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Row Ideals and Fibers of Morphisms

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Affectionately dedicated to Mel Hochster,
who has been an inspiration to us for many years,
on the occasion of his 65th birthday

1. Introduction

In this paper we study the fibers of a rational map from an algebraic point of view. We begin by describing four ideals related to such a fiber.

Let $S = k[x_0, \dots, x_n]$ be a polynomial ring over an infinite field k with homogeneous maximal ideal $\mathbf{m}$, let $I \subset S$ be an ideal generated by an $(r+1)$-dimensional vector space W of forms of the same degree, and let ϕ be the associated rational map $\mathbf{P}^n \to \mathbf{P}^r = \mathbf{P}(W)$. We will use this notation throughout. Since we are interested in the rational map, we may remove common divisors of W and thus assume that I has codimension ≥ 2.

A k-rational point q in the target $\mathbf{P}^r = \mathbf{P}(W)$ is by definition a codimension 1 subspace W_q of W. We write $I_q \subset S$ for the ideal generated by W_q. By a homogeneous presentation of I we will always mean a homogeneous free presentation of I with respect to a homogeneous minimal generating set. If $F \to G = S \otimes W$ is such a presentation, then the composition $F \to G \to S \otimes (W/W_q)$ is called the *generalized row* corresponding to q, and its image is called the *generalized row ideal* corresponding to q. It is the ideal generated by the entries of a row in the homogeneous presentation matrix after a change of basis. From this we see that the generalized row ideal corresponding to q is simply $I_q : I$.

The rational map ϕ is a morphism away from the algebraic set $V(I)$, and we may form the fiber (= preimage) of the morphism over a point $q \in \mathbf{P}^r$. The saturated ideal of the scheme-theoretic closure of this fiber is $I_q : I^\infty$, which we call the *morphism fiber ideal* associated to q.

The rational map ϕ gives rise to a *correspondence* $\Gamma \subset \mathbf{P}^n \times \mathbf{P}^r$, which is the closure of the graph of the morphism induced by ϕ. There are projections

$$\mathbf{P}^n \xleftarrow{\pi_1} \Gamma \xrightarrow{\pi_2} \mathbf{P}^r,$$

and we define the *correspondence fiber* over q to be $\pi_1(\pi_2^{-1}(q))$. Since Γ is $\mathrm{BiProj}(\mathcal{R})$, where $\mathcal{R}$ is the Rees algebra $S[It] \subset S[t]$ of I, it follows that the correspondence fiber is defined by the ideal

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$$(I_q t\mathcal{R} : (It)^\infty) \cap S = \bigcup_i (I_q I^{i-1} : I^i).$$

This ideal describes the locus where I is not integral over I_q. It is clear that our four ideals are contained, each in the next, as follows:

$$\begin{aligned} I_q &\subset I_q : I && \text{(row ideal);} \\ &\subset \bigcup_i (I_q I^{i-1} : I^i) && \text{(correspondence fiber ideal);} \\ &\subset I_q : I^\infty && \text{(morphism fiber ideal).} \end{aligned}$$

In Section 2 we compare the row ideals, morphism fiber ideals, and correspondence fiber ideals.

In Section 3 we use generalized row ideals to give bounds on the analytic spread of I by interpreting the analytic spread as 1 plus the dimension of the image of ϕ.

Many interesting rational maps ϕ are associated as just described to ideals I with linear presentation matrices (see e.g. [HKS]). Thus we are interested in linearly presented ideals and their powers, which arise in the study of the graph. It is known that the powers of a linearly presented ideal need not be linearly presented. The first such examples were exhibited by Sturmfels [St]; for a survey of what is known, see [EHuU]. In Section 3 we also give criteria for birationality of the map or its restriction to a linear subspace of $\mathbf{P}^n$.

In Section 4 we generalize the notion of linear presentation (of an ideal or module) in various directions: A graded S-module M generated by finitely many elements of the same degree has *linear generalized row ideals* if the entries of *every* generalized row of a homogeneous presentation matrix for M generate a linear ideal (i.e., an ideal generated by linear forms). Obviously, any module with a linear presentation has this property, and we conjecture that the two notions are equivalent in the case of ideals. The corresponding conjecture is false for modules, but we prove it for modules of projective dimension 1. The main result of the section implies a weak linearity property of powers; it states that if an ideal I has linear generalized row ideals, then every power of I has a homogeneous presentation all of whose (ordinary) rows generate linear ideals.

2. Comparing the Notions of Fiber Ideals

Recall that the row ideal for a point q is always contained in the correspondence fiber ideal, which in turn is contained in the morphism fiber ideal. If the row ideal is generated by linear forms (or, more generally, is prime) and does not contain I, then these ideals are all equal. But in general the containments are both strict, as the following example shows.

EXAMPLE 2.1. Let

$$S = k[a,b,c,d], \quad J = (ab^2, ac^2, b^2c, bc^2), \quad I = J + (bcd).$$

One can check that I is linearly presented. Computation shows that the row ideal $J : I$ is (b,c) while the correspondence fiber ideal is (a^2,b,c) and the morphism fiber ideal is the unit ideal $J : I^\infty = S$. We have no example of an $\mathbf{m}$-primary ideal (regular morphism) where all three are different: in the examples we have tried, the correspondence fiber is equal to the morphism fiber. (Of course, for any regular map all three are equal up to saturation, but we do not see why any two should be equal as ideals.)

Before stating the next result we recall that an ideal I in a Noetherian ring is said to be *of linear type* if the natural map from the symmetric algebra of I onto the Rees algebra of I is an isomorphism. If I is of linear type, then I cannot be integral over any strictly smaller ideal, as can be seen by applying [NR, Thm. 4, p. 152] to the localizations of I. We say that an ideal is *proper* if it is not the unit ideal.

PROPOSITION 2.2. *If I has linear generalized row ideals, then every proper morphism fiber ideal is equal to the corresponding row ideal and hence is generated by linear forms. If I is also of linear type on the punctured spectrum, then every proper correspondence fiber ideal is equal to the corresponding row ideal.*

Proof. Suppose that the morphism fiber ideal $I_q : I^\infty$ is not the unit ideal. In this case $I_q : I$ does not contain I. The required equality for the first statement is

$$I_q : I = I_q : I^\infty,$$

which follows because $I_q : I$ is linear and thus prime.

Now suppose that I is of linear type on the punctured spectrum and that the correspondence fiber ideal $H := \bigcup_i (I_q I^{i-1} : I^i)$ is proper. Set $K = I_q : I$, the row ideal. We must show $K = H$. Since $K \subset H$ we may safely assume that K is not $\mathbf{m}$, the homogeneous maximal ideal of S. By hypothesis, the row ideal K is generated by linear forms; hence it is prime. Since the localized ideals $(I_q)_K$ and I_K are not equal and since I_K is of linear type, it follows that I_K is not integral over $(I_q)_K$. Therefore, H_K is a proper ideal. Thus $H \subset K$, as required. □

EXAMPLE 2.3. The last statement of Proposition 2.2 would be false without the hypothesis that I is of linear type on the punctured spectrum. This is shown by Example 2.1.

EXAMPLE 2.4. Let Q be a quadratic form in x_0, x_1, x_2, and let F be a cubic form relatively prime to Q. The rational map defined by $x_0 Q, x_1 Q, x_2 Q, F$ has one morphism fiber (and correspondence fiber) ideal (Q), though for a general point in the image both the morphism fiber ideal and the correspondence fiber ideal are linear. This example shows that in [Si, Thm. 4.1] the point p should be taken to be general.

3. How to Compute the Analytic Spread and Test Birationality

The notions of row ideals and fiber ideals provide tests for the birationality of the map ϕ and lead to formulas for the analytic spread of the ideal I. In our setting, the *analytic spread* $\ell(I)$ of I can be defined as 1 plus the dimension of the image of the rational map ϕ. Its ideal-theoretic significance is that it gives the smallest number of generators of a homogeneous ideal over which I is integral or, equivalently, the smallest number of generators of an ideal in $S_{\mathbf{m}}$ over which $I_{\mathbf{m}}$ is integral (see [NR, Cor., p. 151]).

PROPOSITION 3.1. (a) *If q is a point in $\mathbf{P}^r = \mathbf{P}(W)$ such that $I_q : I^\infty \neq S$, then*

$$\ell(I) \geq 1 + \operatorname{codim}(I_q : I^\infty).$$

(b) *If p is a general point in $\mathbf{P}^n$, then*

$$\ell(I) = 1 + \operatorname{codim}(I_{\phi(p)} : I^\infty).$$

(c) *If there exits a point q such that the row ideal $I_q : I$ is linear of codimension n and does not contain I, then ϕ is birational onto its image. Moreover, ϕ is birational onto its image if and only if $I_{\phi(p)} : I^\infty$ is a linear ideal of codimension n for a general point p.*

Proof. Set $J = I_{\phi(p)}$. If the ideal $I_q : I^\infty$ is proper, then it cannot be $\mathbf{m}$-primary and so defines a nonempty fiber of the morphism ϕ. On the other hand, $J : I^\infty$ is the defining ideal of a general fiber of the map. Thus the dimension formula and the semicontinuity of fiber dimension [E, Cor. 14.5, Thm. 14.8(a)] show that

$$\operatorname{codim}(I_q : I^\infty) \leq \operatorname{codim}(J : I^\infty) = \dim \operatorname{im}(\phi).$$

However, the latter dimension is $\ell(I) - 1$, proving parts (a) and (b).

The second assertion in (c) holds because the map is birational onto its image if and only if the general fiber is a reduced rational point.

We reduce the first assertion of (c) to the second one. Assume that the row ideal $I_q : I$ is linear of codimension n and does not contain I. Since $I_q : I$ is a prime ideal not containing I, it follows that $I_q : I^\infty = I_q : I \neq S$. Hence the morphism fiber over q is not empty and there exists a point $p \in \mathbf{P}^n$ with $q = \phi(p)$.

Now let $T_0, \ldots, T_r$ be variables over S and let A_1 denote the linear part of a homogeneous presentation matrix of I. We can write $(T_0, \ldots, T_r) * A_1 = (x_0, \ldots, x_n) * B$ for some matrix B whose entries are linear forms in the variables T_i with constant coefficients. The dimension of the space of linear forms in the row ideal corresponding to any point $\phi(p)$ is the rank of B when the coordinates of $\phi(p)$ are substituted for the T_i; it is therefore semicontinuous in p. Thus, for p general, the dimension of the space of linear forms in the ideal $I_{\phi(p)} : I$ is at least n, and then the same holds for $J : I^\infty$. Because this ideal defines a nonempty fiber, it is indeed linear of codimension n. □

Sometimes one can read off a lower bound on the analytic spread even from a partial matrix of syzygies. The following result is inspired by [HKS, Prop. 1.2].

PROPOSITION 3.2. *With notation as before, suppose that A is a matrix of homogeneous forms each of whose columns is a syzygy on the generators of I. Let A_q be the ideal generated by the elements of the generalized row of A corresponding* *to a point* $q \in \mathbf{P}^r$. *If there exists a prime ideal* $P \in V(A_q)$ *such that* $A \otimes \kappa(P)$ *has rank* r, *then* $I_q : I^\infty \neq S$ *and*

$$\ell(I) \geq 1 + \operatorname{codim} A_q.$$

Proof. Since $A_q \subset I_q : I^\infty$, Proposition 3.1(a) shows that the second claim follows from the first one. To prove the first assertion, $I_q : I^\infty \neq S$, it suffices to verify that $(I_q : I^\infty)_P \neq S_P$.

Since A_P contains an $r \times r$ invertible submatrix and since these relations express each generator of I_P in terms of the one corresponding to q, it follows that A_P is a full presentation matrix of the ideal I_P. Thus $(A_q)_P = (I_q : I)_P$. Furthermore, since I_P is generated by one element and since I has codimension ≥ 2 by our blanket assumption, it follows that $I_P = S_P$, whence $(A_q)_P = (I_q : I)_P = (I_q : I^\infty)_P$. On the other hand, $P \in V(A_q)$, so $(A_q)_P \neq S_P$ and we are done. □

As in [HKS, Prop. 1.2], this gives criteria for birationality as follows.

Corollary 3.3. *As in Proposition 3.2, suppose that* $A \otimes \kappa(P)$ *has rank* r *for some prime ideal* $P \in V(A_q)$. *The map* ϕ *is birational onto its image if* A_q *defines a reduced rational point in* $\mathbf{P}^n$. *The map* ϕ, *restricted to a general* $\mathbf{P}^r \subset \mathbf{P}^n$, *is birational* (*a Cremona transformation*) *if* A_q *defines a reduced linear space of codimension* r *in* $\mathbf{P}^n$.

Proof. Notice that $A_q \subset I_q : I \subset I_q : I^\infty$, where $I_q : I^\infty \neq S$ according to Proposition 3.2. Thus, if A_q defines a reduced rational point in $\mathbf{P}^n$, then the row ideal $I_q : I$ is linear of codimension n and does not contain I. Hence ϕ is birational onto its image by Proposition 3.1(c).

The second assertion follows from the first one applied to the restriction of ϕ. □

For other, related criteria for birationality see [Si].

4. Ideals with Linear Row Ideals and Their Powers

We begin this section by clarifying the relation between these properties of an ideal or module: to have a linear presentation matrix, to have linear generalized row ideals, and to have *some* homogeneous presentation matrix all of whose row ideals are linear. Obviously, if a presention matrix is linear then all its generalized row ideals are linear. However, the converse does not hold, at least for the presentation of modules with torsion. This can be seen by taking the matrix

$$\begin{pmatrix} s & t & t^2 \\ 0 & s & 0 \end{pmatrix},$$

for instance. Even so, we have the following statement.

Proposition 4.1. *If* M *is a graded* S*-module of projective dimension* 1 *generated by finitely many homogeneous elements of the same degree, and if* M *has linear generalized row ideals, then* M *has a linear presentation.*

Proof. Reduce modulo n general linear forms and then use the fundamental theorem for modules over principal ideal domains. □

If an ideal has linear generalized row ideals then obviously there is a presentation matrix with only linear row ideals. Again, the two concepts are not equivalent, as the next example shows.

EXAMPLE 4.2. We consider the ideal $I = (s^4, s^3t, st^3, t^4) \subset S = \mathbf{C}[s,t]$ corresponding to the morphism whose image is the smooth rational quartic curve in $\mathbf{P}^3$. A homogeneous presentation of this ideal is given by

$$S^2(-5) \oplus S(-6) \xrightarrow{\begin{pmatrix} t & 0 & 0 \\ -s & 0 & t^2 \\ 0 & t & -s^2 \\ 0 & -s & 0 \end{pmatrix}} S^4(-4) \xrightarrow{(s^4\ s^3t\ st^3\ t^4)} S.$$

The row ideals of the second and third rows in this presentation are not linear. However, a change of basis in $S^4(-4)$, corresponding to a different choice of generators of I, makes them linear:

$$S^2(-5) \oplus S(-6) \xrightarrow{\begin{pmatrix} t & 0 & 0 \\ 0 & s & 0 \\ s-t & s-t & s^2-t^2 \\ -s+it & -is-t & s^2+t^2 \end{pmatrix}} S^4(-4) \xrightarrow{(F_0 \dots F_3)} S,$$

where

$$F_0 = -s(s-t)(s^2+t^2+(s+t)(s-it)),$$
$$F_1 = -t(s-t)(s^2+t^2+(s+t)(is+t)),$$
$$F_2 = st(s^2+t^2),$$
$$F_3 = -st(s^2-t^2).$$

Whereas powers of linearly presented ideals need not be linearly presented, the next result implies that having a homogeneous presentation with linear generalized row ideals is a weak linearity property that is indeed preserved when taking powers.

THEOREM 4.3. *If I has a homogeneous presentation matrix where at least one row ideal is linear of codimension at least $\ell(I)-1$ and does not contain I, then each power of I has some homogeneous presentation matrix all of whose row ideals are linear of codimension $\ell(I)-1$ and do not contain I.*

Proof. According to Proposition 3.1(b) for general $p \in \mathbf{P}^n$, the morphism fiber ideal $I_{\phi(p)} : I^\infty$ has codimension $\ell(I)-1$; hence the row ideal $I_{\phi(p)} : I$ has codimension at most $\ell(I)-1$. Now one sees, as in the proof of Proposition 3.1(c), that $I_{\phi(p)} : I$ is linear of codimension $\ell(I)-1$ and does not contain I.

Let $E = V(I)$ be the exceptional locus of ϕ. For each $d \geq 1$, the rational map ϕ_d defined by the vector space of forms W^d is regular on $\mathbf{P}^n \setminus E$. For any point $p \in \mathbf{P}^n \setminus E$, the ideal of $\phi(p) \in \mathbf{P}(W)$ is generated by the vector space of linear forms

$W_{\phi(p)}$, so the vector space of forms of degree d that it contains is $W_{\phi(p)}W^{d-1}$. Therefore, $(W^d)_{\phi_d(p)} = W_{\phi(p)}W^{d-1}$ and hence the row ideal corresponding to $\phi_d(p)$ is $I_{\phi(p)}I^{d-1} : I^d$.

We now show that, for general p, the row ideal $I_{\phi(p)}I^{d-1} : I^d$ is linear of codimension $\ell(I) - 1$ and does not contain I. For trivial reasons we have

$$I_{\phi(p)} : I \subset I_{\phi(p)}I^{d-1} : I^d \subset I_{\phi(p)}I^{d-1} : I^\infty \subset I_{\phi(p)} : I^\infty.$$

By the proof's first paragraph, $I_{\phi(p)} : I$ is a linear ideal of codimension $\ell(I) - 1$ and does not contain I. Hence

$$I_{\phi(p)} : I = I_{\phi(p)} : I^\infty$$

and therefore

$$I_{\phi(p)} : I = I_{\phi(p)}I^{d-1} : I^d.$$

Let $\dim W^d = N + 1$. Because the image of ϕ_d is nondegenerate, $N + 1$ general points of $\mathbf{P}^n$ correspond to the $N + 1$ rows of a presentation matrix of I^d, so we are done. □

COROLLARY 4.4. *If I has linear presentation or even just linear generalized row ideals, then every power of I has a homogeneous presentation matrix all of whose row ideals are linear of codimension $\ell(I) - 1$.*

Proof. According to Proposition 3.1(b), the homogeneous presentation matrix of I has a row ideal $I_q : I$ such that $\operatorname{codim}(I_q : I^\infty) = \ell(I) - 1$. In particular, $I_q : I^\infty \neq S$ and so I is not contained in $I_q : I$. Because $I_q : I$ is a linear ideal we conclude that $I_q : I = I_q : I^\infty$, which gives $\operatorname{codim}(I_q : I) = \ell(I) - 1$. Now apply Theorem 4.3. □

PROPOSITION 4.5. *Every ideal has a homogeneous presentation all of whose row ideals are of codimension $\leq \ell(I) - 1$.*

Proof. Take a homogeneous presentation whose rows correspond to the fibers through points of $\mathbf{P}^n$ not in the exceptional locus. The row ideals are contained in the morphism fiber ideals, which by Proposition 3.1(a) have codimension at most $\ell(I) - 1$. □

5. Some Open Problems

We are most interested in answers to the following questions.

1. Can the homogeneous minimal presentation of an ideal I have linear generalized row ideals without actually being linear?
2. If ϕ is a regular map (i.e., if I is $\mathbf{m}$-primary), then are the correspondence fiber ideals equal to the morphism fiber ideals? More generally, when are the correspondence fiber ideals saturated with respect to $\mathbf{m}$?
3. Find lower bounds for the number of linear relations that I^d could have in terms of the number of linear relations on I. How close can one come to the known examples?

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