}$ verifies $\mathbf{k}^{1/2} \subseteq \mathbf{k}$. Then there exists a system of generators $y_1, \ldots, y_h$ of $\mathfrak{n}$ such that*

$$I(\underline{u}) = (\{y_i y_j\}_{1 \le i < j \le h}, \{y_j^2\}_{2 \le j \le \tau}, \{y_i^2 - y_1^s\}_{\tau+1 \le i \le h}).$$

Proof. Since $\mathbf{k}^{1/2} \subseteq \mathbf{k}$, by Lemma 3.1 we can find, for every $i = \tau + 1, \ldots, h$, elements $v_i \in R$ such that $v_i^2 \cong 1/u_i$ modulo $I(\underline{u})$. Hence $v_i \notin \mathfrak{n}$ and so

$$v_i^2 x_i^2 - x_1^s \cong (1/u_i) x_i^2 - x_1^s = (1/u_i)(x_i^2 - u_i x_1^s) \cong 0.$$

This proves that if

$$y_i = \begin{cases} x_i & \text{for } i = 1, \ldots, \tau, \\ v_i x_i & \text{for } i = \tau + 1, \ldots, h, \end{cases}$$

then

$$(\{y_i y_j\}_{1 \le i < j \le h}, \{y_j^2\}_{2 \le j \le \tau}, \{y_i^2 - x_1^s\}_{\tau+1 \le i \le h}) \subseteq I(\underline{u}).$$

Since the two ideals have the same Hilbert function, they must coincide. □

4. Almost Stretched Gorenstein Local Rings

In this section we consider Artinian local rings $(A, \mathfrak{m})$ such that the square of the maximal ideal is minimally generated by two elements. Recall that in Section 1 such a ring A was called almost stretched. If A is almost stretched and Gorenstein, then the Hilbert function of A is given by

0	1	2	...	t	$t+1$	...	s	$s+1$
1	h	2	...	2	1	...	1	0

with $h \geq 2$ and $s \geq t+1 \geq 3$.

The structure result for almost stretched Gorenstein local rings will be a consequence of the following theorem.

THEOREM 4.1. *Let $(A, \mathfrak{m})$ be an Artinian local ring that is Gorenstein with embedding dimension h. If A is almost stretched, then we can find integers $s \geq t+1 \geq 3$ and a minimal basis $x_1, \dots, x_h$ of $\mathfrak{m}$ such that*

$$\begin{cases} x_1 x_j = 0 & \textit{for } j = 3, \dots, h, \\ x_i x_j = 0 & \textit{for } 2 \leq i < j \leq h, \\ x_j^2 = u_j x_1^s & \textit{for } j = 3, \dots, h, \\ x_2^2 = a x_1 x_2 + w x_1^{s-t+1}, \\ x_1^t x_2 = 0 \end{cases}$$

with suitable $w, u_3, \dots, u_h \notin \mathfrak{m}$ and $a \in A$.

Proof. By Proposition 2.3 we may assume that $\mathfrak{m} = (x_1, \dots, x_h)$ with

$$\mathfrak{m}^j = \begin{cases} (x_1^j, x_1^{j-1} x_2) & \text{for } j = 2, \dots, t, \\ (x_1^j) & \text{for } j = t+1, \dots, s. \end{cases}$$

We claim that we may also assume $(x_3, \dots, x_h) \subseteq (0) : x_1$. That is, for $j \geq 3$ we can write $x_1 x_j = b_j x_1^2 + c_j x_1 x_2$, so $x_1(x_j - b_j x_1 - c_j x_2) = 0$. We establish this claim by replacing x_j with $x_j - b_j x_1 - c_j x_2$ for every $j \geq 3$. This means that

$$x_1 x_3 = x_1 x_4 = \cdots = x_1 x_h = 0. \tag{9}$$

Furthermore, since $\mathfrak{m}^{t+1} = (x_1^{t+1})$, for some $c \in A$ we have

$$x_1^t x_2 = c x_1^{t+1}. \tag{10}$$

Let $y_2 := x_2 - c x_1$; then

$$x_1^t y_2 = x_1^t (x_2 - c x_1) = x_1^t x_2 - c x_1^{t+1} = 0.$$

Since x_2 is not involved in (9), we may replace x_2 with y_2 in the generating set of $\mathfrak{m}$. Hence we may assume that

$$x_1^t x_2 = 0. \tag{11}$$

Observe that $x_1^{t-1} x_2 \notin \mathfrak{m}^s$, for otherwise $x_1^{t-1} x_2 \in \mathfrak{m}^{t+1}$, a contradiction to $x_1^{t-1} x_2, x_1^t$ being a minimal basis of $\mathfrak{m}^t$. This implies that $x_1^{t-1} x_2$ cannot be in the socle of A. Since by (11) and (9) we have

$$x_1^{t-1} x_2 \in (0) : (x_1, x_3, \dots, x_h),$$

it follows that

$$x_1^{t-1} x_2^2 \neq 0. \tag{12}$$

We want to prove now the existence of an $a \in A$ and a $w \notin \mathfrak{m}$ such that

$$x_2^2 = ax_1x_2 + wx_1^{s-t+1}.$$

In order to show this, we need the following easy remarks.

Claim 1. If for some $r, p \in A$ and $n \geq 2$ we have $x_2^2 = rx_1x_2 + px_1^n$, then $n \leq s - t + 1$. If also $p \notin \mathfrak{m}$, then $n = s - t + 1$.

Proof of Claim 1. We have

$$x_1^{t-1}x_2^2 = x_1^{t-1}(rx_1x_2 + px_1^n) = px_1^{n+t-1}$$

because, by (11), $x_1^t x_2 = 0$. Since by (12) we have $x_1^{t-1}x_2^2 \neq 0$, this implies that $n + t - 1 \leq s$. We also have $px_1^n = x_2(x_2 - rx_1)$, and thus if $p \notin \mathfrak{m}$ then $x_1^n = vx_2$ for some $v \in A$. As a consequence, $x_1^{n+t} = vx_1^t x_2 = 0$. Since $x_1^s \neq 0$, we have $n + t \geq s + 1$ and so the conclusion follows.

Claim 2. Let $n \geq 2$ and let $a \in A$ and $b \in \mathfrak{m}$. If $x_2^2 = ax_1x_2 + bx_1^n$ then for some $c, d \in A$ we have $x_2^2 = cx_1x_2 + dx_1^{n+1}$.

Proof of Claim 2. This is easy because, by (9), $x_1x_j = 0$ for every $j \geq 3$.

Claim 3. If for some $a, b \in A$ we have $x_2^2 = ax_1x_2 + bx_1^{s-t+1}$, then $b \notin \mathfrak{m}$.

Proof of Claim 3. If (by way of contradiction) $b \in \mathfrak{m}$, then Claims 1 and 2 yield

$$s - t + 2 \leq s - t + 1.$$

Since $\mathfrak{m}^2 = (x_1^2, x_1x_2)$, it follows that $x_2^2 = ax_1x_2 + bx_1^2$ for some $a, b \in A$. Thus we obtain, as a trivial consequence of these three claims, that

$$x_2^2 = ax_1x_2 + wx_1^{s-t+1} \tag{13}$$

for some $a \in A$ and $w \notin \mathfrak{m}$. Now we recall that for every $j \geq 3$, by (9) we have

$$x_j\mathfrak{m}^2 = x_j(x_1^2, x_1x_2) = 0;$$

hence, using the Gorenstein assumption yields

$$x_j\mathfrak{m} \subseteq (0) : \mathfrak{m} = (x_1^s). \tag{14}$$

Let us consider the ideal $J := (x_3, \ldots, x_h)$. By (14), for every $3 \leq i \leq j \leq h$ we have $x_ix_j = u_{ij}x_1^s$ with $u_{ij} \in A$. We remark that if also $x_ix_j = w_{ij}x_1^s$ then $(u_{ij} - w_{ij})x_1^s = 0$, which implies $u_{ij} - w_{ij} \in \mathfrak{m}$.

Hence we may define an inner product in the $\mathbf{k} = A/\mathfrak{m}$-vector space $V := J/J\mathfrak{m}$ by letting

$$\langle \overline{x_i}, \overline{x_j} \rangle := \overline{u_{ij}} \in A/\mathfrak{m}$$

and extending this definition by bilinearity to $V \times V$.

Because the characteristic of $\mathbf{k}$ is not 2, the inner product can be diagonalized. This means that we can find minimal generators $y_3, \ldots, y_h$ of J such that $y_iy_j = 0$ for $i \neq j$. If we replace $x_3, \ldots, x_h$ with $y_3, \ldots, y_h$ in the generating set of $\mathfrak{m}$, it is clear that equations (9), (11), (13), and (14) are still valid. Hence, generators $x_1, \ldots, x_h$ of $\mathfrak{m}$ can be chosen so that

$$x_i x_j = 0 \tag{15}$$

for every i and j such that $3 \le i < j \le h$.

By (14), for every $j \ge 3$ we have

$$x_j^2 = u_j x_1^s$$

with $u_j \in A$. We claim that $u_j \notin \mathfrak{m}$ for every $j \ge 3$.

In order to prove this claim, recall that, again by (14), we have

$$x_2 x_j = a_j x_1^s$$

for every $j \ge 3$ and suitable $a_j \in A$. Fix $j \ge 3$ and let

$$\rho := w x_j - a_j x_1^{t-1} x_2.$$

Since $w \notin \mathfrak{m}$, it is clear that $\rho \notin \mathfrak{m}^2$ and so $\rho \notin \mathfrak{m}^s \subseteq \mathfrak{m}^2$. This implies that ρ *cannot be in the socle of* A. We will use the following equalities:

$x_1 x_j = 0$	for $j \ge 3$	(see (9)),
$x_1^t x_2 = 0$		(see (11)),
$x_2^2 = a x_1 x_2 + w x_1^{s-t+1}$		(see (13)),
$x_j x_k = 0$	for $3 \le j < k \le h$	(see (15)).

Then

$$\rho x_1 = w x_1 x_j - a_j x_1^t x_2 = 0,$$

$$\begin{aligned}\rho x_2 &= w x_2 x_j - a_j x_1^{t-1} x_2^2 = w a_j x_1^s - a_j x_1^{t-1}(a x_1 x_2 + w x_1^{s-t+1}) \\ &= w a_j x_1^s - w a_j x_1^s = 0,\end{aligned}$$

$$\rho x_k = w x_j x_k - a_j x_1^{t-1} x_2 x_k = 0 \quad \text{if } k \ge 3,\ k \ne j,$$

$$\rho x_j = w x_j^2 - a_j x_1^{t-1} x_2 x_j = w u_j x_1^s.$$

Since ρ cannot be in the socle, we must have $u_j \notin \mathfrak{m}$. This completes the proof of Claim 3.

We may therefore assume that, for every $j \ge 3$ and suitable $u_j \notin \mathfrak{m}$,

$$x_j^2 = u_j x_1^s. \tag{16}$$

We come now to the last manipulation of our elements. As a consequence of Claim 3, we may consider the element

$$y_2 := x_2 - \sum_{i=3}^{h} u_i^{-1} a_i x_i.$$

For every $j = 3, \dots, h$, by (15) we have

$$y_2 x_j = x_2 x_j - \sum_{i=3}^{h} u_i^{-1} a_i x_i x_j = a_j x_1^s - u_j^{-1} a_j x_j^2 = a_j x_1^s - u_j^{-1} a_j u_j x_1^s = 0.$$

Furthermore,

$$x_1^t x_2 = x_1^t\left(y_2 + \sum_{i=3}^{h} u_i^{-1} a_i x_i\right) = x_1^t y_2.$$

Finally let $d := x_2 - y_2 = \sum_{i=3}^{h} u_i^{-1} a_i x_i$. Then $d \in J := (x_3, \dots, x_h)$ and so

$$x_1 d = 0, \qquad y_2 d = 0.$$

Since $J\mathfrak{m} \subseteq (x_1^s)$ by (14), we have

$$d^2 = p x_1^s$$

for some $p \in A$. It follows that

$$\begin{aligned} &x_2^2 - a x_1 x_2 - w x_1^{s-t+1} \\ &\quad = (y_2 + d)^2 - a x_1 (y_2 + d) - w x_1^{s-t+1} = y_2^2 + d^2 - a x_1 y_2 - w x_1^{s-t+1} \\ &\quad = y_2^2 - a x_1 y_2 - w x_1^{s-t+1} + p x_1^s = y_2^2 - a x_1 y_2 - (w - p x_1^{t-1}) x_1^{s-t+1}, \end{aligned}$$

where $w - p x_1^{t-1} \notin \mathfrak{m}$.

Thus we may replace x_2 with y_2 and thereby obtain a basis $x_1, \dots, x_h$ for $\mathfrak{m}$ such that

$$\begin{cases} x_1 x_j = 0 & \text{for } j = 3, \dots, h, \\ x_i x_j = 0 & \text{for } 2 \le i < j \le h, \\ x_j^2 = u_j x_1^s & \text{for } j = 3, \dots, h, \\ x_2^2 = a x_1 x_2 + w x_1^{s-t+1}, & \\ x_1^t x_2 = 0 & \end{cases}$$

with suitable $w, u_3, \dots, u_h \notin \mathfrak{m}$ and $a \in A$. □

As a result of Theorem 4.1 we obtain a structure theorem for almost stretched Artinian and Gorenstein local rings.

Corollary 4.2. *Let $(R, \mathfrak{n})$ be a regular local ring of dimension h and let $I \subseteq \mathfrak{n}^2$ be an ideal such that $(A = R/I, \mathfrak{m} = \mathfrak{n}/I)$ is almost stretched Artinian and Gorenstein. Then there is a minimal basis $x_1, \dots, x_h$ of $\mathfrak{n}$ such that I is minimally generated by the elements*

$$\{x_1 x_j\}_{j=3,\dots,h},\ \{x_i x_j\}_{2 \le i < j \le h},\ \{x_j^2 - u_j x_1^s\}_{j=3,\dots,h},\ x_2^2 - a x_1 x_2 - w x_1^{s-t+1},\ x_1^t x_2,$$

where $w, u_3, \dots, u_h \notin \mathfrak{n}$ and $a \in R$.

Proof. By Theorem 4.1 we can find a basis $x_1, \dots, x_h$ of $\mathfrak{n}$ such that the ideal J generated by the elements just listed is contained in I. We need to show that I is indeed equal to J. We first remark that modulo J we have

$$x_1^{s+1} = x_1^t x_1^{s-t+1} \cong x_1^t \frac{x_2^2 - a x_1 x_2}{w} \cong x_1^t x_2 \frac{x_2 - a x_1}{w} \cong 0$$

and so $x_1^{s+1} \in J$.

Passing to the ideals of initial forms in the polynomial ring

$$S = \mathrm{gr}_{\mathfrak{n}}(R) = \bigoplus_{j\geq 0}(\mathfrak{n}^j/\mathfrak{n}^{j+1}) = (R/\mathfrak{n})[X_1, \dots, X_h],$$

we have

$$I^* \supseteq J^* \supseteq K.$$

Here K is the ideal in S generated by the elements

$$\{X_1X_j\}_{j=3,\dots,h},\ \{X_iX_j\}_{2\leq i<j\leq h},\ \{X_j^2\}_{j=3,\dots,h},\ X_1^tX_2,\ X_1^{s+1}$$

and the quadric $Q := X_2^2 - \bar{a}X_1X_2$ for $s \geq t+2$ or $Q := X_2^2 - \bar{a}X_1X_2 - \bar{w}X_1^2$ for $s = t+1$.

In both cases we have $X_jS_1 \subseteq K$ for every $j \geq 3$, so that

$$(K + (X_3, \dots, X_h))_n = K_n$$

for every $n \neq 1$. This implies that, for every $n \neq 1$,

$$H_{S/K}(n) = H_{S/(K+(X_3,\dots,X_h))}(n) = H_{\mathbf{k}[X_1,X_2]/(Q,X_1^tX_2,X_1^{s+1})}(n).$$

Now we compute the Hilbert function of $\mathbf{k}[X_1, X_2]/(Q, X_1^tX_2, X_1^{s+1})$. Let $B := \mathbf{k}[X_1, X_2]$. If $Q = X_2^2 - \bar{a}X_1X_2 = X_2(X_2 - \bar{a}X_1)$ then we have an exact sequence of graded algebras,

$$0 \longrightarrow B/(X_2 - \bar{a}X_1, X_1^t)(-1) \xrightarrow{X_2} B/(Q, X_1^tX_2) \longrightarrow B/(X_2) \longrightarrow 0,$$

which enables us to compute the Hilbert series of $B/(Q, X_1^tX_2)$:

$$\begin{aligned} P_{B/(Q,X_1^tX_2)}(z) &= zP_{B/(X_2-\bar{a}X_1,X_1^t)}(z) + P_{B/(X_2)}(z) \\ &= \frac{z(1-z)(1-z^t) + (1-z)}{(1-z)^2} = \frac{1+z-z^{t+1}}{1-z}. \end{aligned}$$

This yields the Hilbert function

0	1	2	...	t	$t+1$	...	s	$s+1$	$s+2$	...
1	2	2	...	2	1	...	1	1	1	...

Since $X_1^{s+1} \notin (Q, X_1^tX_2)$, the Hilbert function of $\mathbf{k}[X_1, X_2]/(Q, X_1^tX_2, X_1^{s+1})$ is

0	1	2	...	t	$t+1$	$t+2$	...	s	$s+1$
1	2	2	...	2	1	1	...	1	0

and so the Hilbert function of S/K is

0	1	2	...	t	$t+1$	$t+2$	...	s	$s+1$
1	h	2	...	2	1	1	...	1	0

the same as that of S/I^*.

In the case $s = t+1$ we have $Q = X_2^2 - \bar{a}X_1X_2 - \bar{w}X_1^2$ with $\bar{w} \neq 0$. Hence $\{Q, X_1^tX_2\}$ is a regular sequence and $\mathbf{k}[X_1, X_2]/(Q, X_1^tX_2)$ has Hilbert function

0	1	2	...	t	$t+1=s$	$t+2$
1	2	2	...	2	1	0

We remark that in this case $X_1^2 \in (Q, X_2)$, so

$$X_1^{s+1} = X_1^{t+2} = X_1^t X_1^2 \in (Q, X_1^t X_2).$$

In any case we have proven that S/I^* and S/K have the same Hilbert function. This implies that $I^* = J^* = K$, so the Hilbert functions of R/I and R/J are the same. Hence R/I and R/J have the same finite length, which means that the canonical epimorphism $R/J \to R/I$ is an isomorphism and $I = J$ as claimed. □

REMARK 4.3. In the proof of Corollary 4.2 we describe the ideal I^*; it is generated by

$$\{X_1X_j\}_{j=3,\dots,h},\ \{X_iX_j\}_{2\le i<j\le h},\ \{X_j^2\}_{j=3,\dots,h},\ X_1^tX_2,\ X_1^{s+1}$$

and the quadric

$$Q := \begin{cases} X_2^2 - \bar{a}X_1X_2 & \text{when } s \ge t+2,\\ X_2^2 - \bar{a}X_1X_2 - \bar{w}X_1^2 & \text{when } s = t+1,\end{cases}$$

where $\bar{w} \neq 0$ and $\bar{a} \in \mathbf{k}$.

We now wish to prove the converse of Corollary 4.2. Observe that the following lemma does not require a ring to be regular or local.

LEMMA 4.4. *Let B be a ring, and let $t \ge 2$, $h \ge 2$, and $s \ge t+1$. Let $\mathfrak{n} = (x_1, \dots, x_h)$ be an ideal in B and let J be the ideal generated by*

$$\{x_1x_j\}_{j=3,\dots,h},\ \{x_ix_j\}_{2\le i<j\le h},\ \{x_j^2 - u_jx_1^s\}_{j=3,\dots,h},\ x_2^2 - ax_1x_2 - wx_1^{s-t+1},\ x_1^tx_2.$$

If w is a unit in B, then

$$\mathfrak{n}^{s+1} \subseteq J.$$

Proof. For every $i \neq j$ except for $(i,j) = (1,2)$, we have

$$x_ix_j \in J.$$

For every $3 \le j \le h$ we have

$$x_j^2 \in J + (x_1^s)$$

and, since $s - t + 1 \ge 2$,

$$x_2^2 \in J + (x_1^2, x_1x_2).$$

We claim that, for every $r \ge 2$,

$$\mathfrak{n}^r \subseteq J + (x_1^r, x_1^{r-1}x_2).$$

If $r = 2$ then $\mathfrak{n}^2 \subseteq J + (x_1^2, x_1x_2)$ by the three previous inclusions. Now proceed by induction on r. We have

$$\begin{aligned}\mathfrak{n}^{r+1} = \mathfrak{n}\mathfrak{n}^r &\subseteq J + \mathfrak{n}(x_1^r, x_1^{r-1}x_2)\\ &= J + (x_1, x_2)(x_1^r, x_1^{r-1}x_2) = J + (x_1^{r+1}, x_1^rx_2, x_1^{r-1}x_2^2).\end{aligned}$$

The claim follows because $x_2^2 \in J + (x_1^2, x_1x_2)$, so

$$x_1^{r-1}x_2^2 \in J + (x_1^{r+1}, x_1^rx_2).$$

From the claim we have $\mathfrak{n}^{s+1} \subseteq J + (x_1^{s+1}, x_1^s x_2)$. Since $s \geq t$, we obtain $x_1^s x_2 \in (x_1^t x_2) \subseteq J$; on the other hand, since w is a unit,

$$x_1^{s+1} = (x_1^t/w)wx_1^{s-t+1} \cong (x_1^t/w)(x_2^2 - ax_1x_2) \cong 0$$

modulo J. The conclusion follows. □

We come now to a crucial step.

LEMMA 4.5. *Let R be a regular local ring of dimension $h \geq 2$, let $\mathfrak{n} = (x_1, \dots, x_h)$ be the maximal ideal of R, and let $s \geq t+1 \geq 3$ and $a, u_3, \dots, u_h, w \in R$. Let I be the ideal generated by*

$$\{x_1x_j\}_{j=3,\dots,h},\ \{x_ix_j\}_{2\leq i<j\leq h},\ \{x_j^2 - u_jx_1^s\}_{j=3,\dots,h},$$
$$q := x_2^2 - ax_1x_2 - wx_1^{s-t+1},\ x_1^tx_2.$$

If $u_3, \dots, u_h, w \notin \mathfrak{n}$, then

(i) $\overline{x_1}^t, \overline{x_1}^{t-1}\overline{x_2} \in (\mathfrak{n}^t + I)/(\mathfrak{n}^{t+1} + I)$ *are $(R/\mathfrak{n})$-linearly independent elements and*

(ii) $x_1^s \notin I$.

Proof. In order to prove (i) we must show that if $\lambda x_1^t + \mu x_1^{t-1}x_2 \in I + \mathfrak{n}^{t+1}$ then $\lambda, \mu \in \mathfrak{n}$. Clearly, if $\lambda x_1^t + \mu x_1^{t-1}x_2 \in I + \mathfrak{n}^{t+1}$ then

$$\begin{aligned}\lambda x_1^t + \mu x_1^{t-1}x_2 &\in I + \mathfrak{n}^{t+1} + (x_3, \dots, x_h)\\ &= (x_3, \dots, x_h) + (x_1, x_2)^{t+1} + (x_1^s, x_1^tx_2, q)\\ &= (x_3, \dots, x_h) + (x_1, x_2)^{t+1} + (q).\end{aligned}$$

Let's interpret this condition in terms of the two-dimensional regular local ring $T := R/(x_3, \dots, x_h)$, whose maximal ideal is generated by the residue class of x_1 and x_2 modulo $(x_3, \dots, x_h)$. By abuse of notation, we again denote these elements by x_1, x_2 and the maximal ideal of T by $\mathfrak{n}$. Then

$$\lambda x_1^t + \mu x_1^{t-1}x_2 = eq + z,$$

where $z \in \mathfrak{n}^{t+1}$. This implies that $eq \in \mathfrak{n}^t$. If $eq \in \mathfrak{n}^{t+1}$, the conclusion follows by the analytic independence of x_1 and x_2. If $eq \notin \mathfrak{n}^{t+1}$ then, since $q = x_2^2 - ax_1x_2 - wx_1^{s-t+1} \in \mathfrak{n}^2$, we have $e \in \mathfrak{n}^{t-2}$ and $e \notin \mathfrak{n}^{t-1}$. Passing to the associated graded ring $(T/\mathfrak{n})[X_1, X_2]$ of T yields

$$X_1^{t-1}(\bar{\lambda}X_1 + \bar{\mu}X_2) = e^*q^*.$$

Since X_1 is not a factor of q^*, X_1^{t-1} must be a factor of e^*. This is a contradiction because e^* is an homogeneous element of degree $t-2$. The conclusion follows.

We now prove (ii). By way of contradiction, let

$$x_1^s = \sum_{j=3}^h \lambda_jx_1x_j + \sum_{j=3}^h \rho_j(x_j^2 - u_jx_1^s) + \sum_{2\leq i<j\leq h} \mu_{ij}x_ix_j + \sigma x_1^tx_2 + \alpha q.$$

Because $s \geq t+1 \geq 3$, this implies

$$\sum_{j=3}^{h} \lambda_j x_1 x_j + \sum_{j=3}^{h} \rho_j x_j^2 + \sum_{2\le i<j\le h} \mu_{ij} x_i x_j + \alpha(x_2^2 - a x_1 x_2 - w x_1^{s-t+1}) \in \mathfrak{n}^3.$$

By the analytic independence of $x_1, \ldots, x_h$, all the coefficients of the degree-2 monomials in $x_1, \ldots, x_h$ must be in $\mathfrak{n}$. In particular $\rho_j \in \mathfrak{n}$ for every $j = 1, \ldots, h$. This implies that

$$x_1^s \in (x_3, \ldots, x_h) + (x_1^t x_2, q) + \mathfrak{n}^{s+1}.$$

As before, we pass to the two-dimensional regular local ring $T := R/(x_3, \ldots, x_h)$, whose maximal ideal is still denoted by $\mathfrak{n}$ and generated by x_1, x_2. We can write

$$x_1^s = \sigma x_1^t x_2 + \alpha q + \beta, \tag{17}$$

where $\beta \in \mathfrak{n}^{s+1}$. This implies that $x_1^s + \alpha w x_1^{s-t+1} \in (x_2, x_1^{s+1})$, so we can write $x_1^s + \alpha w x_1^{s-t+1} = x_2 a + x_1^{s+1} b$ for some $a, b \in T$. This gives

$$x_1^{s-t+1}(x_1^{t-1} + \alpha w - b x_1^t) = x_2 a.$$

Since x_1^{s-t+1}, x_2 is a regular sequence in T, it follows that $x_1^{t-1} + \alpha w - b x_1^t = x_2 c$ for some $c \in T$. Hence $\alpha w = x_1^{t-1}(b x_1 - 1) + x_2 c$ and, since w is a unit, we finally get

$$\alpha = v x_1^{t-1} + d x_2$$

for some $v, d \in T$, $v \notin \mathfrak{n}$. Using this formula in equation (17) yields

$$x_1^s = \sigma x_1^t x_2 + (v x_1^{t-1} + d x_2) q + \beta, \tag{18}$$

where $\beta \in \mathfrak{n}^{s+1}$ and $v \notin \mathfrak{n}$.

Claim. If for some $r \geq 2$ and $j \geq 2$ we have

$$x_1^j - \sigma x_1^r x_2 - (v x_1^{r-1} + d x_2) q \in \mathfrak{n}^{j+1}$$

as in (18) with $j = s$ and $r = t$, then for suitable $e \in T$ we also have

$$x_1^{j-1} - \sigma x_1^{r-1} x_2 - (v x_1^{r-2} + e x_2) q \in \mathfrak{n}^j.$$

Because $q = x_2^2 - a x_1 x_2 - w x_1^{s-t+1}$, the assumption of our claim implies

$$d x_2^3 \in (x_1) + \mathfrak{n}^{j+1} = (x_1) + (x_2^{j+1}).$$

Now, since $j + 1 \geq 3$ and since x_1, x_2^3 is a regular sequence, $d = e x_1 + f x_2^{j-2}$ for some $e, f \in T$ and so $x_1^j - \sigma x_1^r x_2 - (v x_1^{r-1} + e x_1 x_2) q \in \mathfrak{n}^{j+1}$. Since $\mathfrak{n}^{j+1} \cap (x_1) = x_1 \mathfrak{n}^j$, it follows that

$$x_1^{j-1} - \sigma x_1^{r-1} x_2 - (v x_1^{r-2} + e x_2) q \in \mathfrak{n}^j$$

and the claim is proved.

Starting from (18) with $j = s$ and $r = t$, we apply the claim $t-1$ times to obtain

$$x_1^{s-t+1} - \sigma x_1 x_2 - (v + g x_2) q \in \mathfrak{n}^{s-t+2}$$

for some $g \in T$. This implies

$$(v + g x_2) x_2^2 \in (x_1) + \mathfrak{n}^{s-t+2} = (x_1, x_2^{s-t+2});$$

as a result, since $s-t+2\geq 3$, we get $vx_2^2\in(x_1,x_2^3)$, which is a contradiction because $v\notin\mathfrak{n}$. □

COROLLARY 4.6. *Let R be a regular local ring of dimension $h\geq 2$, let $\mathfrak{n}=(x_1,\dots,x_h)$ be the maximal ideal of R, and let $s\geq t+1\geq 3$ and $a,u_3,\dots,u_h,w\in R$. Let I be the ideal generated by*

$$\{x_1x_j\}_{j=3,\dots,h},\ \{x_ix_j\}_{2\leq i<j\leq h},\ \{x_j^2-u_jx_1^s\}_{j=3,\dots,h},$$
$$q:=x_2^2-ax_1x_2-wx_1^{s-t+1},\ x_1^tx_2.$$

If $u_3,\dots,u_h,w\notin\mathfrak{n}$, then the Hilbert function of R/I is

0	1	2	...	t	$t+1$	$t+2$	...	s	$s+1$
1	h	2	...	2	1	1	...	1	0

Proof. In the proof of Lemma 4.4 we saw that $\mathfrak{n}^r\subseteq J+(x_1^r,x_1^{r-1}x_2)$ for every $r\geq 2$. This proves that all the powers of $\mathfrak{n}/I$ can be generated by two elements. By Lemma 4.5(i) we get $H_{R/I}(t)=2$, which implies (by Macaulay's characterization of Hilbert functions) that $H_{R/I}(j)=2$ for every $2\leq j\leq t$. Since $x_1^tx_2\in I$, we also have $H_{R/I}(t+1)\leq 1$, which implies that $H_{R/I}(j)\leq 1$ for every $j\geq t+1$. The conclusion follows because $x_1^s\notin I$ and $\mathfrak{n}^{s+1}\subseteq I$. □

We are now ready to prove the converse of Corollary 4.2.

THEOREM 4.7. *Let R be a regular local ring of dimension $h\geq 2$, let $\mathfrak{n}=(x_1,\dots,x_h)$ be the maximal ideal of R, and let $s\geq t+1\geq 3$ and $a,u_3,\dots,u_h,w\in R$. Let I be the ideal generated by*

$$\{x_1x_j\}_{j=3,\dots,h},\ \{x_ix_j\}_{2\leq i<j\leq h},\ \{x_j^2-u_jx_1^s\}_{j=3,\dots,h},\ x_2^2-ax_1x_2-wx_1^{s-t+1},\ x_1^tx_2.$$

If $u_3,\dots,u_h,w\notin\mathfrak{n}$, then R/I is an almost stretched Gorenstein local ring with Hilbert function

0	1	2	...	t	$t+1$	$t+2$	...	s	$s+1$
1	h	2	...	2	1	1	...	1	0

Proof. Given Corollary 4.6, we need only prove that R/I is Gorenstein.

Let $\mathfrak{m}:=\mathfrak{n}/I$ and $y_i:=\overline{x_i}\in A=R/I$. By Lemma 4.5 we have $\mathfrak{m}^j=(y_1^j,y_1^{j-1}y_2)$ for every $j=2,\dots,t$ and $\mathfrak{m}^j=(y_1^j)$ for $j=t+1,\dots,s$. We prove the theorem in three steps.

Claim 1. If for some $j\neq 1,t,s$ and some $r\in\mathfrak{m}^j$ we have $ry_1=0$, then $r\in\mathfrak{m}^{j+1}$.

Proof of Claim 1. Let $2\leq j\leq t-1$; then $r=\lambda y_1^j+\mu y_1^{j-1}y_2$. We have

$$0=ry_1=\lambda y_1^{j+1}+\mu y_1^jy_2.$$

Since $y_1^{j+1},y_1^jy_2$ is a minimal basis of $\mathfrak{m}^{j+1}$, it follows that $\lambda,\mu\in\mathfrak{m}$ and $r\in\mathfrak{m}^{j+1}$. The case $t+1\leq j\leq s-1$ is even easier.

Claim 2. If for some $r\in\mathfrak{m}^t$ we have $ry_1=ry_2=0$, then $r\in\mathfrak{m}^{t+1}$.

Proof of Claim 2. Let $r = \lambda y_1^t + \mu y_1^{t-1} y_2$. Since $y_1^t y_2 = 0$, we have $0 = ry_1 = \lambda y_1^{t+1}$. This implies $\lambda \in \mathfrak{m}$. On the other hand,

$$0 = ry_2 = \mu y_1^{t-1} y_2^2 = \mu y_1^{t-1}(\bar{a} y_1 y_2 + \bar{w} y_1^{s-t+1}) = \mu \bar{w} y_1^s.$$

Because $\bar{w}$ is a unit in A, this implies that $0 = \mu y_1^s$ and so $\mu \in \mathfrak{m}$. Thus $r \in \mathfrak{m}^{t+1}$.

Together, Claims 1 and 2 prove that if $r \in \mathfrak{m}^2$ and $ry_1 = ry_2 = 0$ then $r \in \mathfrak{m}^s$.

Claim 3. If $r \in (0) : \mathfrak{m}$ then $r \in \mathfrak{m}^2$, so that $r \in \mathfrak{m}^s$ and A is Gorenstein.

Proof of Claim 3. Let $r \in (0) : \mathfrak{m}$; then $r \in \mathfrak{m}$ and we can write $r = \sum_{i=1}^h \lambda_i y_i$. Since $y_1 y_j = 0$ for every $j \geq 3$, we have

$$0 = ry_1 = \lambda_1 y_1^2 + \lambda_2 y_1 y_2.$$

This implies $\lambda_1, \lambda_2 \in \mathfrak{m}$ so that $r = \sum_{i=3}^h \lambda_i y_i + b$ with $b \in \mathfrak{m}^2$.

Since $y_2 y_j = 0$ for every $j \geq 3$, we obtain $0 = ry_1 = by_1$ and $0 = ry_2 = by_2$; by Claim 2, this implies that $b \in \mathfrak{m}^s$. Since $y_i y_j = 0$ for every $3 \leq i < j \leq h$ and since $\mathfrak{m}^{s+1} = 0$, it follows that

$$0 = ry_j = \lambda_j y_j^2 = \lambda_j \overline{u_j} y_1^s.$$

Since $\overline{u_j}$ is a unit in A, this implies $\lambda_j y_1^s = 0$ so that $\lambda_j \in \mathfrak{m}$ and $r \in \mathfrak{m}^2$. This completes the proof of Claim 3 and hence of the theorem. □

Theorem 4.7, a structure theorem of almost stretched Gorenstein local rings, can be refined under a mild assumption on the residue field of R. This will be crucial for the study of the moduli problem, and it is a consequence of Theorem 4.1 and Lemma 3.6.

PROPOSITION 4.8. *Let $(R, \mathfrak{n}, k)$ be a regular local ring of dimension $h \geq 2$, and let I be an ideal in R such that R/I is almost stretched Artinian and Gorenstein. If $\mathbf{k}^{1/2} \subseteq \mathbf{k}$, then we can find integers $s \geq t+1 \geq 3$, a minimal system of generators $x_1, \ldots, x_h$ of $\mathfrak{n}$, and an element $a \in R$ such that I is generated by*

$$\{x_1 x_j\}_{j=3,\ldots,h},\ \{x_i x_j\}_{2 \leq i < j \leq h},\ \{x_j^2 - x_1^s\}_{j=3,\ldots,h},\ x_2^2 - a x_1 x_2 - x_1^{s-t+1},\ x_1^t x_2.$$

Proof. We know that integers $s \geq t+1 \geq 3$ can be found and a minimal system of generators $y_1, \ldots, y_h$ of $\mathfrak{n}$ can be constructed in such a way that I is generated by

$$\{y_1 y_j\}_{j=3,\ldots,h},\ \{y_i y_j\}_{2 \leq i < j \leq h},\ \{y_j^2 - u_j y_1^s\}_{j=3,\ldots,h},\ y_2^2 - b y_1 y_2 - w y_1^{s-t+1},\ y_1^t y_2,$$

where $w, u_3, \ldots, u_h \notin \mathfrak{n}$ and $b \in R$. Given Lemma 3.6, we can find elements $v, r_3, \ldots, r_h \in R$ such that, modulo I, we have

$$v^2 \cong (1/w), \qquad r_3^2 \cong (1/u_3), \ldots, r_h^2 \cong (1/u_h).$$

From this is clear that $v, r_3, \ldots, r_h$ are units in R and so we can make the following change of minimal generators for $\mathfrak{n}$:

$$x_1 = y_1,\ x_2 = v y_2,\ x_3 = r_3 y_3,\ \ldots,\ x_h = r_h y_h.$$

Then

$$y_2^2 - by_1y_2 - wy_1^{s-t+1} = (x_2^2/v^2) - bx_1(x_2/v) - wx_1^{s-t+1} \in I;$$

hence $x_2^2 - bvx_1x_2 - v^2wx_1^{s-t+1} \in I$. Since $v^2w = 1 + d$ with $d \in I$, for $a := bv$ we have

$$x_2^2 - ax_1x_2 - x_1^{s-t+1} \in I.$$

Furthermore, for every $j = 3, \dots, h$,

$$y_j^2 - u_jy_1^s = (x_j/r_j)^2 - u_jx_1^s \in I;$$

hence $x_j^2 - r_j^2u_jx_1^s \in I$. Since $r_j^2u_j = 1 + e$ with $e \in I$, for every $j = 3, \dots, h$ we have

$$x_j^2 - x_1^s \in I.$$

Therefore, I contains the ideal generated by

$$\{x_1x_j\}_{j=3,\dots,h},\ \{x_ix_j\}_{2\le i<j\le h},\ \{x_j^2 - x_1^s\}_{j=3,\dots,h},\ x_2^2 - ax_1x_2 - x_1^{s-t+1},\ x_1^tx_2.$$

Since by Corollary 4.6 these two ideals have the same Hilbert function, they must coincide. □

5. Classification of Gorenstein Local Algebras with Hilbert Function (1, 2, 2, 2, 1, 1, 1)

We saw in Section 3 that the Cohen–Macaulay type determines the moduli class of stretched Artinian local rings. In the case of almost stretched Artinian local rings, the problem is not so easy, even in the Gorenstein case. For example, in [1] it was proved that if A is Gorenstein with Hilbert function $1, 2, 2, 1$ then we have only two models: the ideals $I = (x^2, y^3)$ and $I = (xy, x^3 - y^3)$. But already in the next case, with symmetric Hilbert function $1, 2, 2, 2, 1$ we have at least three different models: two ideals $I = (x^2, y^4)$ and $I = (xy, x^4 - y^4)$ that are homogeneous and one ideal $I = (x^4 + 2x^3y, y^2 - x^3)$ that is not homogeneous.

Things soon become even more complicated in the complete intersection case of $h = 2$. Here we study the moduli problem for complete intersection local rings with Hilbert function $1, 2, 2, 2, 1, 1, 1$; we will see that, in this case, we have a one-dimensional family.

In what follows, $(R, \mathfrak{n})$ is a two-dimensional regular local ring such that $\mathbf{k} = R/\mathfrak{n}$ has the property $\mathbf{k}^{1/2} \subseteq \mathbf{k}$, and I is an ideal in R such that $A = R/I$ is Gorenstein with Hilbert function $1, 2, 2, 2, 1, 1, 1$. Rather than going into all the details, we simply give a sketch of what is going on.

By the main structure theorem we know that there exists a system of generators y_1, y_2 of $\mathfrak{n}$ and an element $a \in R$ such that, by Proposition 4.8,

$$I = (y_1^3y_2, y_2^2 - ay_1y_2 - y_1^4).$$

Case 1: $a \notin \mathfrak{n}$. Let us change the generators as follows:

$$z_1 = ay_1 - y_2, \qquad z_2 = y_1^3 + ay_2.$$

Then

$$d := \det\begin{pmatrix} a & y_1^2 \\ -1 & a \end{pmatrix} = a^2 + y_1^2 \notin \mathfrak{n},$$

so that z_1, z_2 is a minimal system of generators of $\mathfrak{n}$. We have

$$z_1z_2 = (ay_1 - y_2)(y_1^3 + ay_2) = -a(y_2^2 - ay_1y_2 - y_1^4) - y_1^3y_2 \in I.$$

Since I contains the product of two minimal generators of $\mathfrak{n}$, there exists a system of generators x, y of $\mathfrak{n}$ such that

$$I = (xy, y^4 - x^6).$$

Case 2: $a \in \mathfrak{n}$. In this case, we write $a = by_1 + cy_2$ and choose $v \in R$ such that $1 - cy_1 \cong v^2$ modulo I (see Lemma 3.6). Observe that $v \notin \mathfrak{n}$, so we can change the generators as follows:

$$x_1 = y_1, \qquad x_2 = vy_2.$$

Hence we can prove that

$$I = (x_1^3x_2, x_2^2 - dx_1^2x_2 - x_1^4)$$

with $d = bv^{-1} \in R$.

Case 2a: $d \in \mathfrak{n}$. In this case we write $d = fx_1 + ex_2$ and choose $v \in R$ such that $v^2 \cong 1 - ex_1^2$ modulo I. It is clear that $v \notin \mathfrak{n}$, so we can change the generators of $\mathfrak{n}$ by letting

$$x = x_1, \qquad y = vx_2.$$

Then it is easy to prove that

$$I = (x^3y, y^2 - x^4).$$

Let us now consider the case $d \notin \mathfrak{n}$. We distinguish two subcases, $d^2 + 4 \in \mathfrak{n}$ and $d^2 + 4 \notin \mathfrak{n}$.

Case 2b1: $d^2 + 4 \in \mathfrak{n}$. Here we have, modulo I,

$$(x_2 - (d/2)x_1^2)^2 \cong x_1^4 + (d^2/4)x_1^4 \cong x_1^4(1 + (d^2/4)) = ex_1^5$$

with $e \in R$. It follows that, letting

$$l := x_2 - (d/2)x_1^2 + (e/d)x_1^3 + (e^2/d^3)x_1^4,$$

we have $l^2 \in I$. Then, modulo I,

$$\begin{aligned} x_1^3l &= x_1^3(x_2 - (d/2)x_1^2 + (e/d)x_1^3 + (e^2/d^3)x_1^4) \cong -(d/2)x_1^5 + (e/d)x_1^6 \\ &= x_1^5(-d/2 + (e/d)x_1) = vx_1^5 \end{aligned}$$

with $v \notin \mathfrak{n}$. It follows that $J = (l^2, x_1^3l - vx_1^5) \subseteq I$. Next we prove $J = I$.

Notice that x and l form a minimal system of generators of $\mathfrak{n}$. We denote by L the initial form of l in the associated graded ring $\mathrm{gr}_{\mathfrak{n}}(R)$. In order to prove that $I = J$, we need to show that the Hilbert function of R/J is $1, 2, 2, 2, 1, 1, 1$. Given

$$(X^3L, L^2) \subseteq J^* \subseteq I^*,$$

we must prove that

$$Z^7 \in J.$$

Notice that, modulo J,

$$vx^7 = x^5 l = v^{-1}(x^3 l^2) = 0.$$

Hence $x^7 \in J$, so $(l^2, x_1^3 l - vx_1^5) = I$.

Now

$$(l^2, x_1^3 l - vx_1^5) = (l^2, (x_1^3 l/v) - x_1^5) = ((l/v)^2, x_1^3(l/v) - x_1^5).$$

If we let $x := l/v$ and $y = x_1$, then $\mathfrak{n} = (x, y)$ and

$$I = (x^2, xy^3 - y^5).$$

Case 2b2: $d^2 + 4 \notin \mathfrak{n}$. We can find $c, e \in R \setminus \mathfrak{n}$ such that (modulo I) we have $c^2 \cong d^2 + 4$ and $e^2 \cong -(2/c)$ (see Lemma 3.6). We let $p := d/c$ and change the generators of $\mathfrak{n}$ by letting

$$x = (x_1/e), \qquad y = x_2 + p(x_1/e)^2.$$

Then

$$x_1 = xe, \qquad x_2 = y - px^2$$

and so, modulo I, we have

$$0 \cong x_1^3 x_2 = x^3 e^3 (y - px^2) = e^3(x^3 y - px^5);$$

this implies $x^3 y - px^5 \in I$. Furthermore,

$$\begin{aligned} 0 &\cong x_2^2 - dx_1^2 x_2 - x_1^4 = (y - px^2)^2 - dx^2 e^2 (y - px^2) - x^4 e^4 \\ &= y^2 - x^2 y(2p + de^2) + x^4(p^2 + de^2 p - e^4) \cong y^2 - x^4 \end{aligned}$$

because

$$2p + de^2 = 2(d/c) + de^2 \cong 2(d/c) - 2(d/c) = 0$$

and

$$p^2 + de^2 p - e^4 = (d^2/c^2) + (d/c)d(-2/c) - (4/c^2) = -(d/c)^2 - (2/c)^2 \cong -1.$$

This proves $J := (x^3 y - px^5, y^2 - x^4) \subseteq I$. We remark that

$$p^2 - 1 = (d/c)^2 - 1 = (d^2 - c^2)/c^2 \cong -(2/c)^2,$$

and this implies

$$p^2 - 1 \notin \mathfrak{n}.$$

In order to prove that $I = J$, we need to show that the Hilbert function of R/J is $1, 2, 2, 2, 1, 1, 1$. We have

$$(X^3 Y, Y^2) \subseteq J^* \subseteq I^*;$$

moreover,

$$y(x^3 y - px^5) - x^3(y^2 - x^4) = -pyx^5 + x^7 \in J,$$

which implies $x^5 y - (1/p)x^7 \in J$. Therefore,

$$x^2(x^3 y - px^5) - (x^5 y - (1/p)x^7) = ((1 - p^2)/p)x^7 \in J.$$

From this it follows that $x^7 \in J$ and hence

$$(X^3Y, Y^2, X^7) \subseteq J^* \subseteq I^*.$$

These ideals have the same Hilbert function, so we finally get

$$I = (x^3y - px^5, y^2 - x^4)$$

with

$$p \notin \mathfrak{n}, \qquad p^2 - 1 \notin \mathfrak{n}.$$

We have thus found three models (Case 1, Case 2a, Case 2b1) and a one-dimensional family (Case 2b2). We summarize the models as follows.

Case 1: $I = (xy, y^4 - x^6)$.

Case 2a: $I = (x^3y, y^2 - x^4)$.

Case 2b1: $I = (x^2, xy^3 - y^5)$.

Case 2b2: $I = (x^3y - px^5, y^2 - x^4)$; $p \notin \mathfrak{n}$ and $p^2 - 1 \notin \mathfrak{n}$.

At this point a natural question is whether we can pass from one model to another by changing the generators of $\mathfrak{n}$. For example, the model $I = (xy, y^4 - x^6)$ of Case 1 cannot be reached by any of the other models because, however we choose the element $a \in \mathfrak{n}$, the ideal $(x^3y, y^2 - axy - x^4)$ clearly does not contain the product of two minimal generators of the maximal ideal $\mathfrak{n}$.

We are able to prove that all the models we have found are indeed nonisomorphic. Here we give a proof for the ideals in the family of Case 2b2.

Proposition 5.1. *Let $p, q \in R$ be such that $p, q, p^2 - 1, q^2 - 1 \notin \mathfrak{n}$. If $\mathfrak{n} = (x, y) = (z, v)$ and $(x^3y - px^5, y^2 - x^4) = (z^3v - qz^5, v^2 - z^4)$, then $p^2 - q^2 \in \mathfrak{n}$.*

Proof. Let $I := (x^3y - px^5, y^2 - x^4)$. We will use the equalities $(\mathfrak{n}/I)^3 = (\bar{x}^3, \bar{x}^2\bar{y})$, $(\mathfrak{n}/I)^4 = (\bar{x}^4)$, and $(\mathfrak{n}/I)^5 = (\bar{x}^5)$.

We first use the generators $v^2 - z^4$ to derive that $v^2 \in \mathfrak{n}^4 + I \subseteq (y, x^4)$. This implies $v \in (y, x^2)$ and so $v = ex^2 + by$ with $b \notin \mathfrak{n}$. Since (modulo I) we have

$$v^2 = e^2x^4 + 2ebx^2y + b^2y^2 \cong e^2x^4 + 2ebx^2y + b^2x^4,$$

it follows that $2ebx^2y \in \mathfrak{n}^4 + I$; this gives $e \in \mathfrak{n}$ and, finally,

$$v = ax^3 + by$$

with $a \in R$ and $b \notin \mathfrak{n}$. We also have $z = cx + dy$ with

$$\det\begin{pmatrix} ax^2 & c \\ b & d \end{pmatrix} = adx^2 - bc \notin \mathfrak{n},$$

which implies $c \notin \mathfrak{n}$.

Now, modulo I, we have $0 \cong v^2 - z^4 = b^2x^4 - c^4x^4 + t$ with $t \in \mathfrak{n}^5$, which implies $b^2 - c^4 \in \mathfrak{n}$. We also have

$$0 \cong z^3v - qz^5 = z^3(v - qz^2) \cong c^3bpx^5 - qc^5x^5 + f$$

with $f \in \mathfrak{n}^6$. This implies $c^3bp - qc^5 \in \mathfrak{n}$ and so $bp - qc^2 \in \mathfrak{n}$. Since $b^2 - c^4 \in \mathfrak{n}$, we easily obtain the conclusion $p^2 - q^2 \in \mathfrak{n}$. □

With the methods explained before we can manage also the case with Hilbert function $1, 3, 2, 1$. Because this was the sole remaining unresolved case, we can now classify, up to isomorphism, all Artinian Gorenstein **k**-algebras of degree 7. Thus we can solve Question 4.4. of [1]. We prove that if R/I is Gorenstein with Hilbert function $1, 3, 2, 1$ then, after a possible change of generators of $\mathfrak{n}$, either

$$I = (xy, xz, yz, x^3 - y^3, z^2 - y^3) \quad \text{or} \quad I = (x^3, y^2, yz, xz, z^2 - x^2y).$$

References

[1] G. Casnati and R. Notari, *On some Gorenstein loci in $\mathcal{H}ilb_6(P_k^4)$*, J. Algebra 308 (2007), 493–523.

[2] J. Elias, L. Robbiano, and G. Valla, *Number of generators of ideals,* Nagoya Math. J. 123 (1991), 39–76.

[3] A. Iarrobino, *Hilbert scheme of points: Overview of last ten years,* Algebraic geometry, Bowdoin, 1985 (Brunswick, ME, 1985), Proc. Sympos. Pure Math., 46, pp. 297–320, Amer. Math. Soc., Providence, RI, 1987.

[4] J. C. Rosales, *On numerical semigroups,* Semigroup Forum 52 (1996), 307–318.

[5] ———, *On symmetric numerical semigroups,* J. Algebra 182 (1996), 422–432.

[6] J. C. Rosales and P. A. García-Sánchez, *On numerical semigroups with high embedding dimension,* J. Algebra 203 (1998), 567–578.

[7] M. E. Rossi and G. Valla, *Stretched m-primary ideals,* Beiträge Algebra Geom. 42 (2001), 103–122.

[8] J. D. Sally, *Stretched Gorenstein rings,* J. London Math. Soc. (2) 20 (1979), 19–26.

J. Elias
Departament d'Àlgebra i Geometria
Universitat de Barcelona
Gran Via 585
08007 Barcelona
Spain

elias@ub.edu

G. Valla
Departimento di Matematica
Università di Genova
Via Dodecaneso 35
16146 Genova
Italy

valla@dima.unige.it