The Bergman Projection and Vector-Valued Hardy Spaces

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1. Introduction and Statement of Results

Let S^n be the unit sphere in C^n, and let $d\sigma$ denote the surface area form on S^n normalized so $\int_{S^n} d\sigma = 1$; B_n will denote the unit ball and $d\nu$ will be the normalized volume form on B_n. We assume familiarity with the invariant Poisson integral and nonisotropic metric $d(\zeta,\eta) = |1 - \langle \zeta, \eta\rangle|^{1/2}$ used in the study of function theory on S^n; see [R, Chap. 5]. For $0 < p < \infty$, $H^p(S^n)$ is the usual space of distributions whose invariant Poisson integrals are holomorphic on B_n and whose admissible maximal functions belong to $L^p(d\sigma)$; see [R, Chap. 4]. For a function u defined on B_n and $1 < q < \infty$, let $A_q[u]$ be the *area function*

$$A_q[u](\zeta) = \left(\int_{\Gamma(\zeta)} |u(z)|^q \frac{d\nu(z)}{(1-|z|)^{n+1}}\right)^{1/q},$$

where $\zeta \in S^n$ and $\Gamma(\zeta)$ is the usual approach region

$$\Gamma(\zeta) = \{ z \in B_n : |1 - \langle z, \zeta\rangle| < 1 - |z|^2 \}.$$

For $0 < p < \infty$, the *tent space* $T_q^p(B_n)$ consists of all functions u such that

$$\|u\|_{T_q^p} = \left(\int_{S^n} A[u]^p \, d\sigma\right)^{1/p} < \infty.$$

The tent space $T_q^\infty(B_n)$ consists of those functions u such that $|u(z)|^q \, d\nu(z)/(1-|z|)$ is a Carleson measure; see [CMS].

It is well known that if $0 < p < \infty$ then a distribution F whose invariant Poisson integral is holomorphic belongs to $H^p(S^n)$ if and only if $u(z) = (1-|z|) \times (|F(z)| + |\nabla F(z)|)$ belongs to T_2^p; here $F(z)$ is used to denote the invariant Poisson integral of F evaluated at $z \in B_n$. Another characterization of H^p for $0 < p < \infty$ is given in terms of the "g" function. If u is defined on B_n and $1 < q < \infty$, let

$$g_q[u](\zeta) = \left(\int_0^1 |u(t\zeta)|^q \frac{dt}{1-t}\right)^{1/q}.$$

Then a distribution F with holomorphic Poisson integral belongs to H^p if and only if $g_2(u) \in L^p(d\sigma)$, where again $u(z) = (1-|z|)(|F(z)| + |\nabla F(z)|)$; see [AB]. In each of these characterizations we have norm equivalences:

Received August 20, 1996. Revision received March 14, 1997.
Michigan Math. J. 44 (1997).

$$\|F\|_{H^p} \doteq \|u\|_{T_2^p} \doteq \|g_2(u)\|_{L^p}.$$

Here, the notation $\doteq$ means, for example, that there is a constant $C > 0$ independent of F such that $C^{-1}\|F\|_{H^p} \leq \|u\|_{T_2^p} \leq C\|F\|_{H^p}$. The letter C will denote various numerical constants whose value will change in the different contexts in which it is used.

Combining these characterizations of $H^p(S^n)$ with duality considerations shows that certain linear operators are bounded, at least for $1 < p < \infty$. For f in $L^1(d\sigma)$, let

$$Sf(z) = (1 - |z|) \int_{S^n} f(\zeta) \frac{d\sigma(\zeta)}{(1 - \langle z, \zeta \rangle)^{n+1}}.$$

If F is in $H^p(S^n)$ then $SF(z) = (1 - |z|)(F(z) + \frac{1}{n}RF(z))$, where R denotes the radial derivative of F [R, 6.4.4]. Thus, S is a bounded operator from H^p to T_2^p. For $1 < p < \infty$ the dual of $T_q^p(B_n)$ is $T_{q'}^{p'}$, where $1/r + 1/r' = 1$ and the pairing between the spaces is given by

$$\langle u, v \rangle = \int_{B_n} u(z)\bar{v}(z) \frac{d\nu(z)}{1 - |z|} \tag{1}$$

(see [CMS]).

A calculation shows that if $F \in H^p(S^n)$ and $u \in T_2^{p'}$ then

$$\langle SF, u \rangle = \int_{S^n} F(\zeta) \int_{B_n} \bar{u}(z) \frac{d\nu(z)}{(1 - \langle z, \zeta \rangle)^{n+1}} \, d\sigma(\zeta);$$

that is, the adjoint $S^*\colon T_2^{p'} \to H^{p'}$ is given by the Bergman projection

$$S^*u(\zeta) = Bu(\zeta) = \int_{B_n} u(z) \frac{d\nu(z)}{(1 - \langle \zeta, z \rangle)^{n+1}}. \tag{2}$$

Thus, for $1 < p < \infty$, duality shows that $B\colon T_2^p \to H^p$ is a bounded operator. Although it may seem unusual at first to see the Bergman projection appearing as an operator whose range is Hardy space, this is merely the holomorphic version of the real variable result which states that, if $\psi \in \mathcal{S}(R^n)$, $\psi_t(x) = t^{-n}\psi(x/t)$, and $\int_{R^n} \psi(x)\, dx = 0$, then the linear operator given by

$$Lu(x) = \int_{R^{n+1}} u(y, t)\psi_t(x - y) \frac{dy\, dt}{t}$$

maps the tent space $T_2^p(R^{n+1})$ into $H^p(R^n)$. This holds for $n/(n+1) < p < \infty$; to obtain the result for smaller values of p, more moments of ψ must vanish. See [CMS].

It follows from Theorem 2 in [Co] that if $n/(n+1) < p < \infty$ then $B\colon T_2^p \to H^p$ is a bounded operator, in analogy to the real variable result. (One applies [Co, Thm. 2] with $G(\zeta, z) = (1 - \langle z, \zeta \rangle)$, $r(z) = 1 - |z|$, $a = 1$, $b = 0$, $H = 1$, and $l = 0$ in conjunction with the tent-space characterization of H^p.) The methods of that paper actually apply to tent spaces T_q^p where $1 < q < \infty$; one need only find the right substitute for H^p for $q \neq 2$. It is natural to replace H^p by a *Triebel space* $F_p^{0,q} = F_p^q$; see [T]. Here, in the absence of a Fourier transform that is easy to work with, it is convenient to restrict our attention to holomorphic functions and to define F_p^q as the space of holomorphic functions F such that $u(z) = (1-|z|)(|F(z)| + |\nabla F(z)|) \in T_q^p(B_n)$. Of course, $\|F\|_{F_p^q}$ is defined to be $\|u\|_{T_q^p}$. With this definition, the methods of [Co] yield the following result.

THEOREM A. *If $n/(n+1) < p < \infty$ and $1 < q < \infty$, then there is a constant $c(p,q)$ such that*

$$\|Bu\|_{F_p^q} \le c(p,q)\|u\|_{T_q^p}.$$

The modifications needed to prove Theorem A for $q \neq 2$ will be discussed in the proof of Theorem 1.

In this note our purpose is to find a version of Theorem A in which the tent spaces T_q^p are replaced by spaces based on the g function. As will be seen in the sequel, for $p > 1$ it is a straightforward matter to do this by replacing T_q^p with the *mixed norm* space $L^p(S^n, X_q)$ defined in what follows. We choose to interpret this as an L^p space of vector-valued functions. For $0 < p \le 1$, however, the approach we take here suggests that what is needed is a Hardy space of vector-valued functions. We define such a space, denoted $H_+^p(S^n, X_q)$, and our main result (Theorem 1) is the desired analog of Theorem A. Before proceeding further, however, let us say why we think the spaces $H_+^p(S^n, X_q)$ are interesting and why this is all worth the effort.

First of all, in Theorem 1, where $H_+^p(S^n, X^q)$ replaces $T_q^p(B_n)$, we obtain a bounded linear operator for the full range $0 < p \le 1$; the restriction $n/(n+1) < p$ is unnecessary, in contrast to Theorem A. This is because, as will be seen, the space $H_+^p(S^n, X_q)$ is a space of distributions with cancellation properties—as opposed to the space $T_q^p(B_n)$, where functions need not have those cancellation properties (see Remark 1). Consequently, Theorems A and 1 show that, for certain values of p, the operator B is bounded on both the Hardy-type space $H_+^p(S^n, X_q)$ and the tent space $T_q^p(B_n)$. It is therefore important to understand how the two spaces are related. For $2 \le p < \infty$ it can be checked that $L_q^p(S^n) \subset T_q^p$ and that for $1 < p \le 2$ the containment is reversed. On the other hand, for $0 < p \le 1$, neither space T_q^p nor $H_+^p(S^n, X_q)$ is contained in the other; see the end of Section 2. We think this makes Theorem 1 all the more interesting.

In Theorem 2 it is shown that if $F \in F_p^q$ then the function $SF(z) = (1-|z|) \times (\frac{1}{n}RF(z) + F(z))$ belongs to $H_+^p(S^n, X^q)$. Of course, it is well known that SF belongs to $T_q^p(B_n)$, but Theorem 2 shows in addition that SF belongs to a space where membership is determined not just by absolute value but by cancellation properties as well. In particular, since $SF \in H_+^p(S^n, X_q)$, it has an atomic decomposition into sums of atoms with vanishing moments, unlike the atomic decomposition it has by virtue of its membership in the tent space T_q^p. See Definition 6 and Theorem 5.

Finally, Theorem 3 considers the limiting case of p going to infinity in Theorem 1 and states results in terms of bounded mean oscillation. As a corollary, we obtain a new characterization of the usual space $\mathrm{BMOA}(S^n)$: A holomorphic function F belongs to $\mathrm{BMOA}(S^n)$ if and only if SF belongs to the space $\mathrm{BMO}(S^n, X_2)$. See

Remark 2. This characterization is close to the Carleson measure characterization of BMOA(S^n), and we believe it merits further study.

We now proceed with precise definitions of the spaces to which we have just referred. Let X_q be the space of functions $h(t)$ defined on the unit interval with the norm

$$\|h\|_{X_q} = \left(\int_0^1 |h(t)|^q \frac{dm_n(t)}{1-t} \right)^{1/q},$$

where $dm_n(t) = 2nt^{2n-1}\,dt$. Note that in polar coordinates

$$dv(t\zeta) = dm_n(t)\,d\sigma(\zeta),$$

where $0 \leq t < 1$ and $\zeta \in S^n$. If u is defined on B_n, then the property that $g_q[u] \in L^p(d\sigma)$ is equivalent to the property that, for almost all $\zeta \in S^n$, the formula

$$u(\zeta)(t) = u(t\zeta), \quad 0 \leq t < 1,$$

defines a mapping $u(\zeta)\colon S^n \to X_q$ and

$$\int_{S^n} |u(\zeta)|^p \, d\sigma < \infty,$$

where $|u(\zeta)|$ denotes the norm in X_q. Therefore, if in place of the area function A_q we use the g function g_q, it is natural to replace the tent space T_q^p with the mixed norm space $L_q^p(S^n) = L^p(S^n, X_q)$ of X_q-valued functions u such that

$$\|u\|_{L_q^p(S^n)} = \left(\int_{S^n} |u(\zeta)|^p \, d\sigma \right)^{1/p} < \infty.$$

If $x \in X_q$ and $y \in X_{q'}$, let

$$(x, y) = \int_0^1 x(t)\bar{y}(t) \frac{dm_n(t)}{1-t}$$

be the pairing that gives the duality between X_q and $X_{q'}$. If $1 < p < \infty$ then $(L_q^p(S^n))^* = (L_{q'}^{p'})$ with the pairing

$$\langle u, v \rangle = \int_{S^n} \int_0^1 u(\zeta)(t)\bar{v}(\zeta)(t) \frac{dm_n(t)}{1-t} \, d\sigma(\zeta) = \int_{S^n} (u(\zeta), v(\zeta))\, d\sigma(\zeta);$$

see [BP]. (We have used the measure $dm_n(t)$ to make this pairing coincide with the one given by equation (1).)

If $q = 2$ then, exactly as before, it follows from duality that the g_2-function characterization of H^p implies that, for $1 < p < \infty$, the Bergman projection defines a bounded linear operator $B\colon L_2^p(S^n) \to H^p(S^n)$. For the case $q \neq 2$ it is also true that the Bergman projection defines a bounded linear operator $B\colon L_q^p(S^n) \to F_p^q$. The argument, which is based on duality, will be given in the proof of Theorem 1.

On the other hand, for $p = 1$ it is no longer the case that the Bergman projection defines a bounded linear operator from L_2^1 into H^1. If this were the case then, by duality, the map that takes a function f to $(1-|z|)Rf(z)$ would define a bounded linear operator from BMOA(S^n) to L_2^∞. The function $g(z) = \log(1 - z_1)$ is a counterexample; here $z = (z_1, \ldots, z_n)$.

In the present context, the Bergman projection B_n arises as the adjoint of the operator that essentially takes a function F on S^n to $(1-|z|)RF(z)$. We can regard this operator as taking the scalar function F defined on S^n to the X_q-valued function u, also defined on S^n, given by $u(\zeta)(t) = (1-t)RF(t\zeta)$. The real-variable analog of this is the operator that takes a function f on R^n to $f * \psi_t(x)$, where ψ is a Littlewood–Paley function. It is customary to think of the latter operator as a singular integral operator with vector-valued kernel; see [S, Chap. 1, 6.3]. The Bergman projection may be viewed in a similar way: as a singular integral operator with domain X_q-valued functions (or distributions) on S^n and range scalar-valued functions (or distributions) on S^n. It is determined by the $X_{q'}$-valued kernel $k(\zeta, \eta)$ (where ζ and η are in S^n) given by

$$k(\zeta, \eta)(t) = \frac{1-t}{(1-t\langle\zeta, \eta\rangle)^{n+1}}.$$

The Bergman projection may then be thought of as determined by the formula

$$Bu(\zeta) = \lim_{s\to 1^-} \int_{S^n} (k(s\zeta, \eta), \bar{u}(\eta))\, d\sigma(\eta),$$

where $(\cdot, \cdot)$ denotes the pairing between $X_{q'}$ and X_q. A calculation shows that

$$|k(\zeta, \eta)|_{X_{q'}} \doteq \frac{1}{|1-\langle\zeta, \eta\rangle|^n},$$

as one expects with a singular integral operator. Therefore, if we hope to get a bounded linear operator with range F_p^q for $0 < p \leq 1$, it is natural to replace the domain $L_q^p(S^n)$ by some sort of Hardy space. We therefore make the following definitions.

Let X be a Banach space and X^* its normed dual. We wish to define the space of X^*-valued distributions on S^n. Denote by $\mathcal{D} = \mathcal{C}^\infty(S^n, X)$ the space of $\mathcal{C}^\infty$ functions $\phi\colon S^n \to X$. For a multi-index J let $|J|$ be the nonisotropic order of J as defined in [GL, p. 820], and let D_ζ^J be the differential operator defined at $\zeta \in S^n$ in terms of local coordinates of S^n as described in [GL]. (There is no difficulty in extending the definitions given there in the case where $n = 2$ to the general case.) Give $\mathcal{D}$ the topology induced by the seminorms

$$\|\phi\|_k = \sup_{\zeta\in S^n, |J|\leq k} |D_\zeta^J \phi(\zeta)|_X,$$

where $|\cdot|_X$ denotes the norm in X. Let $\mathcal{D}'$ be the dual of $\mathcal{D}$. We will call $\mathcal{D}'$ the space of X^*-*valued distributions* on S^n. We will say that a distribution $\Lambda \in \mathcal{D}'$ is given by a function $u\colon S^n \to X^*$ if

$$\Lambda(\phi) = \Lambda_u(\phi) = \int_{S^n} u(\zeta)(\phi(\zeta))\, d\sigma(\zeta).$$

For $1 < q < \infty$, we will regard X_q as the dual of $X_{q'}$.

We can now define the Bergman projection of a distribution $\Lambda \in \mathcal{D}'(S^n, X_q)$. It will be convenient to work with holomorphic functions defined on B_n instead of their boundary distributions.

DEFINITION 1. For $z \in B_n$, let k_z be the $X_{q'}$-valued function of $\zeta \in S^n$ given by $k_z(\zeta)(t) = (1-t)/(1-t\langle z, \zeta\rangle)^{n+1}$. With Λ as before, $B\Lambda(z)$ is defined to be

$$B\Lambda(z) = \Lambda(k_z).$$

It can be verified that, if Λ is a X_q-valued distribution determined by a function u defined on B_n according to the formula

$$\Lambda(v) = \Lambda_u(v) = \int_{S^n}\int_0^1 v(\zeta)(t)u(t\zeta)\,\frac{dm_n(t)}{1-t}\,d\sigma = \int_{S^n}(v, \bar{u})\,d\sigma,$$

then $B\Lambda_u$ agrees with the definition of the Bergman projection of u given by equation (2), that is,

$$B\Lambda_u(z) = \Lambda_u(k_z) = Bu(z). \tag{3}$$

We now want to define the ("real") Hardy space $H^p_+(S^n, X^*)$ of X^*-valued distributions on S^n. If $\Lambda \in \mathcal{D}'$ and if $\psi \in C^\infty(S^n)$ is a smooth scalar-valued function, then we define $\Lambda\psi$ to be the element of X^* defined by the equation

$$(\Lambda\psi)(x) = \Lambda(\psi x), \quad x \in X,$$

where ψx is the element of $\mathcal{D}$ that sends ζ to $\psi(\zeta)x$. (The motive for this definition becomes apparent if one considers the case where the distribution Λ is given by a function.) If η and ζ are points on S^n and $0 \le r < 1$, let $P_{r\zeta}(\eta)$ be the function of η given by

$$P_{r\zeta}(\eta) = \frac{(1-r^2)^n}{|1-r\langle\zeta, \eta\rangle|^{2n}}.$$

Thus, $P_{r\zeta}$ is the invariant Poisson kernel.

DEFINITION 2. The Poisson integral of the distribution $\Lambda \in \mathcal{D}'$ at the point $r\zeta \in B$ is the element in X^* given by $\Lambda(P_{r\zeta})$.

It can be verified that, if $\Lambda \in \mathcal{D}'$, then if u_r is the X^*-valued function given by $u_r(\zeta) = \Lambda(P_{r\zeta})$ then $\Lambda_{u_r} \to \Lambda$ as $r \to 1^-$.

For $\Lambda \in \mathcal{D}'$, define the *radial* maximal function

$$\Lambda^+(\zeta) = \sup_{0\le r<1} |\Lambda(P_{r\zeta})|_{X^*},$$

where $|\cdot|_{X^*}$ denotes the norm in X^*.

We also define the admissible maximal function of Λ:

$$\Lambda^*(\zeta) = \sup_{r\eta\in\Gamma(\zeta)} |\Lambda(P_{r\eta})|_{X^*}.$$

DEFINITION 3. A distribution $\Lambda \in \mathcal{D}'$ belongs to $H^p_+(S^n, X^*)$ if

$$\|\Lambda\|^p_{H^p} = \int_{S^n}(\Lambda^+(\zeta))^p\,d\sigma(\zeta) < \infty.$$

Of course, if $X_* = C^1$ is the complex numbers then $H^p_+(S^n, C^1)$ is just the scalar "real" Hardy space studied in [GL]. It can be verified (just as in the scalar case) that, for $1 < p < \infty$, $H^p_+(S^n, X_q) = L^p_q$. Lemma 1 will also show that, for $0 <$

$p < \infty$, the subspace of $H^p_+(S^n, X_q)$ consisting of distributions given by functions of the form $u(t\zeta) = (1-t)g(t\zeta)$, where g is holomorphic on B_n, is a familiar mixed norm space; see Lemma 1.

Our main results can now be stated.

THEOREM 1. *Let $0 < p < \infty$ and $1 < q < \infty$. Then there is a constant $c(p, q)$ such that*

$$\|B\Lambda\|_{F^q_p} \le c(p, q)\|\Lambda\|_{H^p_+(S^n, X_q)}.$$

THEOREM 2. *Let $0 < p < \infty$. Then there is a constant $c(p, q)$ such that*

$$\|SF\|_{H^p_+(S^n, X_q)} \le c(p, q)\|F\|_{F^q_p}.$$

For the limiting case where p goes to ∞, we have the usual BMO results.

THEOREM 3. *There are constants depending on q such that*

(a) $\|B\Lambda\|_{F^q_\infty} \le c(q)\|\Lambda\|_{\mathrm{BMO}(S^n, X^q)}$ *and*
(b) $\|SF\|_{\mathrm{BMO}(S^n, X^q)} \le c(q)\|F\|_{F^q_\infty}$.

The space $\mathrm{BMO}(S^n, X_q)$, defined in the obvious way, is the dual of $H^1_+(S^n, X_{q'})$. Part (b) offers another characterization of the functions in $\mathrm{BMOA}(S^n)$, which we describe in Remark 2 following the proof of Theorem 3.

The proof of Theorem 1 is based on the atomic decomposition for $H^p_+(S^n, X_q)$. To obtain such a decomposition we follow [GL] and [S, Chap. 2, Thm. 2]. We will therefore need to know that $H^p_+(S^n, X^*)$ has equivalent characterizations in terms of radial maximal functions, admissible maximal functions, and grand maximal functions. If $\zeta \in S^n$, let $B(\zeta, \delta)$ be the nonisotropic ball

$$B(\zeta, \delta) = \{\eta : |1 - \langle \eta, \zeta\rangle| < \delta\}.$$

DEFINITION 4. Let $\zeta \in S^n$. The class $\mathcal{F} = \mathcal{F}(\zeta, K)$ is the collection of all (scalar) C^∞ functions ψ such that there is some δ (depending on ψ) where

(i) the support of ψ is contained in $B(\zeta, \delta)$ and
(ii) $\|D^J_\eta \psi\|_\infty \le \delta^{-n-|J|}$ for all $|J| \le K$ and all $\eta \in S^n$.

DEFINITION 5. $\mathcal{M}_{\mathcal{F}_k}\Lambda(\zeta) = \sup_{\psi \in \mathcal{F}(\zeta, k)} |\Lambda(\psi)|_{X^*}$.

THEOREM 4. *If $\Lambda \in \mathcal{D}'$ then the following properties are equivalent*:

(i) $\Lambda \in H^p_+(S^n, X^*)$;
(ii) $\mathcal{M}_{\mathcal{F}_K}\Lambda \in L^p(d\sigma)$ *for some K sufficiently large*;
(iii) $\Lambda^* \in L^p(d\sigma)$.

The proof of Theorem 4 will be outlined in Section 2. The idea, of course, is to adapt the proof of [S, Chap. 3, Thm. 1] to the setting here, where the distributions are vector-valued and there is no Fourier transform. The fact that the distributions are vector-valued causes no real difficulty; duality arguments essentially allow one to substitute the absolute value $|\cdot|$ with $\|\cdot\|_{X^*}$. Finding a replacement for the Fourier transform is considerably more difficult, but fortunately this has already been done in [GL], where the scalar version of the equivalence of (ii) and (iii) in our Theorem 4 is established. We are able to simplify the proof given in [GL], and we also give a proof that (ii) and (iii) are equivalent to (i).

As a corollary of Theorem 4, the methods of [GL] and [S] yield the desired atomic decomposition of $H^p_+(S^n, X^*)$ for $0 < p \le 1$ into (p, ∞) atoms, where the (p, ∞) atoms are X^*-valued functions $a\colon S^n \to X^*$ satisfying the usual support and moment conditions as well as norm bounds. Following [GL], we establish the next definition.

DEFINITION 6. An X^*-valued function on S^n is called a (p, ∞)-atom at $\zeta \in S^n$ if either (case 1) $|a|_{X^*} \le 1$ or (case 2):

(i) a is supported in $B(\zeta, \delta)$;
(ii) $\int_{S^n} a(\eta)\pi_\zeta(\eta)\,d\sigma(\eta) = 0$ for special monomials π_ζ of nonisotropic degree $|J| \le (1/2)[2n(1-1/p]$ (here the monomials π_ζ are those in the variables $\mathrm{Im}\langle\eta, \zeta\rangle$ and $\langle\eta, \omega_j\rangle$, $\langle\omega_j, \eta\rangle$, $j = 2, \dots, n$, where the n vectors $\zeta, \omega_1, \dots, \omega_n$ form an orthonormal basis for C^n);
(iii) $|a(\zeta)|_{X^*} \le \delta^{-n/p}$ for all $\zeta \in S^n$.

Note that, in (ii), the integral is X^*-valued.

THEOREM 5. *If $0 < p \le 1$ and $\Lambda \in H^p_+(S^n, X^*)$, then Λ has atomic decomposition*

$$\Lambda = \sum_j \lambda_j a_j,$$

where a_j is a (p, ∞)-atom and $\sum_j |\lambda_j|^p < \infty$.

REMARK 1. It is interesting to compare atoms in the tent space T^p_q with case-2 atoms in the space $H^p_+(S^n, X_q)$. A function a is a T^p_q-atom if there is some $\zeta \in S^n$ and some $\delta > 0$ such that:

(i) the support of a is contained in $\{\, w \in B_n : |1 - \langle w, \zeta\rangle| < \delta \,\}$;
(ii) $\int_{B_n} |a(w)|^q\,d\nu(w)/(1-|w|) \le \delta^{n-nq/p}$.

On the other hand, an atom a at $\zeta \in S^n$ for $H^p_+(S^n, X_q)$ when regarded as a function of $w \in B_n$ has the following properties:

(i) the support of a is contained in the set $\{\, w : w = t\eta,\ \eta \in S^n,\ 0 \le t < 1,\ |1 - \langle\eta, \zeta\rangle| < \delta \,\}$;
(ii) $\int_{B_n} |a(w)|^q\,d\nu(w)/(1-|w|) \le \delta^{n-nq/p}$;
(iii) $\int_{S^n} a(t\eta)\pi_\zeta(\eta)\,d\sigma(\eta) = 0$ for all $0 \le t < 1$ and all polynomials described as in Definition 1.

Thus, T^p_q-atoms satisfy the same norm bounds as $H^p_+(S^n, X_q)$-atoms but are more restricted in their support. Case-2 $H^p_+(S^n, X_q)$-atoms can have larger support but must satisfy a very strong cancellation condition.

2. Proofs of Theorems 1–4

Proof of Theorem 1

The first thing to do is to settle the case where $p = q$. (This is also the main step in proving Theorem A for the case where $q \neq 2$, since the atomic decomposition for T_q^p and interpolation can be used to prove Theorem A exactly as in [Co], where it is done for the case where $q = 2$.) In this case $H_+^q(S^n, X_q) = L_q^q(S^n) = T_q^q(B_n)$, where the distributions are X_q-valued functions that may be interpreted as functions on B_n. Suppose the distribution Λ is determined by a function u. For $w = r\zeta \in B_n$ write $u(w) = u(\zeta)(r)$. Using (3) and (2) it follows that

$$(1-|z|)(|B\Lambda(z)| + |\nabla B\Lambda(z)|) \leq C \int_{B_n} |u(w)| \frac{(1-|z|)(1-|w|)}{|1-\langle z, w\rangle|^{n+2}} \frac{d\nu(w)}{1-|w|},$$

where $z \in B_n$. Thus, it is enough to show that the operator defined by

$$Ku(z) = \int_{B_n} u(w) \frac{(1-|z|)(1-|w|)}{|1-\langle z, w\rangle|^{n+2}} \, d\mu(w), \tag{4}$$

where $d\mu(w) = d\nu(w)/(1-|w|)$, satisfies the norm bounds

$$\|Ku\|_{L_q^q} \leq C\|u\|_{L_q^q}.$$

This estimate follows from a standard argument using Schur's method in conjunction with the inequality

$$K(h)(z) \leq Ch(z),$$

where $h(z) = (1-|z|)^c$ for c a small negative number. See [R, Thm. 7.1.4].

For the case $1 < p < \infty$ we use a duality argument. What is needed is to show that

$$\|Ku\|_{T_q^p} \leq C\|u\|_{L_q^p(S^n)}.$$

Since $(T_q^p)^* = T_{q'}^{p'}$ (with the pairing $\langle\ ,\ \rangle$ given by equation (1)), and since K is self-adjoint, duality shows that it is enough to prove that

$$\|Ku\|_{L_q^p(S^n)} \leq C\|u\|_{T_q^p}$$

for all $u \in T_q^p$. Without loss of generality, we may assume that u is nonnegative. It is easy to see that Ku satisfies a mean-value inequality

$$Ku(z) \leq C(1-|z|)^{-(n+1)} \int_{Q(z)} Ku(w)\, d\nu(w),$$

where $Q(z) = \{\, w \in B_n : |1-\langle z, w\rangle| \leq (1/2)(1-|z|)\,\}$. Therefore,

$$Ku(z)^q \leq C \int_{Q(z)} Ku(w)^q \frac{d\nu(w)}{(1-|w|)^{n+1}}.$$

Let $z = t\zeta$, and integrate both sides of the last inequality from 0 to 1 against the measure $dt/(1-t)$. An application of Fubini's theorem to the resulting integral on the right gives the pointwise bound of $g_q[Ku](\zeta) \leq CA_q[Ku](\zeta)$, where C is independent of u or ζ. Therefore,

$$\|Ku\|_{L^p_q(S^n)} \le C\|Ku\|_{T^p_q} \le C\|u\|_{T^p_q}$$

by Theorem A.

It remains to prove Theorem 1 for the case $0 < p \le 1$. We will use the atomic decomposition described by Theorem 5. Let a be a case-2 X_q-valued (p, ∞)-atom supported in the ball $B(\omega, \delta)$. (There will be no loss of generality in assuming that a is an atom of this type, as the argument will show.) Let $Ja(z) = (1-|z|)DBa(z)$, where D denotes a first-order derivative. It will be enough to establish the inequality $\|Ja\|_{T^p_q} \le C$, where C is independent of δ or ζ. Without loss of generality, we will assume that $\omega = e_1 = (1, 0, \dots, 0)$.

Using the fact that $|Ja(z)| \le CK|a|(z)$ with K as in (4), estimate first that

$$\begin{aligned}\int_{B(e_1,3\delta)} A_q[Ja]^p\, d\sigma &\le C\delta^{n(1-p/q)}\left(\int_{S^n} A_q[K|a|]^q\, d\sigma\right)^{p/q}\\ &\le C\delta^{n(1-p/q)}\left(\int_{S^n} A_q[|a|]^q\, d\sigma\right)^{p/q}\\ &\le C\delta^{n(1-p/q)}(\delta^{n-qn/p})^{p/q} \le C,\end{aligned}$$

where we have used the $q = p$ result established previously. (Note that, if a were a case-1 atom, then this estimate would suffice since we could take $\delta = 2$.)

To estimate $\int_{S^n - B(e_1,3\delta)} A_q[Ja]^p\, d\sigma$, suppose that $|1-\zeta_1| \ge 3\delta$ and $z \in \Gamma(\zeta)$. Write $DBa(z)$ as the integral

$$DBa(z) = \int_0^1 \int_{S^n} a(t\eta) D\frac{1}{(1-\langle z, t\eta\rangle)^{n+1}}\, dm_n(t)\, d\sigma(\eta).$$

If for each t we expand $D(1/(1-\langle z, t\eta\rangle)^{n+1})$ in a nonisotropic Taylor series in η at e_1, then the cancellation properties of a (see Remark 1) allow the estimate that, for some $k > n(1/p - 1)$, $|Ja(z)|$ is less than a constant times

$$\begin{aligned}&(1-|z|)\delta^k \int_{B_n} |a(w)| \frac{d\nu(w)}{|1-\langle z, w\rangle|^{n+2+k}}\\ &\le C(1-|z|)\delta^k\left(\int_{B_n} |a(w)|^q \frac{d\nu}{1-|w|}\right)^{1/q}\\ &\quad\times\left(\int_{B(e_1,\delta)}\int_0^1 \frac{(1-t)^{q'-1}\, dt}{|1-\langle z, t\eta\rangle|^{(n+2+k)q'}}\, d\sigma(\eta)\right)^{1/q'}\\ &\le C(1-|z|)\delta^{k+n/q-p/q}\left(\int_{B(e_1,\delta)} \frac{d\sigma(\eta)}{((1-|z|)+|1-\zeta_1|)^{(n+1+k)q'}}\right)^{1/q'}\\ &\le C(1-|z|)\delta^{k+n-n/p}\frac{1}{((1-|z|)+|1-\zeta_1|)^{n+k+1}}.\end{aligned}$$

Using this last estimate, integrate over $\Gamma(\zeta)$ to bound $A_q[Ja](\zeta)$. Write

$$\frac{d\nu(z)}{(1-|z|)^{n+1}} = (1-r)^{-n-1} r^{2n-1}\, dr\, d\sigma,$$

where $r = |z|$. Interchange the order to integrate first with respect to $d\sigma$, reducing the power of $(1-r)$ by a factor of n. It follows that

$$A_q[Ja](\zeta) \le C\delta^{n+k-n/p}\left(\int_0^1 \frac{(1-r)^{q-1}\,dr}{((1-r)+|1-\zeta_1|)^{(n+k+1)q}}\right)^{1/q}$$
$$\le C\delta^{n+k-n/p}|1-\zeta_1|^{-n-k}.$$

Therefore,

$$\int_{S^n - B(e_1,3\delta)} A_q[Ja]^p\,d\sigma \le C\delta^{np+kp-n}\int_{S^n-B(e_1,3\delta)} |1-\zeta_1|^{-np-kp}\,d\sigma \le C,$$

which was what we needed to show. □

In order to prove Theorem 2, we will need the following lemma.

LEMMA 1. *Suppose that Λ is an X_q-valued distribution determined by the function u, where $u(\zeta)(t) = (1-t)g(t\zeta)$ with g holomorphic on B_n. Then $\Lambda \in H_+^p(S^n, X_q)$ if and only if*

$$\int_{S^n}\left(\int_0^1 (1-t)^{q-1}|g(t\zeta)|^q\,dm_n(t)\right)^{p/q} d\sigma(\zeta) < \infty.$$

Proof. We first establish the necessity. Let $b = \{h : h \in X_{q'},\ \|h\| = 1\}$ and let b_s be the set of functions in b that are supported on the set $[0, s]$. By duality, if $\zeta \in S^n$ and $0 \le r < 1$ then

$$\begin{aligned}
|\Lambda(P_{r\zeta})| &= \sup_{h\in b}|\Lambda(P_{r\zeta})(h)| \\
&\ge \sup_{h\in b_s}|\Lambda(P_{r\zeta})(h)| \\
&= \sup_{h\in b_s}|\Lambda(P_{r\zeta}h)| \\
&= \sup_{h\in b_s}\left|\int_{S^n}\int_0^s P_{r\zeta}(\eta)h(t)g(t\eta)\,dm_n(t)\,d\sigma(\eta)\right| \\
&= \sup_{h\in b_s}\left|\int_0^s g(tr\zeta)h(t)\,dm_n(t)\right| \\
&= \left(\int_0^s |g(tr\zeta)|^q(1-t)^{q-1}\,dm_n(t)\right)^{1/q}.
\end{aligned}$$

If we take the supremum over s and r and integrate over the sphere, the result follows. To establish the converse, fix $\zeta \in S^n$ and $0 \le r < 1$. Use duality to find $h \in b$ such that

$$|\Lambda(P_{r\zeta})| = \Lambda(P_{r\zeta})(h).$$

It follows that, for s sufficiently close to 1,

$$|\Lambda(P_{r\zeta})| \le 2|\Lambda(P_{r\zeta})(h_s)|,$$

where h_s is the restriction of h to the interval $[0, s]$. Writing the expression in absolute values on the right-hand side as an integral, interchanging the order of integration, and applying Holder's inequality, we have

$$|\Lambda(P_{r\zeta})| \le 2\left(\int_0^1 |g(tr\zeta)|^q (1-t)^{q-1}\, dm_n(t)\right)^{1/q}.$$

It follows easily that

$$\Lambda^+(\zeta) \le C\left(\int_0^1 |g(t\zeta)|^q (1-t)^{q-1}\, dm_n(t)\right)^{1/q}. \qquad \square$$

Proof of Theorem 2

Let u be of the form $u(z) = (1-|z|)f(z)$, where f is holomorphic. If $\zeta \in S^n$ and $0 \le t < 1$, then we have the mean-value inequality

$$|u(t\zeta)|^q \le C(q) \int_{Q(\zeta, \frac{1}{4}(1-t))} |u(w)|^q \frac{d\nu(w)}{(1-|w|)^{n+1}},$$

where $Q(\zeta, \varepsilon) = \{ w \in B_n : |1 - \langle w, \zeta\rangle| \le \varepsilon \}$; see [G] and [ACo, Lemma 3]. Argue as in the proof of Theorem 1 to show that there is an absolute constant (depending on q) such that

$$g_q[u](\zeta) \le CA_q[u](\zeta).$$

Thus, if $F \in F_p^q$ then $g_q[Sf] \le CA_q[SF]$, and Lemma 1 therefore implies that

$$\|SF\|_{H_q^p(S^n, X_q)} \le C\|F\|_{F_p^q}. \qquad \square$$

Proof of Theorem 3

Let $\Lambda \in \mathrm{BMO}(S^n, X_q)$. We must show $B\Lambda \in F_\infty^q$. Letting

$$J\Lambda(z) = (1-|z|)DB\Lambda(z),$$

where D denotes a first-order derivative of $B\Lambda$, this amounts to establishing the norm bound

$$\|J\Lambda\|_{T_q^\infty} \le C\|\Lambda\|_{\mathrm{BMO}}.$$

Since T_q^∞ is the dual of $T_{q'}^1$, we must show

$$\left|\int_{B_n} J\Lambda(z) f(z) \frac{d\nu(z)}{1-|z|}\right| \le C\|\Lambda\|_{\mathrm{BMO}}\|f\|_{T_{q'}^1}.$$

Use the fact that Λ is given by a function on B_n (which we denote by Λ) to write $DB\Lambda(z)$ as an integral, and interchange the order of integration to arrive at the equivalent statement

$$\left|\int_{B_n} \Lambda(w) Kf(w) \frac{d\nu(w)}{1-|w|}\right| \le C\|\Lambda\|_{\mathrm{BMO}}\|f\|_{T_{q'}^1},$$

where

$$Kf(w) = (1-|w|)\int_{B_n} f(z) D_z \frac{1}{(1-\langle z, w\rangle)^{n+1}}\, d\nu(z).$$

By Theorem A, $\|Kf\|_{T_{q'}^1} \le C\|f\|_{T_{q'}^1}$. Since $Kf(w)$ is of the form $(1-|w|)g(w)$, where g is the complex conjugate of a holomorphic function, it follows from Lemma 1 and the argument given in the proof of Theorem 2 that

$$\|Kf\|_{H^1_+(S^n, X_{q'})} \leq C\|f\|_{T^1_{q'}}.$$

Therefore,

$$\left|\int_{B_n} \Lambda(w)Kf(w)\,\frac{d\nu(w)}{1-|w|}\right| \leq C\|\Lambda\|_{\mathrm{BMO}(S^n, X_q)}\|Kf\|_{H^1_+(S^n, X_{q'})}$$
$$\leq C\|\Lambda\|_{\mathrm{BMO}(S^n, X_q)}\|f\|_{T^1_{q'}}$$

as desired.

The second statement of Theorem 3 follows from a standard argument based on duality and the fact that $S^* = B$ and B maps $H^1_+(S^n, X_{q'})$ into $F_1^{q'}$. □

REMARK 2. It is well known that a holomorphic function f belongs to BMOA if and only if $(1-|z|)|Rf(z)|^2\,d\nu(z)$ is a Carleson measure. A corollary of Theorem 3 is that, if f is holomorphic, then f is in F_q^∞ if and only if $g \in \mathrm{BMO}(S^n, X_q)$ with $g(\zeta)(t) = (1-t)Rf(t\zeta)$. The necessity is essentially the second statement of Theorem 3; the sufficiency follows from the first statement of Theorem 3 and the identity

$$f(z) = \int_{B_n} (1-|w|^2)((n+1)f(w) + Rf(w))\,\frac{d\nu(w)}{(1-\langle z, w\rangle)^{n+1}}.$$

One also needs the easily verified fact that if $(1-t)Rf(t\zeta) \in \mathrm{BMO}(S^n, X_q)$ then $(1-t)f(t\zeta) \in \mathrm{BMO}(S^n, X_q)$. For $q = 2$ this means that a holomorphic function f belongs to BMOA if and only if there is a constant C such that, for every ball $Q = B(\eta, \delta)$, there is a function h_Q defined on $[0, 1]$ such that $(1-t)h_Q(t)$ gives a function in X_q with the property that

$$\delta^{-n}\int_Q |(1-t)(Rf(\zeta) - h_Q)|_{X_q}\,d\sigma$$
$$= \delta^{-n}\int_{B(\eta,\delta)}\left(\int_0^1 |Rf(t\zeta) - h_Q(t)|^2(1-t)\,dm_n\right)^{1/2} d\sigma \leq C.$$

The argument of John and Nirenberg (see [S]) allows the exponent "1/2" to be replaced by any positive power. Replacing "1/2" by "1" gives a condition that is close to the Carleson measure condition.

We turn next to the proof of Theorem 4.

Let ψ be a C^∞-function of one complex variable λ. We will need the following substitute for convolution. Let f be a continuous function defined on S^n with values in X^*, the dual of a Banach space X. Then, for $\zeta \in S^n$, $(f * \psi)(\zeta)$ is the element of X^* defined by

$$(f * \psi)(\zeta) = \int_{S^n} f(\eta)\psi(\langle\zeta, \eta\rangle)\,d\sigma(\eta).$$

Of course, this means that

$$(f * \psi)(\zeta)(x) = \int_{S^n} f(\eta)(x)\psi(\langle\zeta, \eta\rangle)\,d\sigma(\eta)$$

for all $x \in X$. More generally, with ψ as before and with $\zeta \in S^n$, let ψ_ζ be the C^∞-function of $\eta \in S^n$ given by $\psi_\zeta(\eta) = \psi(\langle \zeta, \eta \rangle)$. If $\Lambda \in \mathcal{D}'$ then $\Lambda * \psi$ is defined to be the C^∞-mapping from S^n to X^* given by $\Lambda * \psi(\zeta) = \Lambda(\psi_\zeta)$. If the distribution Λ is determined by a function f as before, then the two definitions agree.

The proof of [ACo, Lemma 1] shows that, if ψ_1 and ψ_2 are two functions of a complex variable, then

$$(\Lambda * \psi_1) * \psi_2 = (\Lambda * \psi_2) * \psi_1. \tag{5}$$

Proof of Theorem 4

We first discuss the implication (iii) $\Rightarrow$ (ii). Following [GL], we introduce an auxiliary kernel obtained by averaging the invariant Poisson kernel. Let $1 - r = t$ and $\lambda \in B_1$, and let

$$p_t(\lambda) = \frac{(2-t)^n t^n}{|(1-\lambda) + \lambda t|^{2n}}.$$

Then it is clear that, for $\Lambda \in \mathcal{D}'$, $\Lambda(P_{r\zeta}) = \Lambda * p_t(\zeta)$. Choose a measure μ on the interval $[\frac{1}{4}, \frac{1}{2}]$ such that $\mu([\frac{1}{4}, \frac{1}{2}]) = 1$ and $\int t^k \, d\mu(t) = 0$ for $k = 1, \dots, N_0$, where N_0 is a large integer that will be specified later. Define the auxiliary kernel

$$\tau_t(\lambda) = \int p_{ts}(\lambda) \, d\mu(s).$$

The kernel τ_t is needed because it decays more rapidly as t goes to 0 than the Poisson kernel; see Lemma 2.

For $\zeta \in S^n$ and $0 \le r < 1$, let $\tau_{r\zeta}$ be the function of $\eta \in S^n$ given by $\tau_{r\zeta}(\eta) = \tau_t(\langle \zeta, \eta \rangle)$, where $t = 1 - r$. If Λ is a distribution in $\mathcal{D}'$, define the following maximal functions associated with τ_t:

$$\mathcal{M}_\tau^+ \Lambda(\zeta) = \sup_{0 \le r < 1} |\Lambda(\tau_{r\zeta})|;$$

$$\mathcal{M}_\tau^* \Lambda(\zeta) = \sup_{r\eta \in \Gamma(\zeta)} |\Lambda(\tau_{r\eta})|;$$

$$\mathcal{M}_{\tau,M}^{**} \Lambda(\zeta) = \sup_{r\eta \in B_n} |\Lambda(\tau_{r\eta})| \left(1 + \frac{|1 - \langle \zeta, \eta \rangle|}{1 - r}\right)^M.$$

In the sequel we will drop the subscripts τ and M if doing so causes no confusion.

It is clear that, if a distribution Λ satisfies (iii) in Theorem 4, it follows that $\mathcal{M}^* \Lambda \in L^p(d\sigma)$. We also have the well-known fact that, for M sufficiently large (see [FSt], p. 166]),

$$\int_{S^n} (\mathcal{M}^{**}\Lambda)^p \, d\sigma \le C \int_{S^n} (\mathcal{M}^*\Lambda)^p \, d\sigma.$$

The implication (iii) $\Rightarrow$ (ii) will therefore follow once we show there is a K sufficiently large that the pointwise inequality

$$|\Lambda(\psi)| \le C \mathcal{M}^{**} \Lambda(\zeta)$$

holds for all $\psi \in \mathcal{F}(\zeta, K)$, where C is a constant independent of ψ or ζ.

For this, let $\psi \in \mathcal{F}(\zeta, K)$, and suppose that the support of ψ is contained in $B(\zeta, \delta)$ and that $|D^\alpha \psi(\eta)| \leq \delta^{-n-|\alpha|}$ for all $|\alpha| \leq K$. Because $\psi = \lim_{t\to 0} \psi * \tau_t * \tau_t$ in the topology of C^∞-functions on S^n, it follows that

$$\Lambda(\psi) = \Lambda(\psi * \tau_\delta * \tau_\delta) - \int_0^\delta \frac{d}{dt}\Lambda(\psi * \tau_t * \tau_t)\, dt.$$

Equation (5) then implies that the X^*-valued integral on the right-hand side equals

$$\int_0^\delta \Lambda(\psi * Q_t * \tau_t)\, dt,$$

where $Q_t(\lambda) = 2(d/dt)\tau_t(\lambda)$. Writing the "convolutions" $(\psi * Q_t) * \tau_t$ and $(\psi * \tau_\delta) * \tau_\delta$ as integrals over S^n, and interchanging the order of applying Λ and integration over S^n, yields

$$\Lambda(\psi) = \int_{S^n} \Lambda(\tau_{\delta\eta})(\psi * \tau_\delta)(\eta)\, d\sigma(\eta) - \int_0^\delta \int_{S^n} \Lambda(\tau_{t\eta})\psi * Q_t(\eta)\, d\sigma(\eta)\, dt,$$

where all the integrals are X^*-valued.

Duality and linearity show that $|\Lambda(\psi)|_{X^*}$ is less than or equal to

$$\int_{S^n} |\Lambda(\tau_{\delta\eta})|_{X^*} |(\psi * \tau_\delta)(\eta)|\, d\sigma(\eta) + \int_0^\delta \int_{S^n} |\Lambda(\tau_{t\eta})|_{X^*} |\psi * Q_t(\eta)|\, d\sigma(\eta)\, dt,$$

which in turn is less than or equal to the sum of

$$\mathcal{M}^{**}\Lambda(\zeta) \int_{S^n} |\psi * \tau_\delta(\eta)| \left(1 + \frac{|1 - \langle \eta, \zeta\rangle|}{\delta}\right)^M d\sigma(\eta) \tag{6}$$

and

$$\mathcal{M}^{**}\Lambda(\zeta) \int_0^\delta \int_{S^n} |\psi * Q_t(\eta)| \left(1 + \frac{|1 - \langle \eta, \zeta\rangle|}{t}\right)^M d\sigma(\eta)\, dt. \tag{7}$$

The proof will be complete provided it is shown that the two integrals in (6) and (7) are bounded by absolute constants independent of ψ. The first term is handled easily by the decay properties of τ_t described in the following lemma (see [GL] for a proof).

LEMMA 2. *Let $1 < N_0$. If $|1 - \lambda| > 2t$ then*

$$|\tau_t(\lambda)| \leq C \frac{t^{N_0+n-1}}{|1-\lambda|^{N_0+2n-1}} \quad \textit{and} \quad |Q_t(\lambda)| \leq C \frac{t^{N_0+n-2}}{|1-\lambda|^{N_0+2n-1}}.$$

From Lemma 2 it follows that

$$|\psi * \tau_\delta(\eta)| \leq C\left(\chi_{B(\zeta,3\delta)}(\eta)\delta^{-n} + \chi_{S^n - B(\zeta,3\delta)}(\eta) \frac{\delta^{N_0+n-1}}{|1 - \langle \eta, \zeta\rangle|^{N_0+2n-1}}\right).$$

Here, χ_E denotes the characteristic function of a subset $E \in S^n$. If

$$N_0 + 2n - 1 - M > n$$

then it is easy to see that the integral in (6) is bounded by an absolute constant.

The analysis required to estimate the second term is more complicated, but the basic idea is from [GL]. We present a simplified version of their argument, which takes advantage of the work of [Fo] and homogeneous harmonic polynomials.

LEMMA 3. *Let π be a monomial in $\zeta_1, \dots, \zeta_n$ and $\bar{\zeta}_1, \dots, \bar{\zeta}_n$ of bidegree (p, q). Suppose $N > 0$. If N_0 is sufficiently large then there is a constant $C = C(p, q, N)$ such that $|\pi * Q_t(\zeta)| \le Ct^N$ for all $\zeta \in S^n$.*

Proof. Without loss of generality, we may assume that π is a homogeneous harmonic polynomial; see [R, Thm. 12.1.3, Prop. 12.2.2]. From [Fo] (see also [ACa, p. 7]) it follows that

$$\pi * p_t(\eta) = f(p, q, t)\pi(\eta),$$

where there are functions f and g (depending on p and q), analytic at 0, such that

$$f(p, q, t) = f(t) + t^n g(t) \log(t).$$

Replace t by ts, expand g in a Taylor series in powers of ts, differentiate with respect to t, and average with respect to μ as in the definition of τ_t. This yields the result. □

LEMMA 4. *Let $N > 0$. If N_0 is sufficiently large then there is an absolute constant C such that, for $0 < t < \delta$,*

$$|\psi * Q_t(\eta)| \le C\chi_{B(\zeta,3\delta)}(\eta)\delta^{-N}t^{N-n-1} + C\chi_{B(\zeta,3\delta)^c}(\eta)\frac{t^{N-n-1}}{|1 - \langle\eta, \zeta\rangle|^N}, \tag{8}$$

where E^c denotes the complement of a set $E \subset S^n$.

Assuming Lemma 4, we can finish the proof of (iii) $\Rightarrow$ (ii) in Theorem 4 as follows. If N is chosen sufficiently larger than M then it follows from (8) that

$$\int_{S^n} |\psi * Q_t(\eta)|\left(1 + \frac{|1 - \langle\eta, \zeta\rangle|}{t}\right)^M d\sigma(\eta) \le Ct^{-1}\left(\frac{t}{\delta}\right)^{N-M-n}.$$

The result follows by integrating from 0 to δ with respect to t.

Proof of Lemma 4. We establish the estimate by considering two cases.

Case 1. If $\eta \in S^n - B(\zeta, 3\delta)$ then, by Lemma 2, since ψ is supported on $B(\zeta, \delta)$ and is bounded by δ^{-n} it follows that

$$|\psi * Q_t(\eta)| \le \frac{t^{N_0+n-2}}{|1 - \langle\eta, \zeta\rangle|^{N_0+2n-1}}.$$

If N_0 is sufficiently large then this gives the desired estimate for $\eta \in S^n - B(\zeta, 3\delta)$.

Case 2. This is more difficult. Let $\eta \in B(\zeta, 3\delta)$, let J be a positive integer, and let $T(u) = T^J_\eta(u)$ denote the nonisotropic Taylor expansion of $\psi(u)$ at η of degree J. Then $\psi * Q_t(\eta) = \mathrm{I} + \mathrm{II}$, where

$$\mathrm{I} = \int_{S^n} (\psi(u) - T(u))\chi_{B(\zeta,3\delta)}(u) Q_t(\langle\eta, u\rangle)\, d\sigma(u)$$

and

$$\mathrm{II} = \int_{S^n} T(u)\chi_{B(\zeta,3\delta)}(u) Q_t(\langle \eta, u\rangle)\, d\sigma(u).$$

Write $\mathrm{II} = \mathrm{III} - \mathrm{IV}$, where

$$\mathrm{III} = \int_{S^n} T(u) Q_t(\langle \eta, u\rangle)\, d\sigma(u)$$

and

$$\mathrm{IV} = \int_{S^n - B(\zeta,3\delta)} T(u) Q_t(\langle \eta, u\rangle)\, d\sigma(u).$$

We proceed to bound the terms I, IV, and III.

First we estimate I. For this, observe that (see [GL, Lemma 3.16])

$$|\psi(u) - T(u)| \le C\delta^{-n-1-J}|1 - \langle \eta, u\rangle|^{J+1}$$

and, for any positive integer $L < N_0$,

$$|Q_t(\langle \eta, u\rangle)| \le C \frac{t^{L+n-2}}{|1 - \langle \eta, u\rangle| + t)^{L+2n-1}}.$$

Thus

$$\mathrm{I} \le C\delta^{-n-1-J} t^{L+n-2} \int_{B(\zeta,3\delta)} |1 - \langle \eta, u\rangle|^{J+1-(L+2n-1)}\, d\sigma(u).$$

If $J + 1 - (L + 2n - 1) > -n$ (i.e., if $L < J + 2$) then this last expression is bounded by

$$Ct^{-n-1}(t/\delta)^{L+2n-1}.$$

We may therefore choose N_0 and J such that L may be taken to be sufficiently large that the desired estimate holds. The value for J will now be regarded as fixed, but we will need to allow N_0 to be larger in the next two estimates.

For IV, observe that T is a sum of terms that are bounded by $\delta^{-n-k}|1 - \langle \eta, u\rangle|^k$, where k ranges over integers and half-integers between zero and J. Since

$$|Q_t(\langle \eta, u\rangle)| \le C(t^{N_0+n-2}/|1 - \langle \eta, u\rangle|^{N_0+2n-1}),$$

it follows that IV is bounded by a finite sum of terms of the form

$$\int_{S^n - B(\eta,\delta)} \frac{\delta^{-n-k} t^{N_0+n-2}}{|1 - \langle \eta, u\rangle|^{N_0+2n-1-k}}\, d\sigma(u).$$

If $N_0 + 2n - 1 - J > n$ then this last expression is less than $Ct^{-n-1}(t/\delta)^{N_0+2n-1}$, and the desired estimate holds for N_0 sufficiently large.

It remains to handle III. If $\psi \in \mathcal{F}(\zeta, K)$ with K is sufficiently large, we may estimate that the coefficients of the polynomials in the Taylor series $T^J(u)$ are bounded by $C\delta^{-n-J}$. The desired estimate for III follows from Lemma 3 provided N_0 is sufficiently larger than J. □

We now finish the proof of Theorem 4.

We show that (ii) $\Rightarrow$ (iii). Let $\zeta \in S^n$ and suppose $r\eta \in \Gamma(\zeta)$. By partitioning unity as in [ACo, proof of Thm. 2], we may write $P_{r\eta}(u) = \sum_{k=0}^{\infty} 2^{-k}\psi_k(u)$ where each $\psi_k \in \mathcal{F}(\zeta, K)$. It follows that $\Lambda^*(\zeta) \le C\mathcal{M}_{\mathcal{F}}\Lambda(\zeta)$.

We finally show that (i) $\Rightarrow$ (iii). The proof is a modification of the argument in [S, Chap. 3, 1.5–1.6]. Suppose that Λ satisfies (i). Let $t = 1 - r$ and, for L large and depending on Λ, define the following maximal functions:

$$\mathcal{M}_\varepsilon^*\Lambda(\zeta) = \sup_{r\eta\in\Gamma(\zeta)} \left(\frac{t}{t+\varepsilon}\right)^L |\Lambda(P_{r\eta})|;$$

$$\mathcal{M}_\varepsilon^{**}\Lambda(\zeta) = \sup_{r\eta\in B_n} \left(\frac{t}{t+\varepsilon}\right)^L |\Lambda(P_{r\eta})| \left(1 + \frac{|1-\langle\eta,\zeta\rangle|}{t}\right)^{-M};$$

$$\mathcal{M}_{\mathcal{F}}^\varepsilon\Lambda(\zeta) = \sup|\Lambda(\psi)|;$$

the supremum is over all $\psi \in \mathcal{F}(\zeta, K)$, where the δ associated with ψ is larger than ε. If L is chosen sufficiently large, then each of the first two maximal functions are in L^p and we have

$$\int_{S^n} (M_\varepsilon^{**}\Lambda)^p \, d\sigma \leq C \int_{S^n} (M_\varepsilon^*\Lambda)^p \, d\sigma$$

for a constant independent of ε, L, or Λ.

With these definitions, notice that the argument proving Lemma 4 also gives the pointwise inequality

$$\mathcal{M}_{\mathcal{F}}^\varepsilon\Lambda(\zeta) \leq C\mathcal{M}_\varepsilon^{**}\Lambda(\zeta),$$

where C is a constant independent of ε. The crucial point is that the argument gives an inequality of the form (8), where the power of t is decreased by a factor of L, but an extra factor of ε^L provides sufficient compensation, since the final step is to integrate from zero to δ where $\delta > \varepsilon$.

With this established, argue as in [S, Chap. 3, 1.5] to show that

$$\int_{F_\lambda} (\mathcal{M}_\varepsilon^*\Lambda)^p \, d\sigma \leq C \int_{S^n} \Lambda^+(\zeta)^p \, d\sigma,$$

where C is an absolute constant and

$$F_\lambda = \{\zeta : \mathcal{M}_{\mathcal{F}}^\varepsilon\Lambda(\zeta) \leq \lambda M_\varepsilon^*\Lambda(\zeta)\}.$$

Because Λ^+ is defined in terms of the Poisson kernel, this last step requires a partition-of-unity argument (like the one given in the (ii) $\Rightarrow$ (iii) proof) in order to take advantage of the inequality defining F_λ. If λ is sufficiently large (see [S, Chap. 3, 1.5]), then

$$\int_{S^n} \mathcal{M}_\varepsilon^*\Lambda^p \, d\sigma \leq 2\int_{F_\lambda} \mathcal{M}_\varepsilon^*\Lambda^p \, d\sigma.$$

Since C is a constant independent of ε, we may let ε go to zero and so complete the proof of Theorem 4. □

We conclude by examining the relationship between the spaces $H_+^p(S^n, X_q)$ and T_q^p. We show first that T_q^p is not continuously contained in $H_+^p(S^n, X_q)$ for $0 < p \leq 1$. It suffices to consider functions of the form $F(\zeta)(t) = \chi_Q(\zeta)f(t)$, where $Q = B(\eta, \delta)$ and f is some function in X_q of norm 1 that is supported on the interval $[1-\delta, 1]$. One easily checks that $\|F\|_{T_q^p} \doteq C\|\chi_Q\|_{L^p}$ and that $\|F\|_{H_+^p(S^n,X^q)} = \|\chi_Q\|_{H_+^p}$, where H_+^p is the usual (scalar-valued) Hardy space on S^n. If $T_q^p \subset H_+^p(S^n, X_q)$ then we would have the inequality

$$\|\chi_Q\|_{H_+^p} \leq C\|\chi_Q\|_{L^p}$$

for a constant C independent of Q, which is false; see [S, Chap. 3, 5.6].

To show that the reverse containment is false we look at functions of the form $F(\zeta)(t) = \delta^{-n/p} b(\zeta)\chi_Q(\zeta) f(t)$, where $f \in X_q$ is a nonnegative function with norm 1 and b is a unimodular function chosen so that enough moments of $\chi_Q b$ vanish to make F an X_q-valued (p, ∞)-atom. Let $G(\zeta)(t) = |\delta^{-n/p}\chi_Q(\zeta) f(t)|$. Notice that a function and its absolute value have the same norm in T_q^p. Therefore, if the containment $H_+^p(S^n, X_q) \subset T_q^p$ were valid then we would have the inequality

$$\int_{S^n} A_q[G]^p \, d\sigma \leq C,$$

where C is independent of δ or f. Let $\eta = e_1 = (1, 0, \ldots, 0)$ and let $f(t) = [\ln(4)]^{-1/q}\chi_I(t)$, where I is the interval $\sqrt{\delta} < 1 - t < 4\sqrt{\delta}$. Estimate that, for $|1 - \zeta_1| > 2\delta$,

$$A_q[G](\zeta) \geq C(\delta^{-n/p+n/q})\left(\int_{|1-\zeta_1|}^{1} |f(t)|^q \frac{dt}{(1-t)^{n+1}}\right)^{1/q}.$$

If $\sqrt{\delta} < |1 - \zeta_1| < 2\sqrt{\delta}$ then it follows from the definition of f that

$$A_q[G](\zeta) \geq C\delta^{-n/p+n/q-n/2q}.$$

Thus,

$$\int_{S^n} A_q[G]^p \, d\sigma \geq C\delta^{-n+np/q-np/2q+n/2}.$$

Since this last expression is arbitrarily large for δ small, it follows that $H_+^p(S^n, X_q)$ is not contained in T_q^p. $\square$

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