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Solvable Quotients of Kähler Groups

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1. Introduction

We first recall a definition from [AN].

Definition 1.1. A solvable group Γ has *finite rank* if there is a decreasing sequence $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_{m+1} = \{1\}$ of subgroups, each normal in its predecessor, such that Γ_i/Γ_{i+1} is abelian and $\mathbb{Q} \otimes (\Gamma_i/\Gamma_{i+1})$ is finite-dimensional for all i.

In this paper, $\mathcal{F}_r$ denotes a free group with the number of generators $r \in \mathbb{Z}_+ \cup \{\infty\}$. Our main result is as follows.

Theorem 1.2. *Let M be a compact Kähler manifold. Assume that the fundamental group $\pi_1(M)$ is defined by the sequence*

$$\{1\} \longrightarrow F \longrightarrow \pi_1(M) \xrightarrow{p} H \longrightarrow \{1\},$$

where H is a solvable group of finite rank of the form

$$\{0\} \longrightarrow A \longrightarrow H \longrightarrow B \longrightarrow \{0\}$$

with nontrivial abelian groups A, B such that $\mathbb{Q} \otimes A \cong \mathbb{Q}^m$ and $m \geq 1$. Assume also that $p^{-1}(A) \subset \pi_1(M)$ does not admit a surjective homomorphism onto $\mathcal{F}_\infty$. Then all eigencharacters of the conjugate action of B on the vector space $\mathbb{Q} \otimes A$ are torsion.

In Lemma 3.3 we will show that the condition for $p^{-1}(A)$ holds if F does not admit a surjective homomorphism onto $\mathcal{F}_\infty$.

Using Theorem 1.2, we prove the following result on solvable quotients of Kähler groups.

Theorem 1.3. *Assume that a Kähler group G is defined by the sequence*

$$\{1\} \longrightarrow F \longrightarrow G \xrightarrow{q} H \longrightarrow \{1\},$$

where F does not admit a surjective homomorphism onto $\mathcal{F}_\infty$ and H is a solvable group of finite rank. Then there exist normal subgroups $H_1 \supset H_2$ of H such that

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(a) *H_1 has finite index in H,*
(b) *H_1/H_2 is nilpotent, and*
(c) *H_2 is torsion.*

EXAMPLE 1.4. Let G be an extension of a group H by a finitely generated group F. Suppose that H is an extension of $\mathbb{Z}^n$ by $\mathbb{Z}^m$ $(n, m \geq 1)$ such that not all eigencharacters of the action of $\mathbb{Z}^n$ on $\mathbb{Q} \otimes \mathbb{Z}^m$ are torsion. Then Theorem 1.2 implies that G is not a Kähler group.

REMARK 1.5. This paper was motivated by the pioneering result of Arapura and Nori [AN] and its generalization due to Campana [C]. In order to describe this result, let G be a Kähler group, let $DG = D^1G$ be the derived subgroup of G, and let $D^iG = DD^{i-1}G$.

THEOREM [AN, Thm. 4.9; C, Thm. 2.2]. *Suppose that $H := G/D^nG$ $(n \geq 1)$ is a solvable group of finite rank. Then H satisfies conditions* (a)–(c) *of Theorem 1.3.*

Note that the stated hypothesis for H implies that G does not have any quotient isomorphic to $\mathcal{F}_r$ for $2 \leq r < \infty$. This is the crucial point in the proof. In some sense our results are about groups that admit surjective homomorphisms onto $\mathcal{F}_r$ for $2 \leq r < \infty$. Indeed, let R be the fundamental group of a compact complex curve of genus 2. For any G, F satisfying the hypotheses of Theorem 1.3, define $\tilde{G} = G \times R$ and $\tilde{F} = F \times R$. Then $\tilde{G}$ is a Kähler group defined by

$$\{1\} \longrightarrow \tilde{F} \longrightarrow \tilde{G} \longrightarrow H \longrightarrow \{1\}.$$

Moreover, $\tilde{G}$ has a quotient isomorphic to $\mathcal{F}_2$, and $\tilde{F}$ does not have any quotient isomorphic to $\mathcal{F}_\infty$ (see Lemma 3.3). Theorem 1.3 claims that H satisfies conditions (a)–(c), but any $\tilde{G}/D^n\tilde{G}$ $(n \geq 2)$ is not of finite rank. Hence the conclusion of Theorem 1.3 cannot be obtained solely from the result of [AN] and [C] along with the fact that H is the quotient of some $\tilde{G}/D^n\tilde{G}$, $n \geq 2$.

2. A Preliminary Result

2.1

In what follows, $T_n(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ denotes the Lie group of upper triangular matrices. Let $T_2 \subset T_2(\mathbb{C})$ be the Lie group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \quad (a \in \mathbb{C}^*,\ b \in \mathbb{C}),$$

and let $D_2 \subset T_2$ and $N_2 \subset T_2$ be the groups of diagonal and unipotent matrices. Let M be a compact Kähler manifold. For a homomorphism $\rho \in \mathrm{Hom}(\pi_1(M), T_2)$ we let $\rho_a \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ denote the upper diagonal character of ρ. Assume that $f: M \to C$ is a holomorphic surjective map with connected fibres onto a smooth compact complex curve C of genus $g \geq 1$. As usual, $f_*: \pi_1(M) \to \pi_1(C)$ denotes the induced homomorphism of the fundamental groups. The following result is basic in the proof of Theorem 1.2.

THEOREM 2.1. *Assume that* $\rho \in \operatorname{Hom}(\pi_1(M), T_2)$ *satisfies* $\operatorname{Ker}(f_*) \subset \operatorname{Ker}(\rho_a)$ *but that* $\operatorname{Ker}(f_*) \not\subset \operatorname{Ker}(\rho)$. *Then* ρ_a *is a torsion character.*

The next several subsections contain results used in the proof of Theorem 2.1.

2.2

Let $f: M \to C$ be a holomorphic surjective map with connected fibres of a compact Kähler manifold M onto a smooth compact complex curve C. Consider a flat vector bundle L on C of complex rank 1 with unitary structure group, and let $E = f^*L$ be the pullback of this bundle to M. Let $\omega \in \Omega^1(E)$ be a holomorphic 1-form with values in E (by the Hodge decomposition, ω is d-closed).

LEMMA 2.2. *Assume that* $\omega|_{V_z} = 0$ *for the fibre* $V_z := f^{-1}(z)$ *over a regular value* $z \in C$. *Then* $\omega|_V = 0$ *for any fibre* V *of* f.

If V is not smooth, the lemma asserts that ω equals 0 on the smooth part of V.

Proof of Lemma 2.2. Denote by $S \subset C$ the finite set of nonregular values of f. By Sard's theorem, $f: M \setminus f^{-1}(S) \to C \setminus S$ is a fibre bundle with connected fibres. According to our assumption there is a fibre V_z of this bundle such that $\omega|_{V_z} = 0$. Then $\omega|_V = 0$ for any fibre V of $f|_{M \setminus f^{-1}(S)}$. In fact, for any fibre V there is an open neighborhood O_V of V diffeomorphic to $\mathbb{R}^2 \times V$ such that $E|_{O_V}$ ($\cong E|_V$) is the trivial flat vector bundle (here $E|_V$ is trivial as the pullback of a bundle defined over a point). Any d-closed holomorphic 1-form defined on O_V that vanishes on V is d-exact and so must vanish on each fibre contained in O_V. Starting with a tubular neighborhood O_{V_z} and taking into account that $\omega|_{O_{V_z}}$ can be considered as a d-closed holomorphic 1-form (because $E|_{O_{V_z}}$ is trivial), we obtain that $\omega|_V = 0$ for any fibre $V \subset O_{V_z}$. Then the required statement follows by induction if we cover $M \setminus f^{-1}(S)$ by open tubular neighborhoods of fibres of f and use the fact that $C \setminus S$ is connected.

Consider now fibres over the singular part S. Let $O_x \subset C$ be the neighborhood of a point $x \in S$ such that $O_x \setminus S = O_x \setminus \{x\}$ is biholomorphic to $\mathbb{D} \setminus \{0\}$. In particular, $\pi_1(O_x \setminus S) = \mathbb{Z}$. Without loss of generality we may assume that $V = f^{-1}(x)$ is a deformation retract of an open neighborhood U_x of V and that $W_x := f^{-1}(O_x) \subset U_x$ (such U_x exists, e.g., by the triangulation theorem of the pair (M, V); see [L, Thm. 2]). Since $E|_V$ is trivial, the bundle $E|_{U_x}$ is also trivial; we can therefore regard ω as a d-closed holomorphic 1-form on U_x. Moreover, we have already proved that $\omega|_F = 0$ for any fibre $F \subset W_x \setminus f^{-1}(S)$, so ω equals 0 on each fibre of the fibration $f: W_x \setminus f^{-1}(S) \to O_x \setminus S$. This implies that there is a d-closed holomorphic 1-form ω_1 on $O_x \setminus S$ such that $\omega = f^*(\omega_1)$. Assume now that $\omega|_V \neq 0$. Then integration of $\omega|_{W_x}$ along paths determines a nontrivial homomorphism $h_{\omega,x}\colon \pi_1(W_x) \to \mathbb{C}$ whose image is isomorphic to $\mathbb{Z}$. Indeed, since embedding $W_x \setminus f^{-1}(S) \hookrightarrow W_x$ induces a surjective homomorphism of fundamental groups, we can integrate ω by paths contained in $W_x \setminus f^{-1}(S)$ and so obtain the same image in $\mathbb{C}$. Furthermore, for any path $\gamma \subset W_x \setminus f^{-1}(S)$ we have $\int_\gamma \omega = \int_{\gamma_1} \omega_1$, where γ_1 is a path in $O_x \setminus S$ representing the element $f_*(\gamma) \in \pi_1(O_x \setminus S)$. But $\pi_1(O_x \setminus S) = \mathbb{Z}$ and therefore $h_{\omega,x}(\pi_1(W_x)) \cong \mathbb{Z}$.

Any path in W_x is homotopically equivalent inside U_x to a path contained in V, so we can define $h_{\omega,x}$ by integrating $\omega|_V$. Thus we obtain a homomorphism h of $\Gamma := (\pi_1(V)/D\pi_1(V))/_{\text{torsion}}$ into $\mathbb{C}$ with the same image as for $h_{\omega,x}$ (i.e., $\cong \mathbb{Z}$). We may assume without loss of generality that V is smooth (else we apply the arguments that follow to each irreducible component of a desingularization of V). Integrating holomorphic 1-forms on V along paths, we embed Γ into some $\mathbb{C}^k$ as a lattice of rank $2k$. Then there is a linear holomorphic functional f_ω on $\mathbb{C}^k$ such that $h = f_\omega|_\Gamma$. Since $rk(h(\Gamma)) = 1$, it follows that f_ω is equal to 0 on a subgroup $H \subset \Gamma$ isomorphic to $\mathbb{Z}^{2k-1}$. In particular, $f_\omega = 0$ on the vector space $H \otimes \mathbb{R}$ of real dimension $2k-1$. But $\operatorname{Ker}(f_\omega)$ is a complex vector space and so $f_\omega = 0$ on $\mathbb{C}^k$, which implies that $\omega|_V = 0$. This contradicts our assumption and proves the required statement for fibres over the points of S. $\square$

2.3

Next we prove that the character ρ_a in Theorem 2.1 cannot be nonunitary.

PROPOSITION 2.3. *Let $\rho \in \operatorname{Hom}(\pi_1(M), T_2)$ be such that ρ_a is a nonunitary character. Assume that $\rho_a|_{\operatorname{Ker}(f_*)}$ is trivial. Then $\rho|_{\operatorname{Ker}(f_*)}$ is also trivial.*

Proof. Let $p_t\colon M_t \to M$ be the Galois covering with transformation group $\operatorname{Tor}(\pi_1(M)/D\pi_1(M))$. Then $\rho|_{\pi_1(M_t)}$ determines a C^∞-trivial complex rank-2 flat vector bundle on M_t, because $\rho_a|_{\pi_1(M_t)} = \exp(\rho'|_{\pi_1(M_t)})$ for some $\rho' \in \operatorname{Hom}(\pi_1(M), \mathbb{C})$. In particular, $\rho|_{\pi_1(M_t)}$ can be defined by a flat connection

$$\Omega = \begin{pmatrix} \omega & \eta \\ 0 & 0 \end{pmatrix}, \quad d\Omega - \Omega \wedge \Omega = 0,$$

on $M_t \times \mathbb{C}^2$, where ω is a d-harmonic 1-form on M_t lifted from M. Moreover, since $\rho_a|_{\operatorname{Ker}(f_*)}$ is trivial, ω is the pullback by $f \circ p_t$ of a d-harmonic 1-form defined on C. Let $\omega = \omega_1 + \omega_2$ be the type decomposition of ω into the sum of holomorphic and antiholomorphic 1-forms lifted from C. Denote by E_ρ the rank-1 flat vector bundle on M_t with unitary structure group constructed by the flat connection $\omega_2 - \overline{\omega_2}$, and denote by $E_0 = M_t \times \mathbb{C}$ the trivial flat vector bundle. Observe that E_ρ is the pullback of a flat bundle on C. In particular, $E_\rho|_V$ is trivial for any fibre V of $f \circ p_t$. According to [Br, Thm. 1.2], the equivalence class of $\rho|_{\pi_1(M_t)}$ is determined by a d-harmonic 1-form θ with values in $\operatorname{End}(E_\rho \oplus E_0)$ satisfying the condition that $\theta \wedge \theta$ represents 0 in $H^2(M_t, \operatorname{End}(E_\rho \oplus E_0))$. More precisely,

$$\theta = \begin{pmatrix} \omega_1 & \eta' \\ 0 & 0 \end{pmatrix},$$

where η' is a d-harmonic 1-form with values in E_ρ satisfying the condition that $\omega_1 \wedge \eta'$ represents 0 in $H^2(M_t, E_\rho)$. Let $M_t \xrightarrow{g_1} M_1 \xrightarrow{g_2} C$ be the Stein factorization of $f \circ p_t$, where g_1 is a morphism with connected fibres onto a smooth curve M_1 and g_2 is a finite morphism. Then, for any fibre $V \hookrightarrow M_t$ of g_1, we have that $p_t|_V : V \to p_t(V)$ is a regular covering of the fibre $p_t(V)$ of f with a finite abelian transformation group.

Lemma 2.4. *$\eta'|_V = 0$ for fibres over regular values of g_1.*

We first use this statement to finish the proof of the proposition; then we prove the lemma. From the lemma and Lemma 2.2 it follows that $\eta'|_W = 0$ for any fibre $i: W \hookrightarrow M_t$ of g_1. Thus $\rho|_{\pi_1(M_t)}$ is trivial on $i_*(\pi_1(W)) \subset \pi_1(M_t)$. But $\pi_1(W)$ is a subgroup of a finite index in $\pi_1(p_t(W))$, and by assumption the image of $\rho|_{j_*(\pi_1(p_t(W)))}$ consists of unipotent matrices. (Here $j: p_t(W) \hookrightarrow M$ is embedding.) Hence $\rho|_{j_*(\pi_1(p_t(W)))}$ is trivial for any W. Denote by E the flat vector bundle on M associated to ρ. Then we have proved that $E|_W$ is trivial for any fibre W of f. Further, let $(U_i)_{i\in I}$ be an open cover of C such that (a) $W_i := f^{-1}(U_i)$ is an open neighborhood of a fibre $V_i = f^{-1}(x_i)$, $x_i \in U_i$, and (b) W_i is deformable onto V_i. Since $E|_{V_i}$ is trivial, $E|_{W_i}$ is also trivial. In particular, E is defined by a locally constant cocycle $\{c_{ij}\}$ defined on the cover $(W_i)_{i\in I}$. But then $\{c_{ij}\}$ is the pullback of a cocycle defined on $(U_i)_{i\in I}$ because the fibres of f are connected. This cocycle determines a bundle E' on C such that $f^*E' = E$. Then ρ is the pullback of a homomorphism $\rho' \in \mathrm{Hom}(\pi_1(C), T_2)$ constructed by E'.

This completes the proof of the proposition modulo Lemma 2.4.

Proof of Lemma 2.4. Observe that $\omega_1 \neq 0$ in our definition of θ because ρ_a is nonunitary. We also consider $\eta' \neq 0$; otherwise, the image of ρ consists of diagonal matrices and the required statement is trivial. Let $i: V \hookrightarrow M_t$ be a fibre over a regular value of g_1. For any $\lambda \in \mathbb{C}^*$, consider the form

$$\theta_\lambda = \begin{pmatrix} \lambda\omega_1 & \eta' \\ 0 & 0 \end{pmatrix}.$$

Clearly, $\theta_\lambda \wedge \theta_\lambda$ represents 0 in $H^2(M_t, \mathrm{End}(E_\rho \oplus E_0))$ and thus, by [Br, Prop. 2.4], it determines a representation $\rho_\lambda \in \mathrm{Hom}(\pi_1(M_t), T_2)$ with the upper diagonal character $\rho_{\lambda a}$ defined by the flat connection $\lambda\omega_1 + \omega_2$. In particular, the family $\{\rho_{\lambda a}\}$ contains infinitely many different characters. Assume that $\rho|_{i_*(\pi_1(V))}$ is not trivial; then it can be determined by the restriction $\theta|_{i(V)}$. However, according to our assumption $\omega_1|_{i(V)} = 0$ and $E_\rho|_{i(V)}$ is a trivial flat vector bundle. Thus $\rho|_{i_*(\pi_1(V))}$ is defined by $\eta'|_{i(V)} \in H^1(i(V), \mathbb{C})$. Let $\psi \in \mathrm{Hom}(i(V), \mathbb{C})$ be a homomorphism obtained by integration of $\eta'|_{i(V)}$ along paths generating $\pi_1(i(V))$. Then

$$\rho|_{i_*(\pi_1(V))} = \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}.$$

We also obtain that $\rho_\lambda|_{i_*(\pi_1(V))} = \rho|_{i_*(\pi_1(V))}$ and so it is not trivial for any $\lambda \in \mathbb{C}^*$. We will prove now that the family $\{\rho_{\lambda a}\}$ contains finitely many different characters. This contradicts our assumption and shows that $\rho|_{i_*(\pi_1(V))}$ is trivial and so $\eta'|_V = 0$.

Let $H := i_*(\pi_1(V))$ be a normal subgroup of $\pi_1(M_t)$ (the normality follows from the exact homotopy sequence for the bundle defined over regular values of g_1). Denote by H^{ab} the quotient H/DH. Then H^{ab} is an abelian group of a finite rank. Moreover, H^{ab} is a normal subgroup of $K := \pi_1(M_t)/DH$ and the group $K_1 := K/H^{ab}$ is finitely generated. Note that ρ_λ induces a homomorphism $\hat{\rho}_\lambda \colon K \to T_2$ for any $\lambda \in \mathbb{C}^*$ with the same image as for ρ_λ because the image of $\rho_\lambda|_H$ is abelian. In particular, $\hat{\rho}_\lambda(H^{ab})$ is a normal subgroup of $\hat{\rho}(K)$. Since by assumption $\hat{\rho}_\lambda(H^{ab})$ is a nontrivial subgroup of unipotent matrices, from the identity

$$\begin{pmatrix} c & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & c \cdot v \\ 0 & 1 \end{pmatrix} \quad (c \in \mathbb{C}^*,\ b, v \in \mathbb{C}) \qquad (2.1)$$

it follows that the action of $\hat{\rho}_\lambda(K)$ on $\hat{\rho}_\lambda(H^{ab})$ by conjugation is defined by multiplication of nondiagonal elements of $\hat{\rho}_\lambda(H^{ab})$ by elements of $\rho_{\lambda a}(\pi_1(M_t))$. Note also that $\mathrm{Tor}(H^{ab})$ belongs to $\mathrm{Ker}(\hat{\rho}_\lambda)$ for any λ. Thus, if $H' \cong \mathbb{Z}^s$ is a maximal free abelian subgroup of H^{ab} then $\rho_{\lambda a}(\pi_1(M_t))$ consists of eigenvalues of matrices from $\mathrm{SL}_s(\mathbb{Z})$ obtained by the natural action (by conjugation) of K_1 on H'. Since K_1 is finitely generated, the number of different $\rho_{\lambda a}$ is finite, which is false.

This contradiction completes the proof of the lemma. □

2.4

Let $f \colon M \to C$ be a holomorphic map with connected fibres of a compact Kähler manifold M onto a smooth compact complex curve C of genus $g \geq 1$. Then $\pi_1(M)$ is defined by the exact sequence

$$\{1\} \longrightarrow \mathrm{Ker}(f_*) \longrightarrow \pi_1(M) \longrightarrow \pi_1(C) \longrightarrow \{1\}.$$

Let $G \subset \mathrm{Ker}(f_*)$ be a normal subgroup of $\pi_1(M)$. The quotient $R := \pi_1(M)/G$ is defined by the sequence

$$\{1\} \longrightarrow \mathrm{Ker}(f_*)/G \longrightarrow R \longrightarrow \pi_1(C) \longrightarrow \{1\}\,.$$

Assume that

(1) $H := \mathrm{Ker}(f_*)/G$ is a free abelian group of finite rank $k \geq 1$, and
(2) the natural action of $D\pi_1(C)$ on H is trivial.

From (2) it follows that the action s of $\pi_1(C)$ on H determines an action of $\mathbb{Z}^{2g} \cong \pi_1(C)/D\pi_1(C)$ on H. Identifying H with $\mathbb{Z}^k$, we can think of H as a subgroup (lattice) of $Z^k \otimes \mathbb{C} = \mathbb{C}^k$. Then s determines a representation $s' \colon \pi_1(C) \to \mathrm{GL}_k(\mathbb{C})$ such that $D\pi_1(C) \subset \mathrm{Ker}(s')$ and $s'(g)|_H = s(g)$ for any $g \in \pi_1(C)$. Because s' descends to a representation $\mathbb{Z}^{2g} \to \mathrm{GL}_k(\mathbb{C})$, it admits a decomposition $s' = \bigoplus_{j=1}^m s_j$, where s_j is equivalent to a nilpotent representation $\pi_1(C) \to T_{k_i}(\mathbb{C})$ with a diagonal character ρ_j. Here $\sum_{j=1}^m k_j = k$.

PROPOSITION 2.5. *All characters ρ_j are torsion.*

Proof. By definition, R is an extension of $\pi_1(C)$ by H. It is well known (see e.g. [G, Chap. I, Sec. 6]) that the class of extensions equivalent to R is uniquely defined by an element $c \in H^2(\pi_1(C), H)$, where the cohomology is defined by the action s of $\pi_1(C)$ on H. Let $f \in Z^2(\pi_1(C), H)$ be a cocycle determining c. Then one can define a representative of the equivalence class of extensions as the direct product $H \times \pi_1(C)$ with multiplication:

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 + s(g_1)(h_2) + f(g_1, g_2),\ g_1 \cdot g_2);$$
$$h_1, h_2 \in H, \quad g_1, g_2 \in \pi_1(C).$$

The natural embedding $H \hookrightarrow \mathbb{Z}^k \otimes \mathbb{C}$ $(= \mathbb{C}^k)$ determines an embedding i of R into the group R' defined as $\mathbb{C}^k \times \pi_1(C)$ with multiplication:

$$(v_1, g_1) \cdot (v_2, g_2) = (v_1 + s'(g_1)(v_2) + f(g_1, g_2),\ g_1 \cdot g_2);$$
$$f \in Z^2(\pi_1(C), H), \quad v_1, v_2 \in \mathbb{C}^k, \quad g_1, g_2 \in \pi_1(C).$$

Here we regard f as an element of $Z^2(\pi_1(C), \mathbb{C}^k)$ defined by the action s'. From the decomposition $s' = \bigoplus_{j=1}^m s_j$ it follows that there exists an invariant $\pi_1(C)$-submodule $V_j \subset \mathbb{C}^k$ of $\dim_{\mathbb{C}} V_j = k-1$ such that $W_j = \mathbb{C}^k/V_j$ is a one-dimensional $\pi_1(C)$-module and the action of $\pi_1(C)$ on W_j is defined as multiplication by the character ρ_j. Then, by definition, V_j is a normal subgroup of R' and the quotient group $R_j = R'/V_j$ is defined by the sequence

$$\{1\} \longrightarrow \mathbb{C} \longrightarrow R_j \longrightarrow \pi_1(C) \longrightarrow \{1\}.$$

Here the action of $\pi_1(C)$ on $\mathbb{C}$ is multiplication by the character ρ_j. Further, the equivalence class of extensions isomorphic to R_j is defined by an element $c_j \in H^2(\pi_1(C), \mathbb{C})$ (the cohomology is defined by the action just displayed of $\pi_1(\mathbb{C})$ on $\mathbb{C}$). We will assume that the character ρ_j is nontrivial (for otherwise, ρ_j is clearly torsion). Let us denote by t_j the composite homomorphism $\pi_1(M) \to R \xrightarrow{i} R' \to R_j$.

Let E_{ρ_j} be a complex rank-1 flat vector bundle on C constructed by $\rho_j \in \mathrm{Hom}(\pi_1(C), \mathbb{C}^*)$. Since C is a $K(\pi_1(C), 1)$-space, there is a natural isomorphism of the preceding group $H^2(\pi_1(C), \mathbb{C})$ and the Čech cohomology group $H^2(C, \mathbf{E}_{\rho_j})$ of the sheaf of locally constant sections of E_{ρ_j} (see e.g. [M, Chap. 1, Complement to Sec. 2]). But each flat vector bundle on C is C^∞-trivial, and each homomorphism from $\mathrm{Hom}(\pi_1(C), \mathbb{C}^*)$ can be continuously deformed inside of $\mathrm{Hom}(\pi_1(C), \mathbb{C}^*)$ to the trivial homomorphism. Hence, by the index theorem we have

$$\dim_{\mathbb{C}} H^0(C, \mathbf{E}_{\rho_j}) - \dim_{\mathbb{C}} H^1(C, \mathbf{E}_{\rho_j}) + \dim_{\mathbb{C}} H^2(C, \mathbf{E}_{\rho_j}) = \chi(\pi_1(C)) = 2 - 2g.$$

Note that $H^0(C, \mathbf{E}_{\rho_j}) = 0$ because ρ_j is nontrivial. Furthermore, $H^1(C, \mathbf{E}_{\rho_j})$ is in a one-to-one correspondence with the set of nonequivalent representations $\rho\colon \pi_1(C) \to T_2 \subset T_2(\mathbb{C})$ with the upper diagonal character ρ_j. Using the identity $\prod_{i=1}^g [e_i, e_{g+i}] = e$ for generators $e_1, \dots, e_{2g} \in \pi_1(C)$, we easily obtain that $\dim_{\mathbb{C}} H^1(C, \mathbf{E}_{\rho_j}) = 2g - 2$. Thus we have $H^2(C, \mathbf{E}_{\rho_j}) = 0$. This shows that $H^2(\pi_1(C), \mathbb{C}) = 0$ and that R_j is isomorphic to the semidirect product of $\mathbb{C}$ and $\pi_1(C)$; in other words, $R_j = \mathbb{C} \times \pi_1(C)$ with multiplication:

$$(v_1, g_1) \cdot (v_2, g_2) = (v_1 + \rho_j(g_1) \cdot v_2,\ g_1 \cdot g_2);$$
$$v_1, v_2 \in \mathbb{C}, \quad g_1, g_2 \in \pi_1(C).$$

Let us determine a map ϕ_j of R_j to T_2 by the formula

$$\phi_j(v, g) = \begin{pmatrix} \rho_j(g) & v \\ 0 & 1 \end{pmatrix}.$$

Obviously, ϕ_j is a correctly defined homomorphism with upper diagonal character ρ_j. Hence $\phi_j \circ t_j : \pi_1(M) \to T_2$ is a homomorphism that is nontrivial on $\mathrm{Ker}(f_*)$ by its definition. Now Proposition 2.3 implies that ρ_j is a unitary character. We have thus proved that all characters of the action of $\pi_1(C)$ on H are unitary. This means that each element of the action is defined by a matrix from $\mathrm{SL}_k(\mathbb{Z})$ with unitary eigenvalues. Applying to these eigenvalues the Kronecker theorem, which asserts that an algebraic integer is a root of unity if and only if all its Galois conjugates are unitary (see e.g. [BS, Thm. 2, p. 105]), we obtain that each ρ_j is a torsion character. □

2.5

In this subsection we prove a result about the structure of $\mathrm{Ker}(f_*)$, where $f : M \to C$ is a surjective holomorphic map with connected fibres of a compact Kähler manifold M onto a curve C.

Using the Lojasiewicz triangulation theorem [L, Thm. 2], compactness of M, and the fact that all fibres of f are connected, we can find an open finite cover $(U_i)_{1 \le i \le s}$ of C such that (a) $W_i := f^{-1}(U_i)$ is an open neighborhood of a fibre $V_i = f^{-1}(x_i)$, $x_i \in U_i$, and (b) V_i is a deformation retract of some open $\tilde{W}_i \supset W_i$. Let us fix some points $x_i \in V_i$ and paths γ_i connecting x_1 with x_i, $1 \le i \le s$. By G_i we denote the image of $\gamma_i \cdot \pi_1(V_i, x_i) \cdot \gamma_i^{-1}$ in $\pi_1(M, x_1)$. (Clearly, each G_i is finitely generated.)

LEMMA 2.6. $\mathrm{Ker}(f_*)$ *is the minimal normal subgroup of* $\pi_1(M, x_1)$ *containing all* G_i $(1 \le i \le s)$.

Proof. Let $R \subset \pi_1(M, x_1)$ be the minimal normal subgroup containing all G_i. Clearly, $R \subset \mathrm{Ker}(f_*)$. We will prove the reverse inclusion. Let $p : M_1 \to M$ be the regular covering over M with the transformation group $R_1 := \pi_1(M)/R$. Observe that p is a principal bundle over M with discrete fibre R_1. We will prove that the restriction of M_1 to each $\tilde{W}_i$ is the trivial bundle. Since V_i is a deformation retract of $\tilde{W}_i$, it suffices to prove that, for any closed path $\gamma \subset V_i$ passing through x_i and for any point $y \in p^{-1}(x_i)$, the unique covering path $\gamma' \subset M_1$ of γ passing through y is closed. In order to prove this, let us consider the path $s := \gamma_i \cdot \gamma \cdot \gamma_i^{-1}$. Let $s_1 \subset M_1$ be any path that covers s. Then s_1 is closed, because s represents an element of the *normal subgroup* $R \subset \pi_1(M)$. Let γ' be the covering of γ with endpoints y and y'. Then there exists an element $g \in R_1$ such that $y' = g(y)$. Let h and h' be the unique covering paths of γ_i and γ_i^{-1} passing through y and y', respectively. Clearly $h' = g(h^{-1})$. Now $s' = h \cdot \gamma' \cdot h'$ covers s and so is closed. Then we have $h' = h^{-1}$, implying $g = 1$ and $y = y'$. Thus we have proved that M_1 is trivial over each $\tilde{W}_i$ and, in particular, over each W_i. This means that M_1 is defined by a locally constant cocycle $\{c_{ij}\}$ (with values in R_1) defined on the cover $(W_i)_{i\in I}$ of M. But then $\{c_{ij}\}$ is the pullback of a cocycle defined on $(U_i)_{i\in I}$ because the fibres of f are connected. This cocycle determines a principal bundle C_1 on C with the fibre R_1 such that $f^*C_1 = M_1$. In particular, C_1 determines a homomorphism $q : \pi_1(C) \to R_1$ such that $q \circ f_*$ is the quotient homomorphism $\pi_1(M) \to R_1$. This implies that $\mathrm{Ker}(f_*) \subset R$. □

2.6

We use use Lemma 2.6 to prove the following result.

LEMMA 2.7. *Let $\rho \in \mathrm{Hom}(\pi_1(M), T_2)$ be such that $\mathrm{Ker}(f_*) \subset \mathrm{Ker}(\rho_a)$, where ρ_a is unitary. Then $\rho(\mathrm{Ker}(f_*)) \subset N_2$ is a free abelian group of finite rank.*

Proof. Without loss of generality, we will assume that $\mathrm{Ker}(f_*) \not\subset \mathrm{Ker}(\rho)$. Let $i : V \hookrightarrow \pi_1(M)$ be the fibre over a regular value of f, and let $H = i_*(\pi_1(V)) \subset \pi_1(M)$ be the corresponding normal subgroup. First we prove that $\rho|_H$ is not trivial. Indeed, let E be the flat vector bundle associated to ρ_a. By assumption we have $E = f^*L$, where L is a flat vector bundle on C with unitary structure group. It is well known that the equivalence class of ρ is defined by a harmonic 1-form η with values in E. Since the structure group of E is unitary, the Hodge decomposition implies that $\eta = \omega_1 + \omega_2$, where ω_1 is holomorphic and ω_2 is antiholomorphic. From the proof of Lemma 2.4 we know that ρ_H is defined by integration of η along closed paths of $i(V)$. Assume to the contrary that $\rho|_H$ is trivial. This means that $\eta|_{i(V)}$ represents 0 in $H^1(i(V), \mathbb{C})$ and so $\omega_1|_{i(V)} = \omega_2|_{i(V)} = 0$. But then Lemma 2.2 implies that $\omega_i|_V = 0$ $(i = 1, 2)$ for any fibre V of f. Let us consider a path γ that represents an element of $\pi_1(V_i, x_i)$ (the notation is as in Lemma 2.6). Let $\gamma' = \gamma_i \cdot \gamma \cdot \gamma_i^{-1}$. Then the monodromy argument shows that $\rho(\gamma') = C \cdot A \cdot C^{-1}$, where $C \in T_2$, $A \in N_2$, and A is defined by integration of η along γ. Therefore, $\rho(\gamma') = 1 \in N_2$. Let $[\gamma'] \in \pi_1(M)$ be the element defined by γ'. According to Lemma 2.6, $\mathrm{Ker}(f_*)$ is the minimal normal subgroup containing all such $[\gamma']$ (for any i). The preceding argument then shows that $\rho|_{\mathrm{Ker}(f_*)}$ is trivial, which contradicts our assumption.

Next, we prove that $\rho_a(\pi_1(M))$ consists of algebraic integers. Let $H_f \cong \mathbb{Z}^k$ be the free part of $H^{ab} = H/DH$. In what follows we identify H_f with $\mathbb{Z}^k$. Let $s : \pi_1(M) \to \mathrm{SL}_k(\mathbb{Z})$ be the representation induced by the action of $\pi_1(M)$ on H by conjugation. Since $\rho(H_f) = \rho(H)$ is nontrivial and consists of unipotent matrices, $k \geq 1$ and (2.1) together imply that the action of $\rho(\pi_1(M))$ on $\rho(\mathbb{Z}^k)$ by conjugation is defined by multiplication of nondiagonal elements of $\rho(\mathbb{Z}^k)$ by elements of $\rho_a(\pi_1(M))$. Let $e_1, \ldots, e_k$ be the standard basis in $\mathbb{Z}^k$. Identifying N_2 with $\mathbb{C}$, we obtain a nonzero vector $v = (\rho(e_1), \ldots, \rho(e_k)) \in \mathbb{C}^k$. Then a simple calculation shows that v is a nonzero eigenvector of $s(g)$ with the eigenvalue $\rho_a(g)$, $g \in \pi_1(M)$. This completes the proof.

Now let us finish the proof of the lemma. Let $\{g_j\}$, $1 \le j \le l$, be the *finite family* consisting of all generators of all G_i (as in Lemma 2.6). Then, according to Lemma 2.6, $\rho(\mathrm{Ker}(f_*)) \subset \rho(\pi_1(M))$ is the minimal normal subgroup containing all $h_j := \rho(g_j)$. Since N_2 is abelian, (2.1) shows that $\rho(\mathrm{Ker}(f_*))$ is a discrete abelian group generated by all elements $\rho_a(g) \cdot h_j$ ($g \in H_1(M, \mathbb{Z})$, $1 \le j \le l$). Let $t_1, \dots, t_p$ be the family of generators of $H_1(M, \mathbb{Z})$. Then the foregoing arguments imply that all $\rho_a(t_i)$ and $\rho_a(-t_i)$ ($i = 1, \dots, p$) are roots of polynomials of degree k with integer coefficients and leading coefficients 1. Since any $g \in H_1(M, \mathbb{Z})$ can be written as $\sum_{i=1}^{p} a_i t_i + \sum_{i=1}^{p} b_i(-t_i)$ with $a_i, b_i \in \mathbb{Z}_+$, our previous remarks show that the abelian group $\rho(\mathrm{Ker}(f_*))$ is generated (over $\mathbb{Z}$) by the elements

$$\rho_a(t_1)^{a_1} \cdots \rho_a(t_p)^{a_p} \cdot \rho_a(-t_1)^{b_1} \cdots \rho_a(-t_p)^{b_p} \cdot h_j,$$

where a_i and b_i are nonnegative integers satisfying $0 \le a_i \le k-1$, $0 \le b_i \le k-1$, and $1 \le j \le l$. Because the number of such generators is finite, $\rho(\mathrm{Ker}(f_*))$ is finitely generated. It is also free as a subgroup of the free group N_2.

The proof of the lemma is complete. $\square$

2.7. Proof of Theorem 2.1

Let $\rho \in \mathrm{Hom}(\pi_1(M), T_2)$ be such that $\mathrm{Ker}(f_*) \subset \mathrm{Ker}(\rho_a)$ but $\mathrm{Ker}(f_*) \not\subset \mathrm{Ker}(\rho)$. According to Proposition 2.3, ρ_a is unitary. Then, according to Lemma 2.7, $\rho(\mathrm{Ker}(f_*)) \subset N_2$ is isomorphic to $\mathbb{Z}^r$, $r \ge 1$. We set $G = \mathrm{Ker}(f_*) \cap \mathrm{Ker}(\rho)$ and $R := \pi_1(M)/G$. Then R is defined by the sequence

$$\{1\} \longrightarrow \mathrm{Ker}(f_*)/G \longrightarrow R \longrightarrow \pi_1(C) \longrightarrow \{1\},$$

where $\mathrm{Ker}(f_*)/G \cong \rho(\mathrm{Ker}(f_*)) = \mathbb{Z}^r$. Since $\rho(\mathrm{Ker}(f_*)) \subset N_2$, the displayed definition implies that the natural action of $D\pi_1(C)$ on $\mathrm{Ker}(f_*)/G$ is trivial. Thus, according to Proposition 2.5, all characters of the induced representation $\pi_1(C) \to \mathrm{GL}_r(\mathbb{C})$ are torsion. In particular, ρ_a is torsion as one of these characters. $\square$

3. Proof of Theorem 1.2

We start with the following result.

PROPOSITION 3.1. *Assume that $\pi_1(M)$ is defined by the sequence*

$$\{1\} \longrightarrow F \longrightarrow \pi_1(M) \longrightarrow H \longrightarrow \{1\},$$

where the normal subgroup F does not admit a surjective homomorphism onto $\mathcal{F}_\infty$. Assume that $\rho \in \mathrm{Hom}(\pi_1(M), T_2)$ satisfies $F \subset \mathrm{Ker}(\rho_a)$ but that $F \not\subset \mathrm{Ker}(\rho)$. Then ρ_a is a torsion character.

Proof. Given a character $\xi \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$, let $\mathbb{C}_\xi$ denote the associated $\pi_1(M)$-module. Also, we define $\Sigma^1(M)$ to be the set of characters ξ such that $H^1(\pi_1(M), \mathbb{C}_\xi)$ is nonzero. The structure of $\Sigma^1(M)$ was described in the consequent papers of Beauville [Be], Simpson [Si], and Campana [C].

BSC THEOREM. *There is a finite number of surjective holomorphic maps with connected fibres $f_i\colon M \to C_i$ onto smooth compact complex curves of genus ≥ 1 and torsion characters $\rho_i, \xi_j \in \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ such that*

$$\Sigma^1(M) = \bigcup_i \rho_i f_i^* \,\mathrm{Hom}(\pi_1(C_i), \mathbb{C}^*) \cup \bigcup_j \{\xi_j\}.$$

Further, the group N_2 acts on T_2 by conjugation. Any two homomorphisms from $\mathrm{Hom}(\pi_1(M), T_2)$ belonging to the orbit of this action will be called *equivalent*.

Let $\rho \in \mathrm{Hom}(\pi_1(M), T_2)$ satisfy the conditions of Proposition 3.1. Then it is well known that the class of equivalence of ρ is uniquely defined by an element $c_\rho \in H^1(\pi_1(M), \mathbb{C}_{\rho_a})$ (see e.g. [At, Prop. 2]). In particular, if $c_\rho = 0$ then ρ is equivalent to a representation into D_2. In our case, $c_\rho \neq 0$ because $F \not\subset \mathrm{Ker}(\rho)$. Thus ρ_a satisfies the conditions of the BSC theorem. If ρ_a coincides with one of ξ_j then by this theorem ρ_a is torsion. So assume that $\rho_a = \rho_i f_i^* \phi$ for some torsion character ρ_i and $\phi \in \mathrm{Hom}(\pi_1(C_i), \mathbb{C}^*)$. Let $K := \mathrm{Ker}(\rho_i) \subset \pi_1(M)$ and let $p\colon M_1 \to M$ be the Galois covering of M corresponding to the finite abelian Galois group $\pi_1(M)/K$. Let $M_1 \xrightarrow{g} C \xrightarrow{h} C_i$ be the Stein factorization of $f_i \circ p$, where g is a morphism with connected fibres onto a smooth curve C and h is a finite morphism. Assume, to the contrary, that ρ_a is not torsion. Then we prove the following lemma.

LEMMA 3.2. $F \subset \mathrm{Ker}(g_*)$.

Proof. Set $G := (f_i)_*(F) \subset \pi_1(C_i)$. According to the assumptions of Proposition 3.1, we have either (a) G is a subgroup of finite index in $\pi_1(C_i)$ or (b) G is isomorphic to $\mathcal{F}_r$ with $r < \infty$. Let us consider (a). Since by definition $F \subset \mathrm{Ker}(\rho_a)$ and ρ_i is torsion, it follows that $\phi(G)$ is a finite abelian group. Then $G_1 := \mathrm{Ker}\,\phi \cap G$ is a subgroup of finite index in G. In particular, G_1 is a subgroup of finite index in $\pi_1(C_i)$ and thus is not free. Yet by our assumption, ϕ is not torsion and so $\mathrm{Ker}(\phi) \cong \mathcal{F}_\infty$. Hence G_1 is also free as a subgroup of $\mathcal{F}_\infty$. This shows that (a) never holds.

Consider now (b). Using the fact that $(f_i)_*$ is a surjection, we conclude that G is a normal subgroup of $\pi_1(C_i)$. Let $S \to C_i$ be a regular covering corresponding to $Q := \pi_1(C_i)/G$. If $r \geq 2$ then the group $\mathrm{Iso}(S)$ of isometries of S (with respect to the hyperbolic metric) is finite, and since Q is infinite we have $r \leq 1$. If $r = 1$, then any discrete subgroup of $\mathrm{Iso}(S)$ is virtually cyclic and in particular does not act cocompactly on S. Thus $r = 0$, which means that $G = \{e\}$ and $F \subset \mathrm{Ker}(f_i)_*$. Then the assumption of the proposition implies that $F \subset K = \pi_1(M_1)$.

Now consider $\tilde{\rho} := \rho|_{\pi_1(M_1)}$ with the upper diagonal character $\tilde{\rho}_a := \rho_a|_{\pi_1(M_1)}$. Since $\pi_1(M_1) \subset \pi_1(M)$ is a subgroup of finite index, $\tilde{\rho}_a$ is also not torsion. Set $\tilde{\phi} := h^*\phi$. Then $\tilde{\rho}_a = g^*\tilde{\phi}$. Let $G_1 := g_*(F) \subset \pi_1(C)$. Then the same argument for G_1 as for G (with $\tilde{\rho}_a$ and $\tilde{\phi}$ instead of ρ_a and ϕ) yields $F \subset \mathrm{Ker}(g_*)$. $\square$

Resuming now the proof of Proposition 3.1, by assumption and Lemma 3.2 we have that $\tilde{\rho}|_{\mathrm{Ker}(g_*)}$ is nontrivial and that $\tilde{\rho}_a$ is the pullback of a character from

$\mathrm{Hom}(\pi_1(C), \mathbb{C}^*)$. Then from Theorem 2.1 it follows that $\tilde{\rho}_a$ is torsion, and hence ρ_a is torsion as well. This contradiction proves the proposition. $\square$

We are now ready to prove Theorem 1.2. According to the assumptions of the theorem, there is a homomorphism i of H into the Lie group R of the form

$$\{0\} \longrightarrow \mathbb{C}^m \longrightarrow R \longrightarrow B \longrightarrow \{0\}$$

whose kernel is $\mathrm{Tor}(A)$. Here we identify $\mathbb{C}^m$ with $\mathbb{C} \otimes A$. Consider the action $s\colon B \to \mathrm{GL}_m(\mathbb{C})$ by conjugation. By the definition of $\pi_1(M)$, B is a finitely generated abelian group and so $s = \bigoplus_{j=1}^d s_j$, where s_j is equivalent to a nilpotent representation $B \to T_{m_j}(\mathbb{C})$ with a diagonal character ρ_j. Here $\sum_{j=1}^d m_j = m$. From this decomposition it follows that there exists an invariant B-submodule $V_j \subset \mathbb{C}^m$ of $\dim_{\mathbb{C}} V_j = m-1$ such that $W_j = \mathbb{C}^m/V_j$ is a one-dimensional B-module and the action of B on W_j is defined as multiplication by the character ρ_j. By definition, V_j is a normal subgroup of R and the quotient group $R_j = R/V_j$ is defined by the sequence

$$\{0\} \longrightarrow \mathbb{C} \longrightarrow R_j \longrightarrow B \longrightarrow \{0\}.$$

Here the action of B on $\mathbb{C}$ is multiplication by the character ρ_j. (As before, the associated B-module is denoted by $\mathbb{C}_{\rho_j}$.) Let us denote by t_j the composite homomorphism $\pi_1(M) \to H \xrightarrow{i} R \to R_j$. Also, the equivalence class of extensions of B by $\mathbb{C}$ isomorphic to R_j is defined by an element $c_j \in H^2(B, \mathbb{C}_{\rho_j})$. We assume that the character ρ_j is nontrivial (otherwise, ρ_j is clearly torsion). Then $H^2(B, \mathbb{C}_{\rho_j}) = 0$ (for the proof, see e.g. [AN, Lemma 4.2]). This shows that R_j is isomorphic to the semidirect product of $\mathbb{C}$ and B, that is, $R_j = \mathbb{C} \times B$ with multiplication:

$$(v_1, g_1) \cdot (v_2, g_2) = (v_1 + \rho_j(g_1) \cdot v_2,\ g_1 \cdot g_2);$$
$$v_1, v_2 \in \mathbb{C}, \quad g_1, g_2 \in B.$$

Let us determine a map ϕ_j of R_j to T_2 by the formula

$$\phi_j(v, g) = \begin{pmatrix} \rho_j(g) & v \\ 0 & 1 \end{pmatrix}.$$

Obviously, ϕ_j is a correctly defined homomorphism with upper diagonal character ρ_j. Hence $\phi_j \circ t_j\colon \pi_1(M) \to T_2(\mathbb{C})$ is a homomorphism that is nontrivial on $p^{-1}(A) \subset \pi_1(M)$ by its definition. Also, $p^{-1}(A) \subset \mathrm{Ker}(\rho_j \circ t_j)$. Since by assumption $p^{-1}(A)$ does not admit a surjective homomorphism onto $\mathcal{F}_\infty$, Proposition 3.1 applied to $\phi_j \circ t_j$ implies that ρ_j is torsion. This completes the proof of the theorem. $\square$

We now prove the following result.

LEMMA 3.3. *Assume that a group G is defined by the sequence*

$$\{1\} \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow \{1\},$$

where G_1, G_2 do not admit surjective homomorphisms onto $\mathcal{F}_\infty$. Then also G does not admit such homomorphisms.

Proof. Assume, to the contrary, that there is a surjective homomorphism $\phi\colon G \to \mathcal{F}_\infty$. Then $\tilde{G}_1 := \phi(G_1)$ is a normal subgroup of $\mathcal{F}_\infty$ and $\tilde{G}_2 := \mathcal{F}_\infty/\tilde{G}_1$ is a quotient of G_2. Now the assumption of the lemma implies that $\tilde{G}_1 \cong \mathcal{F}_r$ with $r < \infty$. We may assume without loss of generality that $r \neq 0$; otherwise $\tilde{G}_2 = \mathcal{F}_\infty$, which contradicts the hypothesis. Let X be a complex hyperbolic surface with $\pi_1(X) = \mathcal{F}_\infty$ and let $S \to X$ be the regular covering corresponding to $\tilde{G}_2$. Then $\pi_1(S) = \tilde{G}_1$ and $\tilde{G}_2$ is a discrete subgroup of the group $\mathrm{Iso}(S)$ of isometries of S (defined with respect to the hyperbolic metric). If $r \geq 2$, the group $\mathrm{Iso}(S)$ is finite. Also, for $r = 1$, any discrete subgroup of $\mathrm{Iso}(S)$ is virtually cyclic. This implies that $\tilde{G}_2$ is finitely generated (as well as $\tilde{G}_1$). Therefore, $\mathcal{F}_\infty$ should be finitely generated. This contradiction proves the lemma. □

4. Proof of Theorem 1.3

For a group L, set $L^{ab} := L/DL$. We say that an L-module V is *quasi-unipotent* if there is a subgroup $L' \subseteq L$ of finite index whose elements act unipotently on V. To prove the theorem we will check the following condition from [AN, Lemma 4.8].

Lemma 4.1. *Let $H' \subseteq H$ be a subgroup of finite index. Then H' acts quasi-unipotently on the finite-dimensional vector space $\mathbb{Q} \otimes (H' \cap DH)^{ab}$.*

Proof. We set $K := H'/(H' \cap DH)$, $G' := q^{-1}(H')$, $S := D(H' \cap DH)$, and $S' := q^{-1}(S)$. Here K is a finitely generated abelian group. Indeed, H' is finitely generated as a subgroup of finite index of the finitely generated group H (= the image of the finitely generated group G). Thus K is finitely generated as the image of H'. The fact that K is abelian follows directly from the definition. Furthermore, since H' is a subgroup of finite index in H, it follows that G' is a subgroup of finite index in G; in particular, it is Kähler. Moreover, we have

$$\{1\} \longrightarrow S' \longrightarrow G' \longrightarrow L \longrightarrow \{1\},$$

where L is a solvable group of finite rank defined by the sequence

$$\{0\} \longrightarrow (H' \cap DH)^{ab} \longrightarrow L \longrightarrow K \longrightarrow \{0\}.$$

Now the statement of the lemma is equivalent to the fact that all eigencharacters of the conjugate action of K on $\mathbb{Q} \otimes (H' \cap DH)^{ab}$ are torsion. To prove this, it suffices to (a) show that S' does not admit a surjective homomorphism onto $\mathcal{F}_\infty$ and then (b) apply Lemma 3.3 and Theorem 1.2.

Note that S' is defined by the sequence

$$\{1\} \longrightarrow F \longrightarrow S' \xrightarrow{q} S \longrightarrow \{1\},$$

where S is a solvable group of finite rank. By our assumption, F does not admit a surjective homomorphism onto $\mathcal{F}_\infty$. Thus, by Lemma 3.3, S' satisfies the same property. □

Now Theorem 1.3 is the consequence of Lemma 4.1 and [AN, Lemma 4.8]. □

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