Spaces and Arcs of Bounded Turning

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1. Introduction

A closed subset X of R^n is of *bounded turning* if there is a fixed number $c \geq 1$ such that any two points a and b of X can be joined by a connected subset A of X with the diameter $d(A)$ of A satisfying

$$d(A) \leq c|a-b|. \tag{1}$$

We will abbreviate bounded turning as BT, and A is c-BT if (1) is true with this particular c. The aim of this paper is to prove the following theorem.

THEOREM 1A. *There is $c_0 = c_0(c, n)$ such that any two points in a c-BT set X of R^n can be joined by a c_0-BT arc J of X.*

Originally, the notion of bounded turning was introduced for arcs or topological circles in R^2. The BT condition characterizes such arcs or circles that are images of the standard interval $[0, 1]$ or of the unit circle S^1 under a quasiconformal homeomorphism of the plane; see [A] and [R].

Hence this notion has an honorable standing in the theory of quasiconformal mappings of the plane. In higher dimensions, the BT property is a necessary condition for an arc or a circle to be the image of the standard interval or circle under a quasiconformal map of R^n, but the condition is no longer sufficient. For instance, the Fox–Artin wild arc in R^3 can be made BT. ([Ma] discusses this in the situation where the Fox–Artin arc is fattened so as to obtain a wild sphere; cf. also [T, Sec. 14 and Sec. 17].) On the other hand, we might more modestly want to map only an interval of the real axis onto an arc of R^n using a map that would have the same properties as the restriction of a quasiconformal map of R^n to the interval. Such maps are called *quasisymmetric maps* (see [TV]). Now the BT property characterizes when such an arc or a circle is a quasisymmetric image of a standard interval or circle [TV, Thm. 4.9].

The question of whether any two points of BT space X of R^n can be joined by a BT arc of X was raised by J. Väisälä. I am indebted to him for this very interesting question—whose answer turned out to be much more complicated than anticipated—and for some critical comments on my earlier attempts to prove the

Received January 25, 1996. Revision received June 4, 1996.
Michigan Math. J. 43 (1996).

present result. I also gratefully acknowledge the hospitality of the MSRI during Spring 1995 when I could concentrate on this problem.

We remark that Theorem 1A has been proved [GNV, Thm. 5.12] for such BT sets of the plane whose complement is connected. We also remark that the dependence of c_0 on n is essential. Some examples due to T. Huuskonen (unpublished) indicate that there are c such that $c_0(c, n) \to \infty$ as $n \to \infty$.

It is not too difficult to see that a BT space is arcwise connected. This is the starting point. Another basic observation is that the family of all c-BT sets has a natural compactness property (expressed precisely in Lemma 2B). The c-BT property is unchanged under similarity transformations; that is, we can rescale and change the origin and still move in a compact family of c-BT sets. This suggests that the connection of two points in a BT set can be made by an arc having the same properties so that we move in a compact family of arcs if we rescale and move the origin. However, such an arc must be BT. We will give a more detailed sketch of the proof in the beginning of Section 3.

We will start from a connected and closed set A joining a and b and find a BT-arc in any ε-neighborhood $U_\varepsilon(A) = \{ z : d(z, A) < r \}$ of A. Actually, we will prove at the same time the following more general version of Theorem 1A.

THEOREM 1B. *Let A be a connected subset of a c-BT set X of R^n such that $a, b \in A$. Given $\varepsilon > 0$, there is a c_1-BT arc J in $U_\varepsilon(A) \cap X$ with endpoints a and b and with c_1 depending only on c, n, and $d(A)/\varepsilon$. In addition, there are $r' = r'(c, n) > 0$ and $c_0 = c_0(c, n) > 0$ such that J_{xy} is c_0-BT for the subarc J_{xy} of J with endpoints x and y if $x, y \in J$ and $|x - y| \le r'\varepsilon$.*

As indicated above, the compactness of certain families of sets is crucial. We denote by $\mathcal{C}(X)$ the family of closed and nonempty subsets of a metric space X, and define the Hausdorff distance of two sets $A, B \in \mathcal{C}(X)$ as

$$\varrho(A, B) = \inf\{ r \in \,]0, \infty] : A \subset U_r(B) \text{ and } B \subset U_r(A) \}.$$

Here it is possible that $r = \infty$ and thus the Hausdorff distance may be infinite. However, if the sets A and B are compact then $\varrho(A, B)$ is finite. It is well known (cf. [Mi, 4.7.2]) that if X is compact then the Hausdorff distance makes $\mathcal{C}(X)$ into a compact metric space.

Of course, R^n is not compact. Therefore we often work in $\mathcal{C}(A)$ where $A \subset R^n$ is compact, or in $\mathcal{C}(\bar{R}^n)$ with $\bar{R}^n = R^n \cup \{\infty\}$. In the latter space we use the chordal metric of $\bar{R}^n$ obtained by means of stereographic projection. It is convenient to regard $\mathcal{C}(R^n)$ as a subset of $\mathcal{C}(\bar{R}^n)$ by means of the identification of $A \in \mathcal{C}(R^n)$ and the closure of A in $\bar{R}^n$. Thus we topologize $\mathcal{C}(R^n)$ by means of the chordal metric rather than the Euclidean metric, although we will usually use the Euclidean metric in R^n. Naturally, if $A \subset R^n$ is already compact then it does not matter whether we use the Euclidean or the chordal metric on A to define the topology of $\mathcal{C}(A)$.

Since we identify $\mathcal{C}(R^n)$ with a subset of $\mathcal{C}(\bar{R}^n)$, it is convenient to define $X \in \mathcal{C}(\bar{R}^n)$ as c-BT if $X \cap R^n$ is c-BT with respect to the Euclidean metric. All similarities of R^n are extended to $\bar{R}^n$ by the rule $\infty \mapsto \infty$.

DEFINITIONS AND NOTATION. An *arc* is subset of R^n homeomorphic either to a closed interval or to a point. Thus all arcs are closed unless otherwise stated. If an arc J is a point, then an endpoint of J is simply this point. Allowing points to be arcs will be useful in some limit processes. An *arc family* is a finite union of disjoint closed arcs.

If J is an arc and $x, y \in J$, then $J_{xy} = J[x, y] =$ the subarc of J with endpoints x and y;
$R_+ =$ the set of positive real numbers;
$\bar{R}^n = R^n \cup \{\infty\}$;
$B^n(x, r) = \{ z \in R^n : |z - x| < r \}$;
$B^n(r) = B^n(0, r)$;
$B^n = B^n(1)$;
$d(A) =$ the Euclidean diameter of A;
$d(x, X) =$ the distance of a point x from the set X;
$d(X, Y) =$ the distance between the sets X and Y;
$U_\varepsilon(Z) = \{ z \in R^n : d(z, Z) < \varepsilon \}$;
$\mathcal{C}(X) =$ the family of all closed and nonempty subsets of X;
$\mathcal{D}(K) =$ the set of components of K;
$\operatorname{int} A =$ the interior of A;
$\partial A =$ the boundary of A; and
$\bar{A}$ or $\operatorname{cl} A$ is the closure of A.

2. Auxiliary Results

We present here some general results concerning connectedness, limits in the Hausdorff metric, and arcs.

Connected Sets

The following lemma is obvious.

LEMMA 2A. *Let $X_i \in \mathcal{C}(\bar{R}^n)$ and suppose that $X_i \to X \in \mathcal{C}(\bar{R}^n)$. If the X_i are connected then so is X.*

Lemma 2A has the following consequence, showing that the c-BT property is preserved under limits.

LEMMA 2B. *Let $X_i \in \mathcal{C}(\bar{R}^n)$ be c-*BT *and suppose that $X_i \to X \in \mathcal{C}(\bar{R}^n)$. Then X is c-*BT.

Thus the family of c-BT sets of $\bar{R}^n$ is compact as a closed subset of the compact space $\mathcal{C}(\bar{R}^n)$. Lemma 2B expresses the compactness property of c-BT sets.

Arcs

We aim to prove that points of a BT set can be joined by a BT arc. The proof will make use of some preliminary results concerning arcwise connectivity of BT spaces. For instance, these spaces are arcwise connected as follows by the fact that they are locally connected and hence arcwise connected by a classical topological theorem. This is easy to prove directly, and the next lemma gives a more precise estimate for the diameter of the arc joining two given points.

LEMMA 2C. *Let $X \subset R^n$ be c-BT. Given any $c' > c$, any two points $a, b \in X$ can be joined by an arc J of X such that $d(J) \leq c'|a - b|$.*

Proof. Let $s = (c' - c)|a - b|$. Find a sequence $x_0 = a, \dots, x_p = b$ of points of X such that $|x_i - x_{i-1}| \leq s/2c$ and such that the diameter of $\{x_0, \dots, x_p\} \leq c|a - b|$. This is possible because a and b are contained in a connected set $C \subset X$ with $d(C) \leq c|a-b|$. Next define a map α of $\{0, 1/p, \dots, (p-1)/p, 1\}$ onto $\{x_0, \dots, x_p\}$ by $\alpha(t_i) = x_i$ when $t_i = i/p$. Similarly, connect x_i and x_{i+1} by a sequence $x_{i0} = x_i, \dots, x_{ik_i} = x_{i+1}$ such that the distance of successive elements of the sequence $\leq s/4c$ and that the diameter for each i of $\{x_{ij}\colon j \text{ varies}\} \leq s/2$. Divide $[t_i, t_{i+1}]$ equidistantly by t_{ij} and set $\alpha(t_{ij}) = x_{ij}$. Continuing in this manner, we obtain a map α into X defined in a dense subset of $[0, 1]$; the diameter of $\operatorname{Im}\alpha \leq c'|a - b|$. One easily sees that α is uniformly continuous and hence can be extended to a continuous map $[0, 1] \to X$ with $d(\operatorname{Im}\alpha) \leq c'|a - b|$. It is possible to extract an arc from $\operatorname{Im}\alpha$ and redefine α so that it is an embedding; see [V] for the details of this folklore theorem. □

This result will have the following consequences.

LEMMA 2D. *Let $X \subset R^n$ be c-BT and let $E \subset X$ be a set such that, for some $\varepsilon > 0$, any two points $a, b \in E$ can be joined by a sequence $x_0 = a, x_1, \dots, x_p = b$ of E such that $|x_i - x_{i-1}| \leq \varepsilon$. Then, for every $c' > c$, there is an arc J of X connecting a and b in $U_{c'\varepsilon}(E)$.*

Proof. By the preceding lemma, we can connect x_{i-1} and x_i by an arc J_i in $B^n_{c'\varepsilon}(x_i, c'\varepsilon) \cap X$. Extract an arc J from $J_1 \cup \cdots \cup J_p$ connecting $x_0 = a$ and $x_p = b$. Then $J \subset U_{c'\varepsilon}(E)$. □

LEMMA 2E. *Let I be an arc of R^n with endpoints a and b. Let $X \subset R^n$ be c-BT and suppose that $I \subset U_\varepsilon(X)$. Let $c, d \in X$ be points such that $|a - c| < \varepsilon$ and $|b - d| < \varepsilon$. Then there is an arc J of X with endpoints c and d such that $J \subset U_{4c\varepsilon}(I)$. There is also a map $\alpha\colon J \to I$ such that $\alpha(c) = a$, $\alpha(d) = b$, and, for all $x, y \in J$, $|\alpha(x) - x| \leq 4c\varepsilon$, and*

$$J_{xy} \subset U_{4c\varepsilon}(I_{\alpha(x)\alpha(y)}).$$

Proof. It follows from the assumptions that we can find a sequence $c = x_1, \dots, x_p = d$ of points of X and another sequence $z_0 = a, z_1, \dots, z_p = b$ of points in sequential order on I such that $I[z_{i-1}, z_i] \subset U_\varepsilon(x_i)$. Thus $|x_{i-1} - x_i| \leq |x_{i-1} - z_{i-1}| + |z_{i-1} - x_i| \leq 2\varepsilon$ and hence we can by Lemma 2C find an arc $J_i' \subset X$ with $d(J_i') < 3c\varepsilon$ connecting x_{i-1} and x_i. Then $J' = J_1' \cup \cdots \cup J_p'$ connects c and d.

It is possible to extract a subarc J' from J connecting c and d. This can be done in such a way that $J = J_1 \cup \cdots \cup J_p$, where J_i is either empty or a subarc of J_i'.

In addition, we can assume that $\mathcal{J} = \{ J_i : J_i \neq \emptyset \}$ is a subdivision of J and that the order of $J_i \neq \emptyset$ on J is compatible with the order of J whose smallest point is c. Let $J_i^\circ = J_i \setminus \{a_i\}$, where a_i is the beginning point of J_i induced from the order of J mentioned above. In addition we set $J_0^\circ = \{c\}$. Define the map $\alpha\colon J \to I$ so that $\alpha(x) = z_i$ if $x \in J_i^\circ$. Then $|\alpha(x) - x| \le |x_i - z_i| + d(J_i') \le (1+3c)\varepsilon < 4c\varepsilon$ if $x \in J_i^\circ$ and so $J_i^\circ \subset U_{4c\varepsilon}(z_i)$. Thus, if $x, y \in J$, and if $x \in J_i^\circ$ and $y \in J_k^\circ$ with $i \le k$, then

$$J_{xy} \subset J_i^\circ \cup \cdots \cup J_k^\circ \subset \bigcup_{j=i}^{k} U_{4c\varepsilon}(z_j) \subset U_{4c\varepsilon}(I_{\alpha(x)\alpha(y)}),$$

since $\alpha(x) = z_i$ and $\alpha(y) = z_k$. This also implies that $J \subset U_{4c\varepsilon}(I)$. □

ϱ-Arcs

If ϱ is a homeomorphism of $[0, \infty[$, we call an arc J a *ϱ-arc* if it is true that

$$d(J_{ab}) \le \varrho(|a-b|) \tag{2a}$$

for all $a, b \in J$. Thus a ϱ-arc satisfies a condition that is similar to but not quite as strong as the BT-property. Such arcs have a compactness property given by the next theorem, where the place of ϱ is taken by two decreasing sequences ε_i and δ_i of positive numbers.

THEOREM 2F. *Let ε_i and δ_i be sequences of positive numbers tending to 0 as $i \to \infty$. Suppose that for each i there is a k_i such that*

$$d(J_i[x, y]) \le \varepsilon_k \quad \textit{if} \quad |x - y| \le \delta_k \tag{2b}$$

when $x, y \in J_i$ for all $k \le k_i$. Suppose that $k_i \to \infty$ and that the J_i lie in a compact subset F of R^n. Then there is a subsequence (denoted in the same manner) such that the J_i tend toward a ϱ-arc J in the Hausdorff metric (this includes the case where J is a point) and where ϱ depends only on $d(F)$ and the numbers ε_i and δ_i.

In addition, if the J_i tend toward the arc J and if $x_i, y_i \in J_i$ tend toward $x, y \in J$ as $i \to \infty$, respectively, then

$$J_i[x_i, y_i] \to J_{xy} \tag{2c}$$

in the Hausdorff metric.

Proof. Since $J_i \in \mathcal{C}(F)$ which is compact, it is possible to pass to a subsequence so that the J_i tend toward some compact $J \subset F$ in the Hausdorff metric. Then J is connected by Lemma 2A, and if J is a point then there is nothing to prove. We assume now that X is not a point, and show that in this case J is a nondegenerate arc.

We will first prove that J is an arc, basing our proof on the fact that a compact connected metrizable space X is a closed arc if and only if the removal of any $x \in X$, except for two points that will be the endpoints, makes X nonconnected with two components (cf. [N, 4.10.2]).

Let a_i and b_i be the endpoints of J_i. Pass to a subsequence so that $a_i \to a$ and $b_i \to b$. If $a = b$, then $|a_i - b_i| \le \delta_k$ for big i and hence $d(J_i) \le \varepsilon_k$ for big i; it would follow that J is a point. Since J is not a point, $a \ne b$.

We show that if $c \in J \setminus \{a, b\}$ then $J \setminus \{c\}$ consists of two components. To prove this, pick $c_i \in J_i$ so that $c_i \to c$. We can assume that the numbers $|c_i - a_i|$ and $|c_i - b_i|$ are bigger that some 2^{-n_0}. Pick then for $n \ge n_0$ points $c'_{ni} \in J_i[a_i, c_i]$ and $c''_{ni} \in J_i[b_i, c_i]$ such that

$$|c'_{ni} - c_i| = |c''_{ni} - c_i| = 2^{-n}.$$

Now, let $J'_{ni} = J_i[a_i, c'_{ni}]$ and $J''_{ni} = J_i[c''_{ni}, b_i]$. Then, as follows by (2b),

$$d(J'_{ni}, J''_{mi}) > \delta_k$$

as soon as $\varepsilon_k < \max(2^{-n}, 2^{-m})$ and i is big enough. It is possible to pass to a subsequence so that, for every n, J'_{ni} and J''_{ni} tend toward connected subsets J'_n and J''_n of J as $i \to \infty$, respectively, in the Hausdorff metric. Thus the above inequality implies that

$$d(J'_n, J''_m) \ge \delta_k \tag{2d}$$

as soon as $\varepsilon_k < \max(2^{-n}, 2^{-m})$. Setting

$$J' = \bigcup_n J'_n \quad \text{and} \quad J'' = \bigcup_n J''_n,$$

it follows that J' and J'' are connected and disjoint subsets of J such that $J' \cup J'' = J \setminus \{c\}$. To see that $J' \cup J'' = J \setminus \{c\}$, we need only note that every $x \in J \setminus \{c\}$ is the limit of a sequence x_i with $x_i \in J_i$. Thus, if $x \ne c$, there is m_0 such that $x_i \in J'_{ni} \cup J''_{ni}$ if $n > m_0$ beginning from some i. Thus $x \in J'_n$ or $x \in J''_n$ if $n > m_0$. By (2d), J' and J'' are closed in $J \setminus \{c\}$. Hence J' and J'' are the components of $J \setminus \{c\}$.

On the other hand, if $c = a$ or $c = b$, then a similar limit process would show that $J \setminus \{c\}$ has exactly one component. Hence a and b are the endpoints of J.

Suppose then that $J_i \to J$, and that $x_i, y_i \in J_i$ are such that $x_i \to x$ and $y_i \to y$. We show that $K_i = J_i[x_i, y_i] \to J[x, y]$ for a subsequence. As before, one sees that there is a subsequence (denoted in the same manner) such that the K_i tend toward an arc K with endpoints x and y. Clearly, $K \subset J$ and the only possibility is that $K = J[x, y]$. Thus any subsequence n_i contains a sub-subsequence m_i such that $K_{m_i} \to K$. It follows that $K_i \to K$ even without passing to a subsequence, and (2c) follows.

Next, we show that J satisfies

$$d(J_{xy}) \le \varepsilon_j \tag{2e}$$

whenever $x, y \in J$ and $|x - y| \le \delta_j/2$. To see this, we pick $x_i \in J_i$ and $y_i \in J_i$ such that $x_i \to x$ and $y_i \to y$. If j is fixed, $|x_i - y_i| \le \delta_j$ for big i and hence $d(K_i) \le \varepsilon_j$ for big i. It follows that J satisfies (2e).

Clearly, there is a homeomorphism ϱ of $[0, \infty[$ such that any arc satisfying (2e) for all j is a ϱ-arc. Thus J is a ϱ-arc and, moreover, we see that ϱ depends only on $d(F)$ and on the numbers ε_i and δ_i. $\square$

We need to know that, if J_i and I are ϱ-arcs such that $J_i \to I$, then we can approximate subarcs $J_i[x, y]$ of J_i by arcs $I[x', y']$ where x' and y' are close to x and y, respectively. We will need this in the following form.

THEOREM 2G. *Let I be a ϱ-arc. Given $\varepsilon > 0$, there is an $r > 0$ such that, if J is another ϱ-arc with Hausdorff distance $\varrho(I, J) \leq r$, then there is a map $\alpha\colon J \to I$ with the properties that*

(a) $|\alpha(x) - x| \leq \varepsilon$ *and*
(b) *if $x, y \in J$ then $J_{xy} \subset U_\varepsilon(I_{\alpha(x)\alpha(y)})$.*

In addition, α maps the endpoints of J onto the endpoints of I.

Proof. We first prove that if $\delta > 0$ and $\varepsilon > \varrho(\delta)$ then there is an $r = r(\delta, \varepsilon, \varrho, I) > 0$ such that, if J is a ϱ-arc and $\varrho(I, J) < r$, then

$$J[x', y'] \subset U_\varepsilon(I[x, y]) \tag{2f}$$

whenever $x, y \in I$, $x', y' \in J$, $|x - x'| \leq \delta$, and $|y - y'| \leq \delta$.

Unless this claim is true, there is a sequence J_i of ϱ-arcs tending toward I in the Hausdorff metric as well as $x_i, y_i \in J_i$ and $x_i', y_i' \in I$ with $|x_i - x_i'| \leq \delta$ and $|y_i - y_i'| \leq \delta$ such that (2f) is not true for $x = x_i$, $y = y_i$, $x' = x_i'$, and $y' = y_i'$.

We pass to a subsequence so that $x_i \to x$, $y_i \to y$, $x_i' \to x'$, and $y_i' \to y'$. Since $J_i \to I$, all the points $x, y, x', y' \in I$. By (2c), $J_i[x_i, y_i] \to I[x, y]$. Since $|x - x'| \leq \delta$, $d(I[x, x']) \leq \varrho(\delta)$ and similarly $d(I[y, y']) \leq \varrho(\delta)$. Thus $I[x, y] \subset U_\varepsilon(I[x', y'])$ and it would follow that (2f) is true for big i, contrary to the assumption.

Thus, given $\varepsilon > 0$, we can choose $\delta \in]0, \varepsilon[$ such that $\varepsilon > \varrho(\delta)$ and let $r > 0$ be the number found above. We can assume that $r < \varepsilon$. If J is a ϱ-arc such that $\varrho(I, J) < r$, then we can choose $\alpha\colon J \to I$ to be any map such that $|\alpha(x) - x| \leq \varepsilon$.

Finally, using (2c), we can use a similar compactness argument to show that r can be made so small that an endpoint of J is at a distance $\leq \varepsilon$ from an endpoint of I and the other endpoint of J is also at a distance $\leq \varepsilon$ from the other endpoint of I. Thus α can be chosen to map the endpoints of J onto endpoints of I. □

Arc Families

We will often need to consider the situation that $K \subset R^n$ is the union of disjoint closed arcs such that if $a, b \in K$ and $|a - b| < \delta$, then a and b are in the same component J of K and (2a) is true. Such a union of arcs K is called a *(ϱ, δ)-family.* Note that component arcs of K need not be ϱ-arcs, since now (2a) is required only if $|x - y| < \delta$. However, each component arc will be a ϱ'-arc for ϱ' depending only on ϱ, δ and $d(K)$.

The pertinent aspect of bounded (ϱ, δ)-families is that they are compact.

COROLLARY 2H. *Let ϱ be a homeomorphism of $[0, \infty[$, and let $\delta > 0$. Let $\mathcal{K} \subset \mathcal{C}(F)$ where $F \subset R^n$ is compact and suppose that each $K \in \mathcal{K}$ is a (ϱ, δ)-family. Then the closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ is compact, each $K \in \overline{\mathcal{K}}$ is a (ϱ, δ)-family, and there is a number $N = N(\delta, F)$ such that each $K \in \mathcal{K}$ contains at most N component arcs.*

Proof. The family $\overline{\mathcal{K}}$ is clearly compact as a closed subset of a compact space $\mathcal{C}(F)$. Note that the condition that $d(I, J) \geq \delta$ implies that there are at most $N = N(\delta, F)$ components for each $K \in \mathcal{K}$ and hence, since limits of connected sets are connected, also that each $K \in \overline{\mathcal{K}}$ has at most N components. Otherwise the corollary is a consequence of Theorem 2F and especially of (2c). $\square$

Division of Arc Families

We will later divide (ϱ, δ)-families into two parts depending on four open sets. Let $U_1 \subset U_2 \subset U_3 \subset U_4$ be open sets of R^n such that

$$d(\partial U_i, \partial U_{i+1}) \geq d > 0. \tag{2g}$$

We call J an *arc family* if J is the union of disjoint closed arcs, and will consider divisions of J into two arc families J' and J'' such that $J = J' \cup J''$ and that

$$J \cap \bar{U}_3 \subset J' \subset \bar{U}_4, \qquad J \setminus U_2 \subset J'' \subset J \setminus U_1. \tag{2h}$$

Any pair (J', J'') satisfying these conditions is a *(U_i)-division* of J.

There is a canonical way to obtain a (U_i)-division (J', J'') of J. We call it the *canonical (U_i)-division* of J and define it as follows. The arc family J' will be the maximal arc family such that $J' \subset J \cap \bar{U}_4$ and such that the endpoints of the component arcs of J' are in $\bar{U}_3$. That is, we take all component arcs L of $J \cap \bar{U}_4$ touching $\bar{U}_3$; if an endpoint b of L is not in $\bar{U}_3$ then we remove from L the minimal half-open arc $L_{cb} \setminus \{c\}$ so that $c \in \bar{U}_3$. Similarly, J'' will be the maximal arc family $J'' \subset J \setminus U_1$ such that the endpoints of component arcs of J'' are in $J \setminus U_2$.

We will need the fact that (roughly) J is a (ϱ, δ)-family if and only if J' and J'' are.

THEOREM 2I. *Let ϱ be a homeomorphism of $[0, \infty)$ and let $\delta > 0$ be a number such that $\delta \leq \min(\varrho^{-1}(d), d/2)$, with d as in (2g). If J is a (ϱ, δ)-family and (J', J'') is the canonical (U_i)-division of J, then both J' and J'' are (ϱ, δ)-families. Conversely, if (J, J'') is any (U_i)-division of J and both J' and J'' are (ϱ, δ)-families, then J is also a (ϱ, δ)-family.*

Proof. Assume that J is a (ϱ, δ)-family. Let (J', J'') be the canonical (U_i)-division of J. We will show that both J' and J'' are (ϱ, δ)-families.

We will prove this for J', the other case being similar. We need only prove that if $x, y \in J'$ and $|x - y| < \delta$ then x and y are in the same component of J'. The decisive property is that if $x, y \in J'$ are in a component L of J and if $L_{xy} \not\subset J'$, then L_{xy} contains a point u outside U_4 and a point $v \in \bar{U}_3$ as follows from the definition of the canonical (U_i)-division.

If $x, y \in J'$ and $|x - y| < \delta$, then x and y are in any case in a component L of J. If $x \in \bar{U}_3$, then $d(L_{xy}) \leq \varrho(|x - y|) < \varrho(\delta) \leq \varrho(\varrho^{-1}(d)) = d \leq d(\partial U_3, \partial U_4)$. Hence $L_{xy} \subset \bar{U}_4$ and, since $y \in J'$, $L_{xy} \subset J'$; similarly if $y \in \bar{U}_3$. If $x, y \in \bar{U}_4 \setminus U_3$ and $L_{xy} \not\subset J'$, then L_{xy} contains a point u outside $\bar{U}_4$ and a point $v \in \bar{U}_3$. Hence

$$|u - v| \le d(L_{xy}) \le \varrho(|x - y|) < \varrho(\delta) \le d,$$

and this is impossible since $|u - v| > d$.

In the other direction, assume that (J', J'') is a (U_i)-division of J such that J' and J'' are (ϱ, δ)-families. Suppose that $x, y \in J$ and $|x - y| < \delta$. Now either $x \in U_3$ and $d(x, \partial U_3) \ge d/2$ or $x \notin U_2$ and $d(x, \partial U_2) \ge d/2$. Suppose that we have the first case. Then $x \in \bar{U}_3$ and, since $|x - y| < \delta \le d/2 \le d(x, \partial U_3)$, also $y \in \bar{U}_3$. Thus $x, y \in J'$ by (2h), and so x and y are in the same component of J' and hence in the same component of J. The other case is similar. This is all we need to prove. □

3. The Main Theorem

We now return to our main problem. We assume that X is c-BT and will prove that two points $a, b \in X$ can be joined by a c_0-BT arc $J \subset X$ with $c_0 = c_0(c, n)$. The idea is as follows. As explained earlier, it is beneficial to regard X as a closed subset of $\bar{R}^n$ since we then gain important compactness properties, even if the BT-property is formulated in R^n.

Pick $a, b \in X \cap R^n$. We want to join a and b by a c_0-BT arc in $X \cap R^n$. For simplicity assume that $a = 0$ and $b = e_1 = (1, 0, \dots, 0)$ (this can be obtained by rescaling and moving the origin). Thus a and b can be joined by a connected set $A \subset X$ in $\bar{B}^n(c)$. We can and will assume that A is an arc with endpoints a and b (Lemma 2C), but then we must increase the ball $\bar{B}^n(c)$ so that, say, $A \subset B^n(c + 1)$.

The basic work is to prove that, given $\varepsilon > 0$ and an arc A in $X \cap R^n$ joining a and b, there exists an arc I in $U_\varepsilon(A) \cap X$ joining a and b such that, for some $r > 0$,

$$d(I_{xy}) \le M\varepsilon \tag{3a}$$

whenever $x, y \in I$ and $|x - y| \le r\varepsilon$. Here it is essential that r and M depend only on c and n; the existence of such r and M is based on the compactness of c-BT sets in the Hausdorff metric as well as on the fact that the c-BT property is preserved under rescaling. This enables us to construct I uniformly on scale of ε, implying (3a).

This is the basis of our approach. It allows us to find a number δ $(0 < \delta < 1)$ with which we can construct a sequence $(I_i)_{i>0}$ of arcs joining a and b such that $I_i \subset X \cap U_{\delta^i}(I_{i-1})$ (and $I_0 \subset U_1(A) \cap X \subset B^n(c + 2)$) and such that

$$d(J_i[x, y]) \subset M\delta^i \tag{3b}$$

if $x, y \in I_i$ and $|x - y| \le r\delta^i$ for suitable r and M. We can do this so that the arcs I_i converge toward an arc J of $X \cap R^n$ in such a way that we can use I_i at scale of δ^i to approximate subarcs of J and to obtain from (3b) the estimate that

$$d(J[x, y]) \le M'\delta^i$$

if $x, y \in J$ and $|x - y| \le r'\delta^i$, where M' and r' depend only on c and n. This implies the BT property for J.

To find the arc I in (3a), we let $\mathcal{Q}$ be a tiling of R^n by n-cubes of sidelength of the order of ε. We fix $Q \in \mathcal{Q}$ and change A around Q so that $\operatorname{int} Q \cap A$ will be contained in a finite union I_Q of arcs. If we normalize the situation by means of a similarity map h_Q so that Q becomes the standard cube $Q_0 = [0, 1]^n$, then $h_Q I_Q$ will satisfy a compactness property.

We represent $\mathcal{Q}$ as $\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_N$ ($N = 2^n$), where cubes of $\mathcal{Q}_i$ are disjoint. We can change A around each $Q \in \mathcal{Q}_1$ so that $A \cap \operatorname{int} Q$ is contained in an arc family I_Q as indicated above. We take one cube at a time, but since cubes of $\mathcal{Q}_1$ are disjoint, we can make the changes so that we obtain the same compactness condition for the normalized cube $h_Q Q$ for all $Q \in \mathcal{Q}_1$. At the next step, we change A around $\operatorname{int} Q \cap A$ for $Q \in \mathcal{Q}_2$, and now the normalized $\operatorname{int} Q \cap A$ will be contained in an arc family I_Q satisfying a compactness condition that is not as strict as that of the first stage. We continue and so, at stage N, A will be an arc I satisfying at scale ε a compactness condition; (3a) follows.

We will now carry out this program in detail, and begin by fixing notation as follows. We will use a tiling $\mathcal{Q}$ of R^n by cubes of the same side length with faces parallel to the coordinate axes. Thus $Q \cap P$ is either empty or a common k-subface for distinct $Q, P \in \mathcal{Q}$. We denote by Q° the standard cube $[0, 1]^n$ and by h_Q the similarity of the form $z \mapsto az + b$ ($a > 0$, $b \in R^n$) mapping Q onto Q°. The map h_Q will be used to transform the situation from Q to Q° as just indicated.

If $Q \in \mathcal{Q}$, it may be that A is already changed in some cubes $P \in \mathcal{Q}$ touching Q; we let $\mathcal{P}_Q$ be the set of $P \in \mathcal{Q}$ touching Q such that this is the case.

We denote by tQ the cube with the same center as Q whose side length is $t \times$ (sidelength of Q) and whose faces are parallel to the coordinate planes. We assume that $1 < t < \frac{11}{10}$ is given. Since $t < \frac{3}{2}$, $tQ \cap tP = \emptyset$ if $P, Q \in \mathcal{Q}$ and $P \cap Q = \emptyset$. We adopt the following notation:

$$P_Q = \bigcup \mathcal{P}_Q, \qquad tP_Q = \bigcup \{ tQ : Q \in \mathcal{P}_Q \}.$$

We assume that A is already changed in P_Q so that $A \cap \operatorname{int} P_Q$ is contained in a union of disjoint arcs $J_Q \subset A$; hence $\operatorname{int} P_Q \cap A \subset J_Q \cap A$. In addition, we must assume some kind of compactness condition for the changed part. This is expressed by the requirement that moving to the normalized situation makes $h_Q J_Q$ into a (ϱ, δ)-family as defined in Section 2.

At this stage we change A, or more precisely $h_Q A$, and then transform back to A by means of h_Q^{-1}. The transformation is effected by means of Lemma 3A (below), which makes use of the following terminology.

A set E is *ε-chainable* if any two points $x, y \in E$ can be joined by a sequence $x_0 = x, x_1, \ldots, x_k = y$ in E such that $|x_i - x_{i-1}| \leq \varepsilon$. We say that points a and b are *joinable* by a sequence $F_1, \ldots, F_k$ if $a \in F_1$, $b \in F_k$, and $F_i \cap F_{i+1} \neq \emptyset$; points a and b are *joinable* in a set family $\mathcal{F}$ if they are joinable by a sequence $F_1, \ldots, F_k$ where each $F_i \in \mathcal{F}$. If $\mathcal{F}$ is the union of several families, say $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_p$, we say that a and b can be *joined* by an *alternating sequence* of $(\mathcal{F}_1, \ldots, \mathcal{F}_p)$ if a and b can be joined by a sequence $F_1, \ldots, F_k$ in $\mathcal{F}$ where elements of $\mathcal{F}_i$ alternate; that is, if $F_i \in \mathcal{F}_j$ then $F_{i+1} \in \mathcal{F}_m$ where $m \neq j$. Thus the index set $\{1, \ldots, k\}$ can be represented as a disjoint union $\alpha_1 \cup \cdots \cup \alpha_p$ such that $F_i \in \mathcal{F}_j$ if $i \in \alpha_j$ and no two successive indexes i and $j = i + 1$ are in the same set α_m. In this case we say that $(\alpha_1, \dots, \alpha_p)$ are *alternating indexes* for the sequence F_i.

As indicated above, we will normalize the situation by means of h_Q, mapping Q onto Q°. Thus we will work near Q and near Q° alternatively. In cases where there might be confusion, we use the superscript $^\circ$ to refer to the normalized situation (e.g., X° will be $h_Q X$) or may use the subscript Q when working in the original, nonnormalized situation.

We denote by $\mathcal{Q}^\circ$ the tiling of R^n by means of cubes of side length 1 such that $Q^\circ \in \mathcal{Q}^\circ$. Recall that A was already changed in $\mathcal{P}_Q$. Similarly, we have:

$\mathcal{P}^\circ =$ a subset of $\{ Q \in \mathcal{Q}^\circ : Q \cap Q^\circ \neq \emptyset, Q \neq Q^\circ \}$;
$P^\circ = \bigcup \mathcal{P}$; and
$tP^\circ = \bigcup \{ tP : P \in \mathcal{P}^\circ \}$.

In Lemma 3A we have fixed such a set $\mathcal{P}^\circ$. Note that there is only a finite number of choices for $\mathcal{P}^\circ$, and recall that

$\mathcal{D}(Z) =$ the family of components of a set Z.

The operation of changing $A \cap Q$ into arcs is performed by transforming to Q° and applying the next lemma. The set A° of the lemma will correspond to the arc A above. However, we use a limit process to prove the lemma and limits of sequences of arcs in the Hausdorff metric need not be arcs. Therefore A° in Lemma 3A is a general set, not necessarily a finite union of arcs.

LEMMA 3A. *Let $X^\circ \subset \mathcal{C}(\bar{R}^n)$ be c-BT. Let $e > 0$ be such that $\frac{1}{10} > 2ce$, and let $t = 1 + 2ce < \frac{11}{10}$. Let $\mathcal{E}^\circ$ be a finite family of sets $E \in \mathcal{C}(X^\circ \cap Q^\circ)$ such that each $E \in \mathcal{E}^\circ$ is e-chainable and that $d(E, F) \geq e$ if $E, F \in \mathcal{E}^\circ$ are distinct. Let $\mathcal{D}^\circ$ be a family of closed sets of $(X^\circ \cap \frac{14}{10} Q^\circ) \setminus \operatorname{int}(Q^\circ \cup P^\circ)$. Let J° be a (ϱ, δ)-family of $X^\circ \cap \frac{14}{10} Q^\circ \cap tP^\circ$. Suppose that two points $u, v \in \frac{14}{10} Q^\circ$ are joinable by an alternating sequence $A_1^\circ, \dots, A_p^\circ$ of $(\mathcal{D}^\circ, \mathcal{E}^\circ, \mathcal{D}(J^\circ))$, so that $(\alpha_D, \alpha_E, \alpha_J)$ are alternating indexes for A_i° such that $A_i^\circ \in \mathcal{D}^\circ$ if $i \in \alpha_D$, $A_i^\circ \in \mathcal{E}^\circ$ if $i \in \alpha_E$, and $A_i^\circ \in \mathcal{D}(J^\circ)$ if $i \in \alpha_J$. Let $A^\circ = A_1^\circ \cup \dots \cup A_p^\circ$, and suppose that*

$$A^\circ \cap \operatorname{int}(\tfrac{13}{10} Q^\circ \cap P^\circ) \subset J^\circ. \tag{3c}$$

Under these circumstances there is a (ϱ', δ')-family K° of $X^\circ \cap \frac{14}{10} Q^\circ \cap (tQ^\circ \cup tP^\circ)$, where ϱ' and δ' depend only on $(n, c, e, \varrho, \delta)$, such that u and v are joinable by an alternating sequence $B_1^\circ, \dots, B_q^\circ$ of $(\mathcal{D}^\circ, \mathcal{D}(K^\circ))$ with alternating indexes β_D and β_K such that $B_i \in \mathcal{D}^\circ$ if $i \in \beta_D$ and $B_i \in \mathcal{D}(K^\circ)$ if $i \in \beta_K$. In addition, the following are true when $B^\circ = B_1^\circ \cup \dots \cup B_q^\circ$.

(1°) $B^\circ \cap \operatorname{int}(\frac{13}{10} Q^\circ \cap (Q^\circ \cup P^\circ)) \subset K^\circ$.
(2°) $B^\circ \setminus tQ^\circ \subset A^\circ \setminus tQ^\circ$.
(3°) $K^\circ \setminus tQ^\circ \subset J^\circ \setminus tQ^\circ$.
(4°) *If $i \in \beta_K$, then:* (a) *B_i° intersects B_{i+1}° at an endpoint of B_i° if $i < q$ (if $i = q$ then this endpoint is v) and intersects B_{i-1}° at another endpoint if $i > 1$ (if $i = 1$ then this endpoint is u); and* (b) *$B_i^\circ \cap B_j^\circ = \emptyset$ if $|j - i| > 1$.*

(5°) *The order of sets* B_i° *for* $i \in \beta_D$ *is preserved. That is, there are numbers* $\nu_i \in \alpha_D$ *for* $i \in \beta_D$ *such that* $\nu_i < \nu_j$ *if* $i < j$ *and such that* $B_i^\circ = A_{\nu_i}^\circ$. *Furthermore, if* $1 \in \alpha_D$ *then* $\nu_1 = 1$, *and if* $p \in \alpha_D$ *then* $\nu_q = p$.

We postpone the proof until the next section, and finish the proof of the main theorem assuming Lemma 3A.

LEMMA 3B. *Let the situation be as above, so that we have an arc* $A \subset X$ *with endpoints* a *and* b. *Then there are* $0 < r < 1$ *and a constant* M, *depending only on* c *and* n, *with the following property. Given* $\varepsilon > 0$, *there is an arc* I *with endpoints* a *and* b *such that* $I \subset X \cap U_\varepsilon(A)$ *and such that, whenever* $x, y \in I$ *and* $|x - y| \le r\varepsilon$,

$$d(I_{xy}) \le M\varepsilon. \tag{3d}$$

Moreover, we can associate to any $x \in I$ *a point* $\sigma(x) \in A$ *such that* $|x - \sigma(x)| \le \varepsilon$ *and*

$$I_{xy} \subset U_\varepsilon(A_{\sigma(x)\sigma(y)}). \tag{3e}$$

Proof. We apply Lemma 3A with respect to a tiling $\mathcal{Q}$ of R^n by cubes with faces parallel to the coordinate planes and of sidelength

$$s = \frac{\varepsilon}{2\sqrt{n}2^n}.$$

We choose $e = \frac{1}{22}c$ in Lemma 3A and thus $t = 1 + \frac{1}{11} < \frac{11}{10}$.

We can represent $\mathcal{Q}$ as a disjoint union

$$\mathcal{Q} = \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_N,$$

where $N = 2^n$. If $Q \in \mathcal{Q}_i$, denote

$$\begin{aligned} \mathcal{P}_Q &= \{ P \in \mathcal{Q}_k : k < i \text{ and } P \cap Q \neq \emptyset \}, \\ P_Q &= \bigcup \mathcal{P}_Q, \quad \text{and} \\ tP_Q &= \bigcup \{ tP : P \in \mathcal{P}_Q \}, \end{aligned}$$

so that $\mathcal{P}_Q$ and P_Q are empty if $Q \in \mathcal{Q}_1$. The natural way to construct $\mathcal{Q}_i$ yields sets $\mathcal{P}_Q$ such that, if $Q \in \mathcal{Q}_i$ and we normalize by h_Q, then the following sets do not depend on a particular $Q \in \mathcal{Q}_i$:

$$\begin{aligned} \mathcal{P}_i &= h_Q \mathcal{P}_Q = \{ h_Q P : P \in \mathcal{P}_Q \}; \\ P_i &= \bigcup \mathcal{P}_i = h_Q P_Q; \quad \text{and} \\ tP_i &= \bigcup \{ tP : P \in \mathcal{P}_i \} = h_Q(tP_Q). \end{aligned}$$

Note that $P_1 = \emptyset$.

There is a finite set $\mathcal{Q}_A$ of cubes $Q \in \mathcal{Q}$ such that $A \subset \operatorname{int}(\bigcup \mathcal{Q}_A)$. We will order the cubes $Q \in \mathcal{Q}_A$. Otherwise the order is arbitrary, but if $Q < P$ and $Q \in \mathcal{Q}_i$ and $P \in \mathcal{Q}_k$ then $i \le k$. Let $Q_1 < Q_2 < \cdots < Q_S$ be the cubes of $\mathcal{Q}_A$ enumerated according to this order. We set $I_0 = A$, and successively take each cube $Q = Q_k$ and change A in tQ, so that A is changed to I_k after the change in Q_k.

We will prove the following claim. Recall from Section 2 the definition of a (ϱ, δ)-family. We will say that K is an *s-scaled* (ϱ, δ)*-family* if K is a finite union of disjoint arcs such that, if $x, y \in K$ and $|x - y| \leq s\delta$, then x and y are in the same component L of K and

$$d(L_{xy}) \leq s\varrho\left(\frac{|x-y|}{s}\right).$$

CLAIM. *For each $k \geq 0$, there is an arc $I_k \subset X \cap R^n$ with endpoints a and b such that $I_0 = A$ and if $k > 0$ then there is a subdivision of I_k into subarcs*

$$\text{(i)} \qquad I_k = B_1 \cup \cdots \cup B_q$$

with alternating indexes β_D and β_K such that $B_i \subset \frac{14}{10}Q_k$ if $i \in \beta_K$ and $B_i \subset I_{k-1}$ if $i \in \beta_D$ (with sets B_i and index sets β_D and β_K depending also on k, but this is not shown for simplicity). Consequently,

$$\text{(ii)} \qquad I_k \subset I_{k-1} \cup \tfrac{14}{10}Q_k.$$

In addition, the order of the subarcs B_i $(i \in \beta_K)$ of I_{k-1} as i increases in β_K is compatible with a sequential order of I_{k-1}.

Suppose that $Q_k \in \mathcal{Q}_i$. Then there are ϱ_i and δ_i depending only on c, n, and i as well as an s-scaled (ϱ_i, δ_i)-family $J_k \subset I_k$ such that, setting $J_0 = \emptyset$,

$$\text{(iii)} \qquad J_k \subset \bigcup_{j \leq k} \tfrac{14}{10}Q_j,$$

$$\text{(iv)} \qquad I_k \cap \operatorname{int}\left(\bigcup_{j \leq k} Q_j\right) = J_k \cap \operatorname{int}\left(\bigcup_{j \leq k} Q_j\right),$$

$$\text{(v)} \qquad I_k \setminus J_k \subset I_{k-1} \setminus J_{k-1},$$

$$\text{(vi)} \qquad J_k \setminus \tfrac{12}{10}Q = (J_{k-1} \cap I_k) \setminus \tfrac{12}{10}Q.$$

Proof. We will first say how (ϱ_i, δ_i) are defined. The first pair (ϱ_1, δ_1) is given by Lemma 3A when $P^\circ = P_1 = \emptyset$. If ϱ_i and δ_i are given, then $(\varrho_{i+1}, \delta_{i+1})$ is the pair (ϱ', δ') given by Lemma 3A when $(\varrho, \delta) = (\varrho_i, \delta_i)$ and $P^\circ = P_{i+1}$. In addition, we assume that $\varrho_{i+1} \geq \varrho_i$ and that

$$\delta_i \leq \min(\tfrac{1}{20}, \varrho_i^{-1}(\tfrac{1}{10})). \tag{3f}$$

Since it is always possible to decrease δ_is and increase ϱ_is, we can clearly make this assumption. If this is true then Lemma 2I will allow the following inference. Let $Q \in \mathcal{Q}$ and define U_i for $1 \leq i \leq 4$ as

$$U_i = \operatorname{int}(1 + \tfrac{i}{10})Q,$$

so that $U_i \subset U_{i+1}$ and $d(\partial U_i, \partial U_{i+1}) = s/10$. Thus the sets satisfy the conditions of Lemma 2I with $d = s/10$, and we can use Lemma 2I to show that, if (L', L'') is a suitable (U_i)-division of an arc family L, then L' and L'' are s-scaled (ϱ, δ)-families if L is and vice versa.

We now construct I_k. As already indicated, $I_0 = A$ and $J_0 = \emptyset$ and these satisfy vacuously the required conditions. Now let $k > 0$ and assume that I_i and

J_i have been constructed for $i < k$. Denote $Q = Q_k$ and let U_i refer to the sets above. The set I_k is constructed as follows. Outside tQ we do not want to make any changes, but possibly some components of $I_{k-1} \setminus \operatorname{int} tQ$ will drop away.

We now consider J_{k-1}. We represent J_{k-1} as

$$J_{k-1} = J_Q \cup L_Q, \tag{3g}$$

where (J_Q, L_Q) is the canonical (U_i)-division of J as in Section 2. Thus (cf. (2h)) $J_Q \subset \bar{U}_4$, $L_Q \subset J_{k-1} \setminus U_1$, and

$$J_{k-1} \cap \bar{U}_3 \subset J_Q \quad \text{and} \quad J_{k-1} \setminus U_2 \subset L_Q, \tag{3h}$$

implying by (iv) of the inductive assumption that

$$I_{k-1} \cap U_3 \cap \operatorname{int} P_Q \subset J_Q. \tag{3i}$$

Suppose that $Q \in \mathcal{Q}_i$. If $i = 1$, then the cubes $\frac{14}{10} Q_j$ $(1 \le j < k)$ are disjoint from $\frac{14}{10} Q$ and hence (iii) implies that $J_Q = \emptyset$. If $i > 1$, we will show that J_Q is an s-scaled $(\varrho_{i-1}, \delta_{i-1})$-family. We can see this as follows. Let Q_m be the last cube in $\mathcal{Q}_{i-1}$. We find the canonical (U_i)-subdivision $J_m = J'_Q \cup L'_Q$ of J_m as in (3g). By the inductive assumption, J_m is an s-scaled $(\varrho_{i-1}, \delta_{i-1})$-family and hence also J'_Q is an s-scaled $(\varrho_{i-1}, \delta_{i-1})$-family.

We now examine the transformation of J_m to J_{k-1}. Let $\tilde{Q} = \frac{14}{10} Q_{m+1} \cup \cdots \cup \frac{14}{10} Q_{k-1}$. By (ii), $I_{k-1} \setminus \tilde{Q} \subset I_{k-2} \setminus \tilde{Q} \subset \cdots \subset I_m \setminus \tilde{Q}$. Since I_{k-1} and I_m are arcs with the same endpoints, $I_{k-1} \setminus \tilde{Q}$ is obtained from $I_m \setminus \tilde{Q}$ by omitting some components. Applying now (vi) repeatedly, we find

$$J_{k-1} \setminus \tilde{Q} = J_{k-2} \cap I_{k-1} \setminus \tilde{Q} = J_{k-3} \cap I_{k-2} \cap I_{k-1} \setminus \tilde{Q} = \cdots = J_m \cap I_{k-1} \setminus \tilde{Q}.$$

Consequently, $J_{k-1} \setminus \tilde{Q} = J_{k-1} \cap I_{k-1} \setminus \tilde{Q}$ is obtained from $J_m \setminus \tilde{Q}$ by omitting some components. Since all $Q_j \in \mathcal{Q}_i$ $(m < j \le k)$ are disjoint, the set $\frac{14}{10} Q = \bar{U}_4$ is disjoint from $\tilde{Q}$, and it follows that $J_{k-1} \cap \bar{U}_4$ is obtained from $J_m \cap \bar{U}_4$ by omitting some components. When forming the canonical (U_i)-division (K', K'') of an arc family K, the part lying outside $\bar{U}_4$ has no effect on K'. The final conclusion is that J_Q is obtained from J'_Q by omitting some components. Since J'_Q is an s-scaled $(\varrho_{i-1}, \delta_{i-1})$-family, it follows that also J_Q is an s-scaled $(\varrho_{i-1}, \delta_{i-1})$-family.

Let $\mathcal{D}_Q$ be the set of arcs contained in $I_{k-1} \setminus [U_3 \cap \operatorname{int}(Q \cup P_Q)]$. Let $\mathcal{E}_Q$ be the set of equivalence classes of the relation $\sim$ of $Q \cap X$ such that $x \sim y$ if x and y are es-chainable in $Q \cap I_{k-1}$. So, remembering (iv) and (3i), we see that $I_{k-1} = \bigcup(\mathcal{D}_Q \cup \mathcal{E}_Q \cup \mathcal{D}(J_Q))$.

We order I_{k-1} so that a is the first point of I_{k-1}. Let x_1 be the first point of I_{k-1} that is in some $E_1 \in \mathcal{E}_Q$. Let x_2 be the last point of $I_{k-1} \cap E_1$. We continue and let x_3 be the first point of I_{k-1} after x_2 that is in some $E_2 \in \mathcal{E}_Q$. We let x_4 be the last point of $E_2 \cap I_{k-1}$. We continue in this manner and find distinct sets $E_1, \dots, E_r \in \mathcal{E}_Q$ as well as points $x_1, \dots, x_{2r}$ in increasing order on I_{k-1}. Although it may be that $x_{2i-1} = x_{2i}$, still $x_{2i} \ne x_{2i+1}$ and if we set $L_i = I_{k-1}[x_{2i}, x_{2i+1}]$ then L_i is a nondegenerate arc. Set $L_0 = I_{k-1}[a, x_1]$ and $L_r = I_{k-1}[x_{2r}, b]$. Then $L_0, E_1, L_1, \dots, E_r, L_r$ is an alternating sequence of sets of $\mathcal{E}_Q$ and subarcs of I_{k-1} joining a and b. Note that L_0 and L_r may be points, and in this case we remove L_0 and/or L_r. We note that obviously the L_i are on I_{k-1} in sequential order.

By construction, the arcs L_i are disjoint from Q except possibly for endpoints. Hence, by (3i), J_Q and $\bigcup D_Q$ cover each L_i except possibly for an endpoint x such that $x \in \partial Q$. Let x be an endpoint of L_i such that $x \in \partial Q$. If $x \notin \operatorname{int}(Q \cup P_Q)$, then $x \in I_{k-1} \setminus [U_3 \cap \operatorname{int}(Q \cup P_Q)]$ and so $x \in \bigcup \mathcal{D}_Q$. If $x \in \operatorname{int}(Q \cup P_Q)$, then there is a small nondegenerate subarc L' of L_i with endpoint x such that $L' \setminus \{x\} \subset \operatorname{int} P_Q \cap U_3$. But then $L' \setminus \{x\} \subset J_Q$ and hence, since J_Q is closed, $x \in J_Q$. Thus J_Q and $\mathcal{D}_Q$ cover all of every L_i.

Now J_Q is obtained from a canonical (U_i)-division and hence is finite. Thus each $L_i \setminus J_Q$ has a finite number of components. It follows that we can cover each L_i by an alternating union of nondegenerate arcs $\mathcal{D}(J_Q)$ and of arcs $K \in \mathcal{D}_Q$ such that K is the closure of a component of $L_i \setminus J_Q$.

Replacing each L_i by this sequence covering L_i, we see that there is an alternating sequence $A_1, \ldots, A_p$ joining a and b in $(\mathcal{D}_Q, \mathcal{E}_Q, \mathcal{D}(J_Q))$ with alternating indexes $(\alpha_D, \alpha_E, \alpha_J)$ such that $A_i \in \mathcal{D}_Q$ if $i \in \alpha_D$, $A_i \in \mathcal{E}_Q$ if $i \in \alpha_E$, and $A_i \in \mathcal{D}(J_Q)$ if $i \in \alpha_J$. If $i \in \alpha_D$, then A_i is a subarc of I_{k-1} and obviously, since the L_j are in sequential order on I_{k-1}, the subarcs A_i $(i \in \alpha_D)$ are also in sequential order on I_{k-1}. If $i \in \alpha_D$, then A_i minus endpoints is by construction a component of some $L_k \setminus J_Q$ and hence, if $i, j \in \alpha_D$ and $i \neq j$,

$$\begin{gathered} A_i \cap A_j = \emptyset \quad \text{and} \\ A_k \cap \left(\bigcup_{j \in \alpha_D} A_j \right) \subset \{\text{endpoints of } A_k\} \end{gathered} \tag{3j}$$

if $k \in \alpha_J$.

If $I_{k-1} \cap Q = \emptyset$, we set $I_k = I_{k-1}$ and $J_k = J_{k-1}$ as well as $B_1 = I_k$, $\beta_D = \{1\}$, and $\beta_K = \emptyset$. Since in this case J_{k-1} covers also $\operatorname{int}(\bigcup_{i \leq k} Q_i)$, the Claim is trivial. Otherwise we note that there is more than one element in the sequence $A_1, \ldots, A_p$. Since the elements of $\mathcal{E}_Q \cup \mathcal{D}_Q(J_Q)$ are contained in $\bar{U}_4$, the intersection $A_1 \cap A_2 \subset \bar{U}_4 = \frac{14}{10} Q$ is nonempty; we can thus choose u_Q to be a point of $A_1 \cap \bar{U}_4$. If $a \in \bar{U}_4$, we set $u_Q = a$. Thus, unless $1 \in \alpha_D$, $u_Q = a$. Similarly, we choose a point $v_Q \in A_q \cap \bar{U}_4$ and can assume that $v_Q = b$ unless $q \in \alpha_D$. In addition, we can obviously require that if $u_Q \in J_Q$ or $v_Q \in J_Q$ then u_Q or v_Q (respectively) is an endpoint of a component of J_Q. So we have points u_Q and v_Q in $\bar{U}_4$ joined by the sequence $A_1, \ldots, A_q$.

We now apply Lemma 3A in the following situation. Let $X^\circ = h_Q X$, $u = h_Q(u_Q)$, $v = h_Q(v_Q)$, $\mathcal{D}^\circ = \{ h_Q D \cap \frac{14}{10} Q^\circ : D \in \mathcal{D}_Q \}$, $\mathcal{E}^\circ = h_Q \mathcal{E}_Q = \{ h_Q E : E \in \mathcal{E}_Q \}$, $A_k^\circ = h_Q A_k \cap \frac{14}{10} Q^\circ$, and $J^\circ = h_Q J_Q$. Then $A_1^\circ, \ldots, A_p^\circ$ is an alternating sequence joining u and v in $(\mathcal{D}^\circ, \mathcal{E}^\circ, \mathcal{D}(J^\circ))$ with alternating indexes $(\alpha_D, \alpha_E, \alpha_J)$ and

$$A^\circ = A_1^\circ \cup \cdots \cup A_p^\circ = h_Q(I_{k-1} \cap \bar{U}_4). \tag{3k}$$

Note that the intersection of successive elements of the sequence A_i° is indeed nonempty since always $A_i \cap A_{i+1} \subset \bar{U}_4$.

We will now check that the assumptions of Lemma 3A are true. Obviously, $\mathcal{E}^\circ$ is a family of e-chainable closed subsets of Q° such that $d(E, F) \geq e$ if $E, F \in \mathcal{E}^\circ$ are distinct. Similarly, since $J_Q \subset \bar{U}_4$ is an s-scaled $(\varrho_{i-1}, \delta_{i-1})$-family, J° is a $(\varrho_{i-1}, \delta_{i-1})$-family of $X^\circ \cap \frac{14}{10}Q^\circ$. Since for no $i \in \alpha_D \cup \alpha_E$ does A_i intersect with int $P_Q \cap U_3$, we have $J_Q \cap \operatorname{int} P_Q \cap U_3 = I_{k-1} \cap \operatorname{int} P_Q \cap U_3$ and hence the set A° of Lemma 3A satisfies (3c).

Thus we can apply Lemma 3A and obtain a (ϱ_i, δ_i)-family K° of $X^\circ \cap \frac{14}{10}Q^\circ \cap (tQ^\circ \cup tP^\circ)$ such that u and v can be joined by an alternating sequence $B_1^\circ, \dots, B_q^\circ$ of $(\mathcal{D}^\circ, \mathcal{D}(K^\circ))$ with alternating indexes (β_D, β_K) such that $B_i^\circ \in \mathcal{D}^\circ$ if $i \in \beta_D$ and $B_i^\circ \in \mathcal{D}(K^\circ)$ if $i \in \beta_K$. The sets B_i° and $B^\circ = B_1^\circ \cup \dots \cup B_q^\circ$ satisfy conditions (1°)–(5°) of Lemma 3A.

Now it is time to change back to Q. If $i \in \beta_K$, we set $B_i = h_Q^{-1} B_i^\circ$. If $i \in \alpha_D$, then $B_i^\circ = A_{\nu_i}^\circ$ and we would like to set $B_i = A_{\nu_i}$. The problem is that although B_i° can, by (4°), intersect only the preceding and succeeding B_j° at the endpoints of B_i° if $i \in \beta_K$, this might not be true if $i \in \beta_D$. We noted in (3j) that A_i, $i \in \alpha_D$, are disjoint. Hence, remembering (4°), it follows that A_{ν_i}, $i \in \beta_K$, are disjoint and that A_{ν_i} can intersect B_j, $j \in \beta_D$, only if $|j - i| = 1$. Thus we can find a subarc B_i of A_{ν_i} for $i \in \beta_D$ such that each B_i, $i \in \beta_D \cup \beta_K$, intersects only the preceding and succeeding B_j at the endpoints of B_i. In addition, we can require that a and $B_1 \cap B_2$ are endpoints of B_1, as can be seen by (5°) if $1 \in \beta_D$ and by (4°) if $1 \in \beta_K$. Similarly, b and $B_{q-1} \cap B_q$ are the endpoints of B_q. Thus, if we set

$$I_k = B_1 \cup \dots \cup B_q$$

then I_k is an arc with endpoints a and b. We note that

$$I_k \cap \tfrac{14}{10}Q = I_k \cap \bar{U}_4 \subset h_Q^{-1} B^\circ. \tag{3l}$$

Thus $B_i = h_Q^{-1} B_i^\circ \subset \frac{14}{10}Q$ if $i \in \beta_K$, and if $i \in \beta_D$ then $B_i \subset A_{\nu_i}$ and is thus a subarc of I_{k-1}. The arcs A_i, $i \in \alpha_D$, were in sequential order on I_{k-1} and, by (4°), the subarcs B_i of I_{k-1} for $i \in \beta_D$ are still in sequential order on I_{k-1}. Consequently, the first paragraph of the Claim is true.

We now define J_k, but will need some preliminary estimates. If $x \in (I_k \cap \bar{U}_4) \setminus tQ$, then $h_Q(x) \in B^\circ \setminus tQ^\circ \subset A^\circ$ by (2°) and hence $x \in h_Q^{-1} A^\circ \subset I_{k-1}$ by (3k). On the other hand, if $x \in I_k \setminus \bar{U}_4$ then $x \in B_i \subset A_{\nu_i}$ for some $i \in \beta_D$, and such a B_i is a subarc of I_{k-1}. Hence

$$I_k \setminus tQ \subset I_{k-1} \setminus tQ \quad \text{and so} \quad I_k \subset I_{k-1} \cup tQ. \tag{3m}$$

Since I_k and I_{k-1} are arcs with the same endpoints, it follows that if L is a subarc of $I_{k-1} \setminus tQ$ such that $L \cap I_k \neq \emptyset$, then $L \subset I_k$.

In particular, this is true if L is a component of L_Q which is a subset of $I_{k-1} \setminus U_1 = I_{k-1} \setminus tQ$; thus, if we define

$$\begin{aligned} L'_Q &= \bigcup \{ L \in \mathcal{D}(L_Q) : L \cap I_k \neq \emptyset \} \quad \text{then} \\ L'_Q &= \bigcup \{ L \in \mathcal{D}(L_Q) : L \subset I_k \} = L_Q \cap I_k. \end{aligned} \tag{3n}$$

Thus L'_Q is obtained by dropping some components from L_Q. It follows that L'_Q is an s-scaled $(\varrho_{i-1}, \delta_{i-1})$-family and hence an s-scaled (ϱ_i, δ_i)-family, since L_Q is an s-scaled $(\varrho_{i-1}, \delta_{i-1})$-family.

Let $K'_Q = h_Q^{-1} K^\circ = \bigcup_{i \in \beta_K} B_i$ (obviously we can assume that $K^\circ = \bigcup_{i\in\beta_K} B_i^\circ$, possibly by dropping some components of K°). Then K'_Q is an s-scaled (ϱ_i, δ_i)-family since K° is a (ϱ_i, δ_i)-family. We define

$$J_k = K'_Q \cup L'_Q. \tag{3o}$$

Now $K'_Q \subset \bar{U}_4$ and $L'_Q \subset L_Q \subset J_{k-1} \setminus U_1 \subset \bigcup_{j \le k-1} \frac{14}{10} Q_j$ since (iii) is true for $k-1$. This implies (iii) for k. Since J_k is a finite union of subarcs of I_k, it is a union of disjoint closed arcs.

We show that J_k is an s-scaled (ϱ_i, δ_i)-family. By Lemma 2I and (3f), it suffices to show that (K'_Q, L'_Q) is a (U_i)-division of J_k. We already know that $K'_Q \subset \bar{U}_4$ and $L'_Q \subset J_k \setminus U_1$. Thus, in order to have (2h), it suffices to prove that

$$L'_Q \cap \bar{U}_3 \subset K'_Q \quad \text{and} \quad K'_Q \setminus U_2 \subset L'_Q. \tag{3p}$$

We prove the first inclusion of (3p). Let $x \in L'_Q \cap \bar{U}_3 \subset L_Q \cap \bar{U}_3 \subset J_{k-1} \cap \bar{U}_3$ and so $x \in J_Q$ by (3h). Suppose that $x \in A_i$; then $i \in \alpha_J \cup \alpha_D$ since $x \notin tQ$. Our construction of A_i for $i \in \alpha_J \cup \alpha_D$ implies that $i \in \alpha_J$; if x is an endpoint of A_i then there is also a $j \in \alpha_D$ such that x is an endpoint of A_j. Since x persists to I_k, it follows that $x \in B_r$ for some $r \in \beta_D \cup \beta_K$. If $r \in \beta_K$ then $x \in K'_Q$ and so we are done. If $r \in \beta_D$ then B_r is a subarc of A_{ν_r}, and since A_k $(k \in \alpha_D)$ are disjoint except for endpoints from A_l $(l \in \alpha_J)$, x must be an endpoint of B_r. But then x is an endpoint of some B_s where $s \in \beta_K$ unless $x = a$ or $x = b$, and so $x \in K'_Q$. If $r \in \beta_D$ and $x = a$ or $x = b$, then by (5°) x is the endpoint of some A_i where $i \in \alpha_D$ not intersecting with any A_l with $l \in \alpha_J$; hence $x \notin J_Q$ and, since we know that $x \in J_Q$, this case is impossible.

Similarly, suppose that K' is a component of $K'_Q \setminus U_2$. By (3°), $K' \subset h_Q^{-1} J^\circ \setminus U_2 = J_Q \setminus U_2$ and hence $K' \subset J_{k-1} \setminus U_2$ and so is contained in a component L of L_Q. We have seen (cf. (3n)) that either $L \subset I_k$ or $L \cap I_k = \emptyset$. It follows that a component of $K'_Q \setminus U_2$ is contained in a component of L'_Q, implying the inclusion $K'_Q \setminus U_2 \subset L'_Q$ and the second inequality of (3p) follows. Thus (K'_Q, L'_Q) is a (U_i)-division of J_k and so J_k is an s-scaled (ϱ_i, δ_i)-family by Theorem 2I. We also obtain

$$J_k \setminus \bar{U}_2 = L'_Q \setminus \bar{U}_2 = L_Q \cap I_k \setminus \bar{U}_2 \subset L_Q \setminus \bar{U}_2 = J_{k-1} \setminus \bar{U}_2,$$

using (3n) and the fact that (J_Q, L_Q) is a (U_i)-division of J_{k-1}. This implies (vi).

We have now only to prove (iv) and (v). Let $U = \operatorname{int}(\bigcup_{i \le k} Q_i)$. By (3l), $I_k \cap \bar{U}_4 \subset h_Q^{-1} B^\circ$ and so (1°) implies

$$\begin{aligned} I_k \cap U_3 \cap U &\subset h_Q^{-1}[B^\circ \cap \operatorname{int}(\tfrac{13}{10} Q^\circ \cap (Q^\circ \cup P^\circ))] \\ &\subset h_Q^{-1} K^\circ \cap U_3 \cap \operatorname{int}(Q \cup P_Q) = K'_Q \cap U_3 \cap \operatorname{int}(Q \cup P_Q) \subset J_k. \end{aligned}$$

On the other hand, using (3m), the fact that (iv) is known for $k-1$, as well as (vi),

$$I_k \cap (U \setminus \bar{U}_2) = I_k \cap I_{k-1} \cap (U \setminus \bar{U}_2) \subset I_k \cap J_{k-1} \cap (U \setminus \bar{U}_2) = J_k \cap (U \setminus \bar{U}_2).$$

These two inclusions prove (iv).

We already have (vi). To obtain (v) we observe that, by (3m) and (vi), $(I_k \setminus J_k) \setminus \bar{U}_2 \subset [I_{k-1} \setminus (I_k \cap J_{k-1})] \setminus \bar{U}_2 \subset I_{k-1} \setminus J_{k-1}$. Consider then the situation in $\bar{U}_3$. For technical reasons it is simpler to prove $\operatorname{cl}(I_k \setminus J_k) \cap \bar{U}_3 \subset \operatorname{cl}(I_{k-1} \setminus J_{k-1}) \cap \bar{U}_3$, and this of course implies the inclusion without the closure. We first use (3l) to transfer the situation to $\frac{13}{10}Q^\circ$, remembering that $J_k \cap \bar{U}_3 = K'_Q \cap \bar{U}_3 = h_Q^{-1}K^\circ \cap \bar{U}_3$. Then we use (4°) in the first equality below and (3j) in the last equality:

$$\begin{aligned}\operatorname{cl}(I_k \setminus J_k) \cap \bar{U}_3 &\subset h_Q^{-1}[\operatorname{cl}(B^\circ \setminus K^\circ) \cap \tfrac{13}{10}Q^\circ] = h_Q^{-1}\left[\left(\bigcup_{i \in \beta_D} B_i^\circ\right) \cap \tfrac{13}{10}Q^\circ\right] \\ &\subset h_Q^{-1}\left[\left(\bigcup_{i \in \alpha_D} A_i^\circ\right) \cap \tfrac{13}{10}Q^\circ\right] = \left(\bigcup_{i \in \alpha_D} A_i\right) \cap \bar{U}_3 \\ &= \operatorname{cl}\left[\left(\bigcup_{i \in \alpha_D} A_i\right) \setminus J_Q\right] \cap \bar{U}_3 \subset \operatorname{cl}(I_{k-1} \setminus J_{k-1}) \cap \bar{U}_3,\end{aligned}$$

since $J_Q \cap \bar{U}_3 \supset J_{k-1} \cap \bar{U}_3$. Our Claim follows. □

Proof of Lemma 3B (conclusion). After the last cube $P = Q_S$, we have an arc $I = I_S$ with endpoints a and b such that, by (v), $I_S \setminus J_S \subset I_{S-1} \setminus J_{S-1} \subset \cdots \subset I_0 \setminus J_0 = A$. Since $A \subset \operatorname{int}(\bigcup Q_i)$, it follows by (iv) that $I_S = J_S$ implying that $I = J_S$ is an s-scaled (ϱ_N, δ_N)-family. Since $s = \varepsilon/2\sqrt{n}N = \varepsilon/2\sqrt{n}2^n$, it follows that (3d) is true for some r and M depending only on n and c.

The arc I is also contained in $U_\varepsilon(A)$. To see this, let $k_1, \dots, k_N = P$ be the numbers such that Q_{k_i} is the last cube of $\mathcal{Q}_i$. Then, by (ii),

$$I_{k_i} \subset I_{k_{i-1}} \cup \left(\bigcup_{k_{i-1} < j \le k_i} \tfrac{14}{10}Q_j\right) \subset U_{2\sqrt{n}s}(I_{k_{i-1}})$$

(where $I_0 = A$ and $k_0 = 0$) and so $I \subset U_{2N\sqrt{n}s}(A) = U_\varepsilon(A)$.

We must still prove (3e). As before, let k_i be the index such that Q_{k_i} is the last cube of $\mathcal{Q}_i$. When we changed $A = I_0$ first to I_1, then to I_2, and finally to I_{k_1}, we basically changed A first around Q_1 to obtain I_1, then around Q_2 to obtain I_2, and so forth. The arc I_1 could be subdivided into subarcs $I'_1, \dots, I'_k$ where alternatively either $I'_j \subset \frac{14}{10}Q_1$ or I'_j is a subarc of $I_0 = A$ (they are the sets B_i of the Claim). By the Claim, the order of subarcs of A in the sequence I'_j is the same as their order on $I_0 = A$.

Similarly, we changed I_1 to I_2 by changing I_2 around tQ_2, and obtained a similar subdivision of I_2 into arcs alternatively contained in $\frac{14}{10}Q_2$ and into subarcs of I_1 so that the order of these latter arcs in the sequence is compatible with a sequential order of I_1. Similarly, we change from I_2 to I_3, and so on, until we reach I_{k_1}. Since the cubes $\frac{14}{10}Q$, $Q \in \mathcal{Q}_1$, are disjoint, it follows that there is a subdivision of I_{k_1} into arcs $J_1, \dots, J_r$ such that alternatively either $J_i \subset \frac{14}{10}Q$ for some $Q \in \mathcal{Q}_1$ (say, this happens if $i \in \alpha_0$) or J_i is a subarc $I_0 = A$ (and this happens if $i \in \alpha_1$). In addition, the subarcs J_i such that $J_i \subset I_0$ are in sequential order on $I_0 = A$.

Define $\sigma_1 \colon I_{k_1} \to I_0 = A$ by $\sigma_1(t) = t$ if $t \in J_i$ and $i \in \alpha_1$ so that J_i is a subarc of I_0; if $i \in \alpha_0$ then define $\sigma_1(t)$ as an endpoint of J_i; in this case $J_i \subset \frac{14}{10}Q$

for some $Q \in \mathcal{Q}_1$. Now $|\sigma_1(x) - x| \le \frac{14}{10}s\sqrt{n} < \varepsilon/N$ and, since the order of the arcs $I_j \subset I_0$ is preserved, one easily sees that (3e) is valid for $\sigma = \sigma_1$, $I = I_{k_1}$, and $A = I_0$ if ε is replaced by ε/N.

Exactly in the same manner, one sees that there is a map $\sigma_i \colon I_{k_i} \to I_{k_{i-1}}$ such that (3e) is true for $\sigma = \sigma_i$, $I = I_{k_i}$, and $A = I_{k_{i-1}}$ if ε is replaced by ε/N. It follows that $\sigma = \sigma_N \circ \cdots \circ \sigma_1 : I = I_S \to A$ satisfies (3e). □

Proof of Theorems 1A and 1B

Now we have at our disposal the tools needed to prove the main theorems. We will first consider Theorem 1A. Let a and b be two elements in a c-BT set X. We claim that we can connect a and b by a c_0-BT arc J of X where $c_0 = c_0(c, n)$. Since we can transform the situation by a similarity, we can assume that $a = 0$ and $b = e_1 = (1, 0, \ldots, 0)$. Thus we can connect a and b by a set $A \subset \bar{B}^n(c) \cap X$. We can assume by Lemma 2C that A is an arc if we allow the connection to be made in a slightly bigger ball, say $A \subset B^n(c+1)$.

Let $r = r(c, n)$ and $M = M(c, n)$ be as in Lemma 3B. We now choose $\delta = r/5 < 1/5$. Thus $\delta = \delta(c, n)$. We will apply Lemma 3B repeatedly so that ε in Lemma 3B will assume the values δ^i, $i \ge 0$. By Lemma 3B (using $\delta^0 = 1$ as ε), we can connect a and b by an arc I_0 in $B^n(c+2)$ such that $d(I_0[x, y]) \le M$ if $x, y \in I_0$ and $|x - y| \le r$.

The arc I_0 starts induction. Assume that we have found $I_0, \ldots, I_k$ such that (3d) and (3e) are true with $I = I_{p+1}$, $A = I_p$, and $\varepsilon = \delta^{p+1}$ if $p + 1 \le k$. In this situation we apply Lemma 3B with $\varepsilon = \delta^{k+1}$ and with $A = I_k$, and let I_{k+1} be the arc I given by Lemma 3B. In this manner we find $I_k \subset U_{\delta^k}(I_{k-1})$ for all $k > 0$ such that if $x, y \in I_k$ and $|x - y| \le r\delta^k$ then

$$d(I_k[x, y]) \le M\delta^k. \tag{3q}$$

Suppose now that $x, y \in I_k$ and $i < k$. We apply (3e) repeatedly and find points $x', y' \in I_i$ such that $|x - x'| \le \delta^k + \cdots + \delta^{i+1} < 2\delta^{i+1}$ and similarly $|y - y'| < 2\delta^{i+1}$, and such that

$$I_k[x, y] \subset U_{2\delta^{i+1}}(I_i[x', y']). \tag{3r}$$

In particular, since $I_0 \subset B^n(c+2)$, it follows that

$$I_k \subset B^n(c + 2 + 2\delta).$$

We claim that there is $c_0 = c_0(c, n)$ such that if $x, y \in I_k$ and $|x - y| \ge \delta^{k+2}$, then

$$d(I_k[x, y]) \le c_0|x - y|. \tag{3s}$$

Assuming this, it follows by Lemma 2F that there is a subsequence I_{k_i} tending toward an arc J in the Hausdorff metric. In addition it follows by (3s) and (2c) that $d(I_{xy}) \le c_0|x - y|$ for all $x, y \in J$ and hence J is c_0-BT.

Thus we have only to prove (3s). If $|x - y| \ge \delta$, then (3s) is true with $c_0 = (c + 2 + 2\delta)\delta^{-1}$. Otherwise, if $|x - y| \ge \delta^{k+2}$, choose $i \le k$ such that $\delta^{i+2} \le |x - y| \le \delta^{i+1}$. As we have seen, there are $x', y' \in I_i$ such that $|x - x'| \le 2\delta^{i+1}$

and $|y-y'| \le 2\delta^{i+1}$ and such that (3r) is true. Thus $|x'-y'| \le 5\delta^{i+1} = r\delta^i$ and so $d(I_i[x',y']) \le M\delta^i$ by (3q) and consequently

$$d(I_k[x,y]) \le M\delta^i + 4\delta^{i+1} \le (M+4\delta)\delta^i \le (M+4\delta)\delta^{-2}|x-y|$$

by (3r), proving our claim if $c_0 = c_0(c,n)$ is the maximum of numbers $(M+4\delta)\delta^{-2}$ and $(c+2+\delta)\delta^{-1}$.

This proves Theorem 1A. The proof of Theorem 1B is the same. As above we can assume that A is an arc. We can also normalize the situation so that, say, $\varepsilon = 4$. Then the foregoing proof finds an arc I in $U_2(A)$ such that $d(I_{xy}) \le c_0|x-y|$ if $|x-y| \le \delta$. This easily implies Theorem 1B. □

4. Proof of Lemma 3A

The proof involves two steps. In the first step we prove the lemma except for the part that claims that K° is a (ϱ', δ')-family with ϱ' and δ' depending on $(n, c, e, \varrho, \delta)$ as claimed. We refer to this situation repeatedly and so, in order to avoid clumsy language, will say that K° satisfies Lemma 3A* when K° is a union of disjoint arcs satisfying Lemma 3A except that K° need not be a (ϱ', δ')-family with ϱ' and δ' depending only on $(n, c, e, \varrho, \delta)$. Of course, each such family K° is a (ϱ', δ')-family for some ϱ' and δ', and our basic idea is that we move in a compact set defined by $(n, c, e, \varrho, \delta)$ and hence can choose ϱ' and δ' uniformly.

Before we start, we make the following observation. Since $d(E,F) \ge e$, the number of elements of $\mathcal{E}^\circ$ is bounded by a number $N_0 = N_0(n,e)$. Similarly, the distance of components of J° is at least δ, and hence the number of components of J° is bounded by a number $N_1 = N_1(n,\delta)$. Because, in the sequence A_i°, the sets A_i° alternate in $\mathcal{D}^\circ$, $\mathcal{E}^\circ$, and $\mathcal{D}(J^\circ)$ so that two successive elements are in different families, we will consider only sequences A_i° such that no set of $\mathcal{E}^\circ$ or of $\mathcal{D}(J^\circ)$ occurs twice in the sequence A_i°. (However, a set of $\mathcal{D}^\circ$ may occur more than once; this is needed since we will go to a limit and different sets of $\mathcal{D}^\circ$ might converge to the same set.) Thus we can assume that

$$p \le N \tag{4a}$$

for some $N = N(n, e, \delta)$.

Step 1: Step 1 proves Lemma 3A*. Let $\{1, \dots, p\} = \alpha_D \cup \alpha_E \cup \alpha_J$ be as in Lemma 3A; that is, $A_i^\circ \in \mathcal{D}^\circ$ if $i \in \alpha_D$, and so forth. Here $p \le N$ as in (4a). If $i \in \alpha_E$, then $A_i^\circ \subset X^\circ \cap Q^\circ$ and is e-chainable, and it follows by Lemma 2D that we can find arcs $L_i \subset U_{2ce}(A_i^\circ) \cap X^\circ \subset \operatorname{int} tQ^\circ$ for $i \in \alpha_E$ such that L_i intersects A_{i-1}° and A_{i+1}°. (However, if $i = 1$ and $1 \in \alpha_E$, then L_1 intersects A_2° and $u \in A_1^\circ$; similarly, if $i = p$ and $p \in \alpha_E$, then L_p intersects A_{p-1}° and $v \in L_p$.) Let $\mathcal{L}$ be the set of components of $J^\circ \cup \{L_i : i \in \alpha_E\}$. Then each $L \in \mathcal{L}$ is a finite union of arcs of $(tQ^\circ \cup tP^\circ) \cap \frac{14}{10}Q^\circ \cap X^\circ$, and u and v are joinable in $\mathcal{D}^\circ \cup \mathcal{L}$ by a sequence $A_1', \dots, A_q'$ which can be constructed as follows.

Set $A_1' = A_1^\circ$. Suppose, for instance, that $1 \in \alpha_D$. Then we put 1 to β_D and define $\nu_1 = 1$. If $p \notin \alpha_D$ and there is an $L \in \mathcal{L}$ such that $L \cap A_1' \ne \emptyset$ and $v \in L$, we set $A_2' = L$, put 2 to β_K, and end the construction so that we have obtained a two-element sequence A'_1, A'_2. Otherwise we choose $L \in \mathcal{L}$ with the property that $L \cap A'_1 \neq \emptyset$ and such that if $k = k_L \in \alpha_D$ is the maximal number such that $A^\circ_k \cap L \neq \emptyset$, then L is chosen to make k as big as possible. We set $A'_2 = L$, put $2 \in \beta_K$, set $A'_3 = A^\circ_k$, put 3 to β_D, and define $\nu_3 = k$. We continue and the process ends when $\nu_q = p$ (if $p \in \alpha_D$) or $v \in L$.

Clearly, we obtain in this manner a sequence $A'_1, \ldots, A'_q$ such that each A'_i intersects only the preceding and succeeding element. In addition, every other A'_i is in $\mathcal{D}^\circ$ and every other is in $\mathcal{L}$ so that $\{1, \ldots, q\} = \beta_D \cup \beta_K$, where $A'_i \in \mathcal{D}^\circ$ if $i \in \beta_D$ and $A'_i \in \mathcal{L}$ if $i \in \beta_K$ and where i alternates in β_D and β_K. We have also constructed an increasing sequence $\nu_i \in \alpha_D$ for $i \in \beta_D$ such that $A'_i = A^\circ_{\nu_i}$ if $i \in \beta_D$. Also, the construction gives that $\nu_1 = 1$ if $1 \in \alpha_D$ and $\nu_q = p$ if $q \in \beta_D$. Since A'_i will be B°_i for $i \in \beta_D$, we have (5°). Clearly, still $q \leq N$.

If $i \in \beta_K$, then $A'_i \in \mathcal{L}$ and hence A'_i intersects only A'_{i-1} if $i > 1$ (if $i = 1$ then $u \in A'_i$) and A'_{i+1} if $i < q$ (if $i = q$ then $v \in A'_i$). Thus it is possible to find a subarc $A''_i \subset A'_i$ such that $A''_i \cap A'_{i-1}$ and $A''_i \cap A'_{i+1}$ are endpoints of A''_i (or, if $i = 1$ or $i = p$, then one endpoint of A''_i is u or v, respectively) and such that A''_i intersects no other A''_i or A'_j. If K° is the union of the arcs A''_i, then the components of K° are just the arcs A''_i and so K° is a union of disjoint arcs in $(tQ^\circ \cup tP^\circ) \cap \frac{14}{10}Q^\circ \cap X^\circ$ and u and v are joinable in $\mathcal{D}^\circ \cup \mathcal{D}(K^\circ)$. We obtain the B°_i-sequence of Lemma 3A° from the A'_i-sequence if we replace A'_i by A''_i for $i \in \beta_K$ so that $B^\circ_i = A'_i$ if $i \in \alpha_D$ and $B^\circ_i = A''_i$ if $i \in \beta_K$. Thus (4°) is true.

The remaining parts (1°)–(3°) of Lemma 3A are obvious by construction. Furthermore, we will later make use of the fact that each arc $L \in \mathcal{D}(K^\circ)$ can be subdivided into successive subarcs in such a way that

$$L = K_1(L) \cup \cdots \cup K_{p_L}(L) \tag{4b}$$

where alternatively either $K_i(L) \subset \operatorname{int} tQ^\circ$ or $K_i(L)$ is a subarc of some $J' \in \mathcal{D}(J^\circ)$. Furthermore, we can orient each $L = B^\circ_i \in \mathcal{D}(K^\circ)$ so that the beginning point of L is either u (if $i = 1$) or is in B°_{i-1} and the ending point of L is either v (if $i = q$) or is in B°_{i+1}. Thus, in the induced orientation for $K_i(L)$, the endpoint of $K_i(L)$ is the beginning point of $K_{i+1}(L)$.

If K° satisfies these conditions then K° will be called *admissible.* Note that component arcs of K° will be oriented as indicated above.

Step 2: We now show that we can choose K° in Lemma 3A to be such a (ϱ', δ')-family of arcs as claimed.

We first introduce some terminology, and define a condition for such arc families as considered in Lemma 3A. Assume that there is given a positive number r and two k-tuples $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k)$ and $\bar{\delta} = (\delta_1, \ldots, \delta_k)$ of positive numbers. Here $k \geq 0$, and if $k = 0$ we are given only r. Let K be a family of disjoint closed arcs. We say that K is an $(r, \bar{\varepsilon}, \bar{\delta})$-family if

(α) $d(L, L') \geq r$ for distinct $L, L' \in \mathcal{D}(K)$, and

(β) if $x, y \in L \in \mathcal{D}(K)$ and $|x - y| \leq \delta_i$, then $d(L_{xy}) \leq \varepsilon_i$.

Thus we can think of $\tau = (r, \bar{\varepsilon}, \bar{\delta})$ as an element of $(R_+)^{2k+1}$; such a triple τ is called a k-triple.

We will first prove a lemma whose formulation depends on a number k and is different for $k = 0$ and for $k > 0$.

LEMMA 4A. (a) $[k = 0]$ *There is an* $r' = r'(n, c, e, \varrho, \delta) > 0$ *such that there is an admissible* (r')*-family* K° *satisfying Lemma 3A*.*

(b) $[k > 0]$ *Suppose that, for some fixed* n, c, e, ϱ, *and* δ *in Lemma 3A, there is a* $(k-1)$*-triple* $\tau = (r, \bar{\varepsilon}, \bar{\delta}) \in (R_+)^{2k+1}$ *such that the family* K° *in Lemma 3A* (with these fixed* n, c, e, ϱ, δ*) can always be chosen to be an admissible* τ*-family. Let* r', $\varepsilon'_1, \ldots, \varepsilon'_k$, *and* $\delta'_1, \ldots, \delta'_{k-1}$ *be positive numbers such that* $r' < r$, $\varepsilon_i < \varepsilon'_i$, *and* $\delta_i > \delta'_i$ *if* $i \leq k-1$. *Then there is a* $\delta_k > 0$, *depending only on* $(n, c, e, \varrho, \delta, \tau, r', \varepsilon'_1, \ldots, \varepsilon'_k, \delta'_1, \ldots, \delta'_{k-1})$, *such that if* $\bar{\varepsilon}' = (\varepsilon'_1, \ldots, \varepsilon'_k)$ *and* $\bar{\delta}' = (\delta'_1, \ldots, \delta'_k)$ *then* K° *in Lemma 3A can be chosen to be an admissible* τ'*-family for the k-triple* $\tau' = (r', \bar{\varepsilon}', \bar{\delta}')$.

The proof of the lemma is based on Step 1, which finds the family K° for Lemma 3A*, and on the fact that we move in Lemma 3A in a compact situation where compactness is governed by $(n, c, e, \varrho, \delta)$. We will prove the lemma under the assumption that $k > 0$. (The proof for $k = 0$ is much the same but some details are slightly different; we give the changes for $k = 0$ inside brackets [ ].) Since the number of cubes of $\mathcal{Q}^\circ$ touching Q° is finite, it suffices to consider the situation that $\mathcal{P}^\circ$ and P° are fixed in Lemma 3A.

Suppose that, given n, c, e, ϱ, and δ, Lemma 4A is not true for a particular k and $(k-1)$-triple $\tau = (r, \bar{\varepsilon}, \bar{\delta})$. First of all this means that we can always choose K° in Lemma 3A* (with these n, c, e, ϱ, δ) to be a τ-family. It also means that there are $0 < r' < r$, $0 < \delta'_i < \delta_i$, and $\varepsilon'_i > \varepsilon_i$ for $i = 1, \ldots, k-1$ as well as $\varepsilon'_k > 0$ such that, for every $\delta'_k > 0$, there is a situation where the assumptions of Lemma 3A are true *but* K° is not an admissible τ'-family whenever the admissible family K° satisfies Lemma 3A*. [If $k = 0$, then for every $r' > 0$ there is a situation such that, however the admissible K° satisfying Lemma 3A* is chosen, there exist $L, L' \in \mathcal{D}(K^\circ)$ such that $d(L, L') < r'$.]

Let τ_i be the k-triple, where r', ε'_i for $i \leq k$, and δ_i for $i \leq k-1$ are the numbers found above when $\delta_k = 2^{-i}$. [If $k = 0$, then $r = 2^{-i}$.]

Thus there are c-BT sets $X_i \in \mathcal{C}(\bar{R}^n)$, families $\mathcal{D}_i$ of closed subsets of $(X_i \cap \frac{14}{10}Q^\circ) \setminus \operatorname{int}(Q^\circ \cup P^\circ)$, families $\mathcal{E}_i$ of e-chainable sets of $X_i \cap Q^\circ$ such that $d(E, F) \geq e$ for distinct $E, F \in \mathcal{E}_i$, as well as (ϱ, δ)-families J_i of $tP^\circ \cap \frac{14}{10}Q^\circ$ such that there is an alternating sequence $A_{i1}, \ldots, A_{ip_i}$ in $(\mathcal{D}_i, \mathcal{E}_i, \mathcal{D}(J_i))$ joining u_i and v_i such that, if $A_i = A_{i1} \cup \cdots \cup A_{ip_i}$, then

$$A_i \cap \operatorname{int}(P^\circ \cap \tfrac{13}{10}Q^\circ) \subset J_i \cap \operatorname{int}(P^\circ \cap \tfrac{13}{10}Q^\circ),$$

and such that if the τ-family K_i is admissible and if $K^\circ = K_i$ satisfies Lemma 3A* for $X^\circ = X_i$, and so on, then K_i is not a τ_i-family, however the τ-family K_i is chosen.

We derive a contradiction from this. Let $\alpha_{iD} \cup \alpha_{iE} \cup \alpha_{iJ}$ be the decomposition of $\{1, \ldots, p_i\}$ into alternating indexes so that $A_{ij} \in \mathcal{D}_i$ if $j \in \alpha_{iD}$, and so on. By (4a), we can assume that $p_i \leq N = N(n, e, \delta)$ and hence can pass to a subsequence so that $p_i = p$ is independent of i. We can also pass to a subsequence so that $\alpha_{iD} = \alpha_D$, $\alpha_{iE} = \alpha_E$, and $\alpha_{iJ} = \alpha_J$ independently of i. By compactness, we can pass to a subsequence so that $X_i \to X^\circ \in \mathcal{C}(\bar{R}^n)$ and $A_{ik} \to A_k^\circ$ as $i \to \infty$. By Lemma 2B, X° is c-BT.

Let now $\mathcal{D}^\circ = \{ A_k^\circ : k \in \alpha_D \}$. If we denote $\mathcal{E}_i = \{E_{i1}, \dots, E_{iq}\}$ and $\mathcal{D}(J_i) = \{J_{i1}, \dots, J_{ir}\}$ (as before, since the number of elements of $\mathcal{E}_i$ and $\mathcal{D}(J_i)$ are uniformly bounded, we can pass to a subsequence so that q and r are independent of i), then we can assume that $E_{ij} \to E_i^\circ$ and $J_{ij} \to J_i^\circ$ as $i \to \infty$. We easily see that $\mathcal{E}^\circ = \{E_1^\circ, \dots, E_q^\circ\}$ is a disjoint family of sets of $X \cap Q^\circ$ satisfying the conditions of Lemma 3A. Similarly, $J^\circ = \bigcup_i J_i^\circ$ is a (ϱ, δ)-family of $X^\circ \cap \frac{14}{10} Q$ by Corollary 2H. We can still assume that $u_i \to u$ and $v_i \to v$ and thus $A_1^\circ, \dots, A_p^\circ$ is an alternating sequence in $(\mathcal{D}^\circ, \mathcal{E}^\circ, \mathcal{D}(J^\circ))$ joining u and v. Then $(\alpha_D, \alpha_E, \alpha_J)$ are the corresponding alternating indexes.

We now have a situation where the assumptions of Lemma 3A are true. Hence, by the assumptions of Lemma 4A, we can in any case find $K^\circ \subset (tQ^\circ \cup tP) \cap \frac{14}{10} Q^\circ \cap X^\circ$ as in Lemma 3A* such that K° is an admissible τ-family. [If $k = 0$, we find such a K° by Step 1.] In addition, in this particular instance there is certainly a $\delta_k > 0$ such that if $x, y \in L \in \mathcal{D}(K^\circ)$ and $|x - y| \leq 2\delta_k$ then

$$d(L_{xy}) \leq \varepsilon_k/2. \tag{4c}$$

This of course gives a contradiction. Namely, if X_i is close enough to X°, and J_i is close enough J°, and so on, we can find for big i a set K_i such that $K_i \to K^\circ$ as $i \to \infty$ and that $K_i = K^\circ$ satisfies Lemma 3A*, where $X^\circ = X_i$, $J^\circ = J_i$, $u_i = u$, and so on. However, if K_i is close enough to K°, then K_i must satisfy (4c) with ε_k and δ_k if K° satisfies (4c) for $\varepsilon_k/2$ and $2\delta_k$. Similarly, for big i,

$$d(L_{xy}) \leq \varepsilon_j' \tag{4d}$$

if $x, y \in L \in \mathcal{D}(K_i)$ and $|x - y| \leq \delta_j'$ if $j \leq k - 1$. In addition, if $L, L' \in \mathcal{D}(K_i)$ are distinct then $d(L, L') > r'$ for big i. Thus the family K_i is a τ_i-family for big i, contrary to the assumption.

More formally, the argument runs as follows. Let $B_1^\circ \cup \cdots \cup B_q^\circ$ be the alternating sequence as in Lemma 3A* joining u and v in $(\mathcal{D}^\circ, \mathcal{D}(K^\circ))$, with $\beta_D \cup \beta_K$ the division of $\{1, \dots, q\}$ into alternating indexes. Thus $B_j^\circ = A_{\nu_j}^\circ$ if $j \in \alpha_D$, and we set $B_{ij} = A_{i\nu_j}$ if $j \in \beta_D$. Our tasks are to define B_{ij} for big i if $j \in \beta_K$ so that the sequence $B_{i1}, \dots, B_{iq}$ (with $\nu_{ij} = \nu_j$ as above not depending on i) satisfies Lemma 3A* and to show that $K_i = \bigcup\{ B_{ij} : j \in \beta_K \}$ is a τ_i-family for big i, giving a contradiction.

We use the fact that each $L \in \mathcal{D}(K^\circ)$ is admissible and hence can be divided into subarcs as in (4b). Thus subarcs of components of J° and arcs in $\operatorname{int} tQ^\circ$ alternate in the sequence $K_j(L)$ of (4b). In the following, subarcs of components of J° are called subarcs of J°.

We fix $L = B_m$ where $m \in \beta_K$, and use the notation $K_j(L)$ as in (4b). Let us consider the situation where $K_j(L)$ is a subarc of J°. Now J_i and J° are (ϱ, δ)-families and $J_i \to J^\circ$. Hence by Lemma 2F (applied to suitable components of J_k and of J°) there is a subarc K_{ij} of J_i such that $K_{ij} \to K_j(L)$ as $i \to \infty$. Now both $K_j(L)$ and K_{ij} are $\tilde{\varrho}$-arcs where $\tilde{\varrho}$ depends on ϱ and δ, and hence we can apply Theorem 2G to find a map $\sigma_{ij}\colon K_{ij} \to K_j(L)$ for big i sending the endpoints K_{ij} onto the endpoints $K_j(L)$, so that there are numbers d_{ij} such that

$$|\sigma_{ij}(x) - x| \le d_{ij} \quad \text{and} \quad K_{ij}[x, y] \subset U_{d_{ij}}(K_j(L)[\sigma(x), \sigma(y)]) \tag{4e}$$

where, for each j, $d_{ij} \to 0$ as $i \to \infty$. In addition, we orient K_{ij} so that the beginning point is sent by σ_{ij} to the beginning point of $K_j(L)$.

We do this for all big i and all arcs $K_j(L)$ such that $K_j(L)$ is a subarc of some component of J°, that is, for every other $K_j(L)$ in the sequence (4b). The other case is that $K_j(L) \subset X^\circ \cap \operatorname{int} tQ^\circ$. In this case, by Lemma 2E we can find for big i an arc $K_{ij} \subset X_i \cap \operatorname{int} tQ^\circ$ such that there is a map $\sigma_{ij}\colon K_{ij} \to K_i(L)$ satisfying (4e) with suitable d_{ij} tending to 0 as $i \to \infty$. In addition, we can assume K_{ij} oriented in such a way that the beginning point of K_{ij} is the ending point of $K_{i,j-1}$ if $j > 1$. If $j = 1$, then the beginning point is a point of $B_{i,m-1}$ if $L = B_m$ (or the point u if $m = 1$). Similar rules apply for the ending point of K_{ij}.

Define $K_{i0} = B_{i,m-1}$ (or $K_{i0} = \{u_i\}$ if $m = 1$) and $K_{i,p_L+1} = B_{i,m+1}$ (or $K_{i,p_L+1} = \{v_i\}$ if $m = q$). Define similarly $K_0(L)$ and $K_{p_L+1}(L)$. Then the intersections $K_j(L) \cap K_{j-1}(L)$ and $K_j(L) \cap K_{j+1}(L)$ are the endpoints of $K_j(L)$ for $1 \le j \le p_L$. We would like the corresponding statement to be true also for K_{ij}, at least if i is big. However, (4e) implies that in any case we can have for big i that K_{ij} intersects only the preceding and succeeding K_{im} and that, if there are multiple intersections, they are near the endpoints. It follows that we can shorten the arcs K_{ij} for $1 \le j \le p_L$ (K_{i0} and K_{i,p_L+1} are unchanged) so as to obtain arcs $K'_{ij} \subset K_{ij}$ such that K'_{ij}, $1 \le j \le p_L$, intersects only $K'_{i,j-1}$ and $K'_{i,j+1}$ at endpoints and such that

$$\sup\{\, d(K) : K \text{ is a component of } K_{ij} \setminus K'_{ij} \text{ for some } j \,\} \to 0 \tag{4f}$$

as $i \to \infty$. Thus

$$L_i = K'_{i1} \cup \cdots \cup K'_{ip_L}$$

will be an arc. Setting $B_{im} = L_i$, it follows that, for big i, B_{im} intersects B_{ij} only if $j = m - 1$ or $j = m + 1$ at the endpoints (or if $m = 1$ or $m = q$, an endpoint of B_{im} is u_i or v_i, respectively), as follows from the fact that the corresponding statement is true for $B_m = L$ and that the B_{im} are contained in smaller and smaller neighborhoods of B_m as $i \to \infty$.

We can now define $\sigma_i : L_i \to L$ by the rule $\sigma_i(t) = \sigma_{ij}(t)$ if $t \in K'_{ij} \subset K_{ij}$ (if $t \in K'_{ij} \cap K'_{i,j+1}$ we use σ_{ij} in the definition of $\sigma_i(t)$). It follows by (4e) and (4f) that there are positive d_i tending to 0 as $i \to \infty$ such that, for big i,

$$|\sigma_i(x) - x| \le d_i \quad \text{and} \quad L_i[x, y] \subset U_{d_i}(L[\sigma_i(x), \sigma_i(y)]).$$

Since $d_i \to 0$, it follows that $K_i = \bigcup_i L'_i$ will be a τ_i-family for big i. In addition, K_i is admissible and satisfies Lemma 3A*. This contradicts the choice of X_i, J_i, and so on. $\square$

Proof of Lemma 3A (conclusion). The proof is an inductive construction using Lemma 4A in the following manner. In the construction we will use a descending sequence $a_0 = 1 > a_1 > \cdots > 1/2$, and we fix such a sequence (a_i).

We now fix n, c, and e, as well as ϱ and δ, and prove Lemma 3A with these fixed values. In the first step we use the Lemma 4A with $k = 0$ and find $r > 0$ such that the family K° in Lemma 3A* can always be chosen to be an admissible τ-family for $\tau = (r)$; that is, $d(L', L) \geq r$ for distinct components L and L' of K°. In the next step, we apply Lemma 4A for $k = 1$ and find $\delta_1 > 0$ such that K° in Lemma 3A* can be chosen to be an admissible $(a_1 r, 2^{-1}, \delta_1)$-family. The next step uses Lemma 4A with $k = 2$, and the conclusion is that there is a $\delta_2 > 0$ such that we can always find K° for Lemma 3A* such that K° is an admissible $(a_2 r, ((a_2 2)^{-1}, 2^{-2}), (a_2\delta_2, \delta_2))$-family.

Since $(a_i 2)^{-k} \leq 2^{-k+1}$ and $a_i \delta^k \geq \delta_k/2$, the inductive construction finds $\delta_i > 0$ such that if $\varepsilon^{(k)} = (1, \ldots, 2^{k-1})$ and $\delta^{(k)} = (\delta_1/2, \ldots, \delta_k/2)$, then for every k there is a τ_k-family K_k for Lemma 3A* with $\tau_k = (r/2, \varepsilon^{(k)}, \delta^{(k)})$. Thus given A_i° ($i \in \alpha_D \cup \alpha_E \cup \alpha_J = \{1, \ldots, p\}$) of Lemma 3A, there is a an alternating sequence B_{ki}° ($i \in \beta_{kD} \cup \beta_{kK} = \{1, \ldots, q_k\}$) in $\{ A_i^\circ : i \in \alpha_{kD} \} \cup \mathcal{D}(K_k)$ satisfying Lemma 3A* and where K_k is a τ_k-family. Since $p \leq N$ by (4a), part (5°) of Lemma 3A implies that also $q_k \leq N$ and hence we can pass to a subsequence so that $q_k = q$ does not depend on k. We can also assume that $\beta_{kD} = \beta_D$ and $\beta_{kK} = \beta_K$ do not depend on k. Furthermore, for each k there are given numbers $\nu_{ki} \in \alpha_D$ ($i \in \beta_D$) as in (5°) of Lemma 3A such that $A_{\nu_{ki}} = B_{ki}$ and where ν_{ki} is increasing in i. Again, we can pass to a subsequence so that $\nu_{ki} = \nu_i$ does not depend on k.

Now, every K_k is contained in $\frac{14}{10} Q^\circ$ and the distance of distinct components of K_k is at least $r/2$. Thus the number of components of K_k is bounded and we can apply Theorem 2F to single components. It follows that we can still pass to a subsequence so that $K_k \to K^\circ$, where K° is a (ϱ', δ')-family with ϱ' and δ' depending only on n, c, e, r, and ε_i and δ_i—that is, depending only on $(n, c, e, \varrho, \delta)$. It follows that if $i \in \beta_K$ then $B_{ki} \to B_i^\circ$ for some $B_i^\circ \in \mathcal{D}(K^\circ)$ as $i \to \infty$. If $i \in \beta_D$ then we set $B_i^\circ = A_{\nu_i}$. Thus $B_1^\circ, \ldots, B_r^\circ$ joins u and v in $\mathcal{D}^\circ \cup \mathcal{D}(K^\circ)$.

It is possible that the sequence B_i° we have obtained does not satisfy (4°). For instance, after passing to the limit, it might happen that B_i° ($i \in \beta_K$) intersects B_j° even if $|i - j| > 1$. Therefore we may have to change the sequence B_i. Suppose, for example, that $1 \in \beta_D$. Let j be the biggest index in β_K such that B_j° intersects B_1°, and redefine B_2° to be B_j°. After that, we choose the biggest $k \in \beta_D$ such that the old B_j° intersects the new B_2°. We continue in this manner, possibly dropping some B_j° between the old B_1° and the old B_q°. We change β_D and β_K appropriately and note that we can redefine the increasing sequence of ν_is so that $B_i^\circ = A_{\nu_i}^\circ$ if $i \in \beta_D$. The new B_i° is still an alternating sequence joining u and v with alternating indexes β_D and β_K.

We have now obtained that if $i \in \beta_K$ then B_i° intersects B_j° only if $|i - j| = 1$; however, it could still be that if $i \in \beta_K$ then the intersection of B_i° with the succeeding or the preceding B_j° is not a point. If this is the case, we shorten the arc B_i° at the ends so that these intersections will be points. After this we redefine $K^\circ = \bigcup_{i \in \beta_K} B_i^\circ$, which is still a (ϱ, δ)-family, and so obtain the final sequence $B_1^\circ, \ldots, B_q^\circ$.

All these operations have had the effect that (4°) is now true. Parts (1°)–(3°) and (5°) are easily seen to be true. Now we finally have a sequence B_i° satisfying Lemma 3A and not only Lemma 3A*. □

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