Interpolating Sequences for Weighted Bergman Spaces of the Ball

MIROLJUB JEVTIĆ, XAVIER MASSANEDA, & PASCAL J. THOMAS

0. Introduction

Let B^p_α be the space of holomorphic functions f in the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ such that $f \in L^p((1-|z|^2)^{\alpha-1/p}dm)$, where $0 < p \leq \infty$ and $\alpha \geq 0$ (weighted Bergman space). In this paper we study the interpolating sequences for various B^p_α. The limiting cases $\alpha = 0$ and $p = \infty$ are respectively the Hardy spaces H^p and $A^{-\alpha}$, the spaces of holomorphic functions with polynomial growth of order α, which have generated particular interest. Note that the class of spaces we are considering is invariant under restriction to balls of lower complex dimension, which justifies the choice of those special weights.

As far as we know, for $n > 1$ the first research on this subject was carried out by Amar [Am] for the classical Bergman spaces, which in our notation correspond to the case $\alpha = 1/p$. Amar's main result states that separated sequences (in terms of the Gleason invariant distance) can be written as finite unions of interpolating sequences for $B^p_{1/p}$.

A sufficient condition due to Berndtsson [Be] is known for the case H^∞. Also, Varopoulos [Va] showed that if $\{a_k\}_k$ is H^∞-interpolating then $\sum_k (1-|a_k|^2)^n \delta_{a_k}$ is a Carleson measure. Later, Thomas [Th1] proved that the same necessary condition holds for H^1 and that it actually characterizes the finite unions of H^1-interpolating sequences.

On the other hand, after Seip's characterization of $A^{-\alpha}$-interpolating sequences in the unit disc ([Se1], see another proof in [BO]), Massaneda [Ma] obtained some results for the case $n > 1$. In particular, $\{a_k\}_k$ is a finite union of $A^{-\alpha}$-interpolating sequences if and only if $\sum_k (1-|a_k|^2)^{n+1} \delta_{a_k}$ is an $(n+1)$-Carleson measure or, equivalently, if and only if $\{a_k\}_k$ is a finite union of separated sequences.

It is worth noting that in [Se1], Seip also implicitly gives a characterization of interpolating sequences for all weighted Bergman spaces in the disk. In Section 5 we spell out the details for the reader's convenience.

Here we deal with different aspects concerning B^p_α-interpolating sequences. In Section 1 we first collect some definitions and well-known facts about weighted Bergman spaces and then introduce the natural interpolation problem, along with

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some basic properties. In Section 2 we describe, in terms of α and p, the inclusions between B^p_α spaces; in Section 3 we show that most of these inclusions also hold for the corresponding spaces of interpolating sequences. Unfortunately, our proof does not capture the intuitive conjecture given in [Th1] to the effect that, for $p' < p$, every H^p-interpolating sequence is also $H^{p'}$-interpolating. Section 4 is devoted to sufficient conditions for a sequence to be B^p_α-interpolating, expressed in the same terms as the conditions given in [Th1] for the Hardy spaces and in [Ma] for $A^{-\alpha}$. In particular we show, under some restrictions on α and p, that finite unions of B^p_α-interpolating sequences coincide with finite unions of separated sequences.

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1. Definitions and First Properties

1.1 Notation

For $z, w \in \mathbb{C}^n$, we set $z\bar{w} := \sum_{j=1}^n z_j\bar{w}_j$, $|z|^2 := z\bar{z}$, the unit ball $\mathbb{B}^n := \{ z \in \mathbb{C}^n : |z| < 1 \}$, and the unit sphere $S := \partial\mathbb{B}^n$.

Given $a \in \mathbb{B}^n$, φ_a is the involutive automorphism of the ball exchanging 0 and a (see [Ru1, 2.2.2]). For $a, b \in \mathbb{B}^n$, $d(a, b) := |\varphi_a(b)| = |\varphi_b(a)|$ is the invariant distance between a and b. Recall that

$$1 - d(a, b)^2 = 1 - |\varphi_a(b)|^2 = \frac{(1 - |a|^2)(1 - |b|^2)}{|1 - a\bar{b}|^2}.$$

We call the sets $E(z, r) := \{ \zeta \in \mathbb{B}^n : d(z, \zeta) < r \}$ *hyperbolic balls*.

The normalized Lebesgue measures on the ball and the sphere will be denoted by dm and $d\sigma$ respectively (dm_{2n} and $d\sigma_{2n-1}$ when we want to stress the dimension). The measure $d\tau(z) := (1 - |z|^2)^{-(n+1)}dm(z)$ is invariant under the automorphisms of the ball [Ru1, 2.2.6]. In particular, $\tau(E(z, r))$ depends only on r.

We write $A \preceq B$, or equivalently $B \succeq A$, when there is a constant C such that $A \le CB$, and $A \simeq B$ when $A \preceq B$ and $B \preceq A$.

Throughout this paper we will use the following estimates.

LEMMA 1.1. *Let $a, b \in \mathbb{B}^n$, $c > 0$, and $t > -1$. Then*:

(a) $$\int_{\mathbb{B}^n} \frac{(1 - |z|^2)^t}{|1 - \bar{a}z|^{n+1+c+t}}\, dm(z) \simeq (1 - |a|^2)^{-c};$$

(b) $$\int_S \frac{d\sigma(\zeta)}{|1 - \bar{a}\zeta|^{n+c}}\, dm(z) \simeq (1 - |a|^2)^{-c}; \text{ and}$$

(c) $$\int_{\mathbb{B}^n} \frac{(1 - |z|^2)^t}{|1 - z\bar{a}|^{n+1+c+t}|1 - z\bar{b}|^{n+1+c+t}}\, dm(z)$$
$$\preceq |1 - a\bar{b}|^{-(n+1+c+t)}\{\min(1 - |a|^2, 1 - |b|^2)\}^{-c}.$$

Proof. (a) and (b) are given in [Ru1, 1.4.10]. To prove (c), first split into the cases $|1 - z\bar{a}| \geq \frac{\sqrt{2}}{2}|1 - a\bar{b}|$ and $|1 - z\bar{a}| \leq \frac{\sqrt{2}}{2}|1 - a\bar{b}|$, which implies $|1 - z\bar{b}| \geq \frac{\sqrt{2}}{2}|1 - a\bar{b}|$ by the triangle inequality [Ru1, 5.1.2(i)]; then apply (a). □

1.2. Weighted Bergman Spaces

For $p > 0$ and $\alpha \in \mathbb{R}$, let $L^p_\alpha(\mathbb{B}^n)$ be the space of all measurable complex-valued functions on $\mathbb{B}^n$ such that $f \in L^p((1 - |z|^2)^{\alpha - 1/p} dm)$; that is, for $p < \infty$,

$$\|f\|^p_{p,\alpha} := \int_{\mathbb{B}^n} |f(z)|^p (1 - |z|^2)^{\alpha p - 1}\, dm(z) < \infty,$$

and for $p = \infty$,

$$\|f\|_{\infty,\alpha} := \sup_{z \in \mathbb{B}^n} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

For $p > 0$ and $\alpha > 0$, and denoting by $H(\mathbb{B}^n)$ the space of holomorphic functions in the ball, the *weighted Bergman space* $B^p_\alpha(\mathbb{B}^n) := H(\mathbb{B}^n) \cap L^p_\alpha(\mathbb{B}^n)$.

For $\alpha \leq 0$, the above condition holds only for the zero function, but we sometimes will use the limiting case of the *Hardy spaces*:

$$B^p_0 := H^p(\mathbb{B}^n) = \Big\{ f \in H(\mathbb{B}^n) : \|f\|_{H^p} = \|f\|^p_{p,0} := \sup_{0<r<1} \int_{\partial\mathbb{B}^n} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty \Big\}.$$

The facts presented in this section are essentially well-known, but we recap them here for the reader's convenience, and also to write them in notation inspired by Seip [Se1; Se2], which differs from that in Horowitz [Ho], Coifman, and Rochberg [CR; Ro].

The following statement is an immediate consequence of [Ru1, p. 14].

LEMMA 1.2. *If $l \in \mathbb{Z}_+$ and $f \in B^p_{\alpha - l/p}(\mathbb{B}^{n+l})$, then its restriction to $\mathbb{B}^n \times \{0\}$ lies in $B^p_\alpha(\mathbb{B}^n)$; conversely, whenever $g \in B^p_\alpha(\mathbb{B}^n)$, its trivial extension (constant along the vertical directions) must be in $B^p_{\alpha - l/p}(\mathbb{B}^{n+l})$.*

LEMMA 1.3. *For $p > 0$ and $\alpha \geq 0$, there exists a constant $c = c(\alpha, p, n) > 0$ such that, for all $z \in \mathbb{B}^n$,*

$$|f(z)| \leq c\|f\|_{p,\alpha}(1 - |z|^2)^{-(n/p + \alpha)},$$

and this is the best possible exponent.

Proof. Lemma 1.2 allows us to reduce this to the case $n = 1$ by considering the disk through 0 and z. Then use the mean value inequality on the disk $D(z, \frac{1}{2}(1 - |z|))$. That the estimate is sharp can be seen by considering the functions $f_{N,a}(z) := (1 - z\bar{a})^{-N}$ for $a \in \mathbb{B}^n$ and $N > n/p + \alpha$. Lemma 1.1 shows that $\|f_{N,a}\|_{p,\alpha} \simeq (1 - |a|^2)^{n/p + \alpha - N}$. □

Lemma 1.3 says that $B^p_\alpha \subset B^\infty_{n/p+\alpha}$. A more complete catalog of inclusions will be given in Section 2.

LEMMA 1.4. *There exists $c' = c'(\alpha, p, n) > 0$ such that, for any $a, b \in \mathbb{B}^n$ with $d(a,b) < 1/2$,*

$$|f(a) - f(b)| \le c' \|f\|_{p,\alpha}(1 - |a|^2)^{-(n/p+\alpha)} d(a,b).$$

Proof. We apply the generalized Schwarz lemma (see [Ru1, 8.1.4]) over $E(a, \frac{1}{2})$ together with the estimate from Lemma 1.3. □

1.3. The Interpolation Problem

We call a sequence $\{a_k\}_k \subset \mathbb{B}^n$ *interpolating* for the space B^p_α if, given arbitrary values $\{v_k\}_k$ subject to some reasonable restrictions, there exists $f \in B^p_\alpha$ such that $f(a_k) = v_k$ for all $k \in \mathbb{Z}_+$. It is not so easy in general to determine what the restrictions of holomorphic functions to a sequence of points are; see [BNØ] for the case of $H^p(\mathbb{D})$.

Here we want to impose growth restrictions only on the sequence $\{v_k\}_k$; more precisely, we shall work with weighted l^p spaces.

DEFINITION. For $p > 0$ and $\beta \in \mathbb{R}$, let

$$l^p_\beta = l^p_\beta(\{a_k\}) := \left\{ \{v_k\}_k \subset \mathbb{C} \ : \ \{(1 - |a_k|^2)^\beta v_k\}_k \in l^p \right\},$$

$\|v\|^p_{p,\beta} := \sum_k [(1 - |a_k|^2)^\beta |v_k|]^p$, and $\|v\|_{\infty,\beta} := \sup_k [(1 - |a_k|^2)^\beta |v_k|]$.

Lemma 1.3 motivates the following.

DEFINITION. We say that $\{a_k\}_k$ is an *interpolating sequence* for B^p_α, denoted by $\{a_k\} \in \mathrm{Int}(B^p_\alpha)$, iff, for any $\{v_k\} \in l^p_{n/p+\alpha}$, there exists $f \in B^p_\alpha$ such that $f(a_k) = v_k$ for all k.

This definition appeared in [SS] for the case $\alpha = 0$ and $n = 1$, and occurs under various guises in [Am; Ro; Se1; Th1; Th2]. There is an immediate necessary condition.

DEFINITION. We say that a sequence $\{a_k\}_k$ is *separated* iff there exists $\delta > 0$ such that, for all $j \neq k$, $d(a_j, a_k) \ge \delta$.

LEMMA 1.5. *For any $p > 0$ and $\alpha \ge 0$, if $\{a_k\}_k$ is an interpolating sequence for B^p_α then $\{a_k\}_k$ is separated.*

Proof. Set $v^j_k := (1 - |a_j|^2)^{-(n/p+\alpha)}\delta_{jk}$, where δ_{jk} is the Kronecker symbol. All the v^j are in the unit ball of $l^p_{n/p+\alpha}$. Applying Baire's theorem to the closure of the images under the restriction map of balls of arbitrarily large radius, we see that there is an $M > 0$ such that, for any $\varepsilon > 0$, there exist $f_j \in B^p_\alpha$ such that $\|f_j\|_{p,\alpha} \le M$ and $\|f_j(a_k) - v^j_k\|_{p,n/p+\alpha} < \varepsilon$.

Applying Lemma 1.4 to the function f_j,

$$(1 - |a_j|^2)^{-(n/p+\alpha)} - \varepsilon \le c' M (1 - |a_j|^2)^{-(n/p+\alpha)} d(a_j, a_k),$$

and we get the result by taking ε small enough. □

PROPOSITION 1.6. *For $\alpha > 0$ or $p \geq 1$, if $\{a_k\}_k \in \operatorname{Int}(B^p_\alpha)$ then*

(a) *the restriction map $f \mapsto \{f(a_k)\}_{k \in \mathbb{Z}_+}$ is bounded from B^p_α to $l^p_{n/p+\alpha}$, and*

(b) *there exists a constant $M > 0$ such that the interpolating function f can be chosen with the additional condition $\|f\|_{p,\alpha} \leq M\|v\|_{p,n/p+\alpha}$.*

The best such constant M is called the *constant of interpolation of* $\{a_k\}_k$.

Proof. If we can prove the first statement, then the second will follow by applying the open mapping theorem. For $p = \infty$, boundedness of the restriction is trivial.

In the case where $\alpha > 0$, notice that Lemma 1.5 implies that for some $\delta > 0$ the hyperbolic balls $E(a_k, \delta)$ are pairwise disjoint. Thus

$$\begin{aligned}\|f\|^p_{p,\alpha} &\geq \sum_k \int_{E(a_k,\delta)} (1-|z|^2)^{\alpha p-1}|f(z)|^p\, dm(z)\\ &\geq \delta^{n+1} \sum_k (1-|a_k|^2)^{\alpha p+n}|f(a_k)|^p,\end{aligned}$$

using the plurisubharmonicity of $|f|^p$.

For $\alpha = 0$ and $p \geq 1$, applying Baire's theorem as above to rectify the proof of [Th1, Thm. 2.2], we can see that $\{a_k\}_k \in \operatorname{Int}(H^p)$ implies that $\sum_k (1-|a_k|^2)^n \delta_{a_k}$ is a Carleson measure (see [Th2] for a direct proof when $p > 1$), and this implies the boundedness of the restriction mapping by Lemma 3.1 ([Hr]; see the beginning of Section 3 for a definition of Carleson measures). □

REMARK. The condition that $\{a_k\}_k$ be separated implies that it has finite density, if one defines the density as (say) the upper limit of the number of sequence points in a hyperbolic ball of fixed radius as its center tends to the boundary. (For a more rigorous definition, adapt the one in Section 5, which is taken from [Se1]). However, the density we would get from Lemma 1.5 depends on the constant M of interpolation. We conjecture there is some necessary condition for a sequence to be interpolating, in terms of its density, depending only on α and p but not on the constant M, as was proved in [Se1] when $n = 1$.

1.4. Invariance under Automorphisms and Restriction to Subspaces

For any automorphism (holomorphic self-map) φ of the ball, let

$$T_\varphi f(z) := \left(\frac{1-|\varphi^{-1}(0)|^2}{(1-z\overline{\varphi^{-1}(0)})^2}\right)^{n/p+\alpha} f \circ \varphi(z).$$

LEMMA 1.7.

(a) *T_φ is an isometry of B^p_α.*

(b) *If φ is an automorphism of the ball and $\{a_k\}_k$ is an interpolating sequence for B^p_α, then so is $\{\varphi(a_k)\}_k$, with the same constant of interpolation.*

Proof. (a) is trivial when $p = \infty$. Any automorphism is a composition of a map φ_a and a rotation, and the result is immediate in the latter case. For the former, in the case where $p < \infty$ and $\alpha > 0$,

$$\begin{aligned}\int_{\mathbb{B}^n} |T_{\varphi_a} f(z)|^p (1-|z|^2)^{\alpha p-1}\, dm(z) &= \int_{\mathbb{B}^n} (1-|\varphi_a(z)|^2)^{n+\alpha p} |f(\varphi_a(z))|^p\, d\tau(z) \\ &= \int_{\mathbb{B}^n} (1-|\zeta|^2)^{n+\alpha p} |f(\zeta)|^p\, d\tau(\zeta) \\ &= \|f\|_{p,\alpha}^p,\end{aligned}$$

which finishes the proof since $(T_\varphi)^{-1} = T_{\varphi^{-1}}$, as can be seen by an elementary calculation using [Ru1, 2.2.5] and the fact that $|\varphi^{-1}(0)| = |\varphi(0)|$.

(b) Take $v \in l^p_{n/p+\alpha}(\{\varphi(a_k)\})$. Then

$$\left\{ \left(\frac{1-|\varphi^{-1}(0)|^2}{(1-a_k\overline{\varphi^{-1}(0)})^2} \right)^{n/p+\alpha} v_k \right\} \in l^p_{n/p+\alpha}(\{a_k\}),$$

with the same norm, so there is an F such that $\|F\|_{p,\alpha} \le M\|v\|_{p,n/p+\alpha}$ and

$$F(a_k) = \left(\frac{1-|\varphi^{-1}(0)|^2}{(1-a_k\overline{\varphi^{-1}(0)})^2} \right)^{n/p+\alpha} v_k.$$

Then $G := T_{\varphi^{-1}} F$ solves the original problem, with $\|G\|_{p,\alpha} \le M\|v\|_{p,n/p+\alpha}$. □

The next lemma, which follows immediately from Lemma 1.2, has been used in [Am], and provides some necessary conditions for a sequence to be interpolating.

LEMMA 1.8. *Suppose* $\{a_k\} \subset \mathbb{B}^n \times \{0\} \subset \mathbb{B}^{n+l}$, *where* $l \in \mathbb{Z}_+^*$. *Let* $\alpha \ge l/p$. *Then* $\{a_k\} \in \operatorname{Int}(B^p_\alpha(\mathbb{B}^n))$ *if and only if* $\{a_k\} \in \operatorname{Int}(B^p_{\alpha-l/p}(\mathbb{B}^{n+l}))$.

1.5. Stability under Perturbation and Finite Sets

The proof of the following lemma was sketched in [Lu, Sec. 6.II].

LEMMA 1.9. *For* $0 < p \le \infty$ *and* $\alpha > 0$ *or for* $1 \le p \le \infty$ *and* $\alpha \ge 0$, *let* $\{a_k\} \in \operatorname{Int}(B^p_\alpha)$ *and let* $\{a'_k\}_k$ *be another sequence in* $\mathbb{B}^n$. *There exists* $\delta > 0$ *such that if*

$$d(a_k, a'_k) < \delta \quad \forall k \in \mathbb{Z}_+$$

then $\{a'_k\} \in \operatorname{Int}(B^p_\alpha)$.

Proof. Case $\alpha > 0$. Let $v \in l^p_{n/p+\alpha}$. Denote $a = \{a_k\}_k$, $a' = \{a'_k\}_k$, and $v^0 = v$. By hypothesis there exists $f_0 \in B^p_\alpha$ such that $f_0(a) = v^0$ ($f_0(a_k) = v^0_k$ for all k) and $\|f_0\|_{p,\alpha} \le M\|v^0\|$, where M denotes the constant of interpolation of $\{a_k\}_k$. Consider now $v^1 := v^0 - f_0(a')$.

Claim: For δ small enough, $\|v^1\| \le \gamma\|v^0\|$, with $\gamma < 1$.

To see this, we use a general estimate for holomorphic functions that is a refinement of Lemma 1.4. Let f be holomorphic and let $z, w \in \mathbb{B}^n$ with $d(z,w) < r < 1$. The plurisubharmonicity of $|f(z) - f(w)|^p$ as a function of z, together with a gradient estimate, shows that there exists a constant $C = C(r) > 0$ such that (see [Lu, Lemma 3.1] or [Th2, Lemma 2.4.4]):

$$|f(z) - f(w)|^p \leq Cd^p(z, w) \int_{E(w,r)} |f(\zeta)|^p \, d\tau(\zeta)$$

for any $r > \frac{2}{3}d(z, w)$. With this estimate applied to f_0 we have, provided that r is chosen small enough so that the invariant balls $E(a_k, r)$ are pairwise disjoint:

$$\begin{aligned}
\|v^1\|^p &= \sum_k (1 - |a_k|^2)^{n+\alpha p} |f_0(a_k) - f_0(a_k')|^p \\
&\leq C \sum_k (1 - |a_k|^2)^{n+\alpha p} d^p(a_k, a_k') \int_{E(a_k,r)} |f_0(\zeta)|^p \, d\tau(\zeta) \\
&\leq C\delta^p \sum_k \int_{E(a_k,r)} |f_0(\zeta)|^p (1 - |\zeta|^2)^{\alpha p - 1} \, dm(\zeta) \leq C\delta^p \|f_0\|_{p,\alpha}^p \\
&\leq C\delta^p M^p \|v^0\|^p.
\end{aligned}$$

Choosing δ so that $\gamma^p := C\delta^p M^p < 1$, the claim is proved.

Take now $f_1 \in B_\alpha^p$ with $f_1(a) = v^1$ and $\|f_1\|_{p,\alpha} \leq M\|v^1\|$, and define $v^2 = v^1 - f_1(a')$. An iteration of this construction provides functions $f_j \in B_\alpha^p$ with $f_j(a) = v^j = v^{j-1} - f_{j-1}(a')$ and $\|f_j\|_{p,\alpha} \leq M\|v^j\| \leq M\gamma^j \|v^0\|$. Finally, the function $f = \sum_j f_j$ solves the interpolation problem for $\{a_k'\}_k$.

Case $\alpha = 0$, $p \geq 1$. We can use Luecking's estimate, together with the fact that, when $\{a_k\} \in \text{Int}(H^p)$, $p \geq 1$, the measure $\sum_k (1 - |a_k|^2)^n \delta_{a_k}$ is a Carleson measure [Th1, Thm. 2.2], and therefore so is the measure $\sum_k (1 - |a_k|^2)^{-1} \chi_{E(a_k,r)} dm$. Then, applying Lemma 3.1 (see Section 3) we get $\sum_k (1 - |a_k|^2)^n |f(a_k) - f(a_k')|^p \lesssim \|f\|_{H^p}^p$ for any function for which the right-hand side is finite. The proof then proceeds as before. □

The results described above also hold for the corresponding interpolating sequences for the spaces b_α^p of $\mathcal{M}$-harmonic (instead of holomorphic) functions in L_α^p. This is so because the main ingredients used above, namely Lemma 1.3 and the separation of the interpolating sequences, can be proven likewise in the $\mathcal{M}$-harmonic case.

It is also important to know that the interpolation property does not depend on changing a finite number of points. Note that the union of two B_α^p-interpolating sequences is not in general B_α^p-interpolating [Am].

THEOREM 1.10. *The union of a B_α^p-interpolating sequence and a finite number of points is again B_α^p-interpolating.*

Proof. It is enough to show that the union of a B_α^p-interpolating sequence and one point is B_α^p-interpolating and—by invariance under automorphisms of the B_α^p-interpolating sequences—we can assume that this point is 0. Let then $\{a_k\}_k$ be the original sequence and let $\delta > 0$ be such that $|a_k| \geq \delta$ for all k.

We claim first that it is enough to find $f \in B_\alpha^p$ with $f(a_k) = 0$ for all k and $f(0) \neq 0$. To see this, let $\{v_k\} \cup v_0 \in l_{n/p+\alpha}^p(\{a_k\} \cup 0)$ and let $g \in B_\alpha^p$ be such that $g(a_k) = v_k$. Then the function

$$F(z) = g(z) + \frac{v_0 - g(0)}{f(0)} f(z)$$

belongs to B^p_α, and $F(0) = v_0$ and $F(a_k) = v_k$ for all k.

Suppose that all $f \in B^p_\alpha$ with $f(a_k) = 0$ for all k have $f(0) = 0$. This implies that, for any $f \in B^p_\alpha$, the value $f(0)$ is determined by the values $f(a_k)$, since the difference of two functions with the same values on $\{a_k\}_k$ vanishes at 0.

Assume $1 \le p < \infty$ and define the functional $\Lambda : l^p \to \mathbb{C}$ by

$$\Lambda(\{v_k\}) = f(0),$$

where $f \in B^p_\alpha$ is such that $f(a_k) = (1 - |a_k|^2)^{-(n/p+\alpha)} v_k$ for all k. Since $f(0)$ is determined only by these values, which are actually independent of f, we have that Λ is linear. It is also continous:

$$|\Lambda(b)| = |f(0)| \le c\|f\|_{p,\alpha} \le cM\|\{(1 - |a_k|^2)^{-(n/p+\alpha)} v_k\}\|_{p,n/p+\alpha} = cM\|v\|_p,$$

where M denotes the interpolation constant of $\{a_k\}_k$. So $\Lambda \in l^q$, in the sense that there exists $\{c_k\}_k \in l^q$ such that

$$\Lambda(v) = \sum_{k=1}^{\infty} v_k c_k \quad \forall v = \{v_k\} \in l^p.$$

Consider now the sequences $v^j = \{\delta_{jk}\}_k \in l^p$ and a function $f_j \in B^p_\alpha$ with $f_j(a_k) = (1 - |a_k|^2)^{-(n/p+\alpha)} \delta_{jk}$. By definition,

$$\Lambda(v^j) = f_j(0) = \sum_{k=1}^{\infty} \delta_{jk} c_k = c_j.$$

Take now the functions $F_j(z) = (|a_j|^2 - \bar{a}_j z) f_j(z)$. Obviously $F_j \in B^p_\alpha$ and $F_j(a_k) = 0$ for all k. Therefore $F_j(0) = |a_j|^2 f_j(0) = |a_j|^2 c_j = 0$, and hence $c_j = 0$. This shows that $\Lambda \equiv 0$, which is evidently false, since there are many functions in B^p_α not vanishing at 0.

The case $0 < p < 1$ is solved in the same way, using that the dual of l^p is l^∞. For the case $p = \infty$ we can restrict the functional Λ to the subspace $c_0 \subset l^\infty$ of sequences with limit 0 and apply the same argument. □

2. Inclusions for B^p_α and l^p_β

Our purpose is now to describe, in terms of the values α and p, the relationship between B^p_α spaces.

LEMMA 2.1.

(a) *If $p \le p'$, then $B^p_\alpha \subset B^{p'}_{\alpha'}$ if and only if $\alpha + n/p \le \alpha' + n/p'$.*
(b) *If $p > p'$ and $\alpha' > 0$, then $B^p_\alpha \subset B^{p'}_{\alpha'}$ if and only if $\alpha < \alpha'$.*

In particular, if $\alpha < \alpha' + n(1/p' - 1/p)_+$ then $B^p_\alpha \subset B^{p'}_{\alpha'}$, and if $B^p_\alpha \subset B^{p'}_{\alpha'}$ then $\alpha \le \alpha' + n(1/p' - 1/p)_+$. Those results have been obtained in the case $n = 1$ by Horowitz [Ho]. Seip [Se2] has shown that for $n = 1$, $\alpha > 0$, and $p > p'$, the zero sets for $B_\alpha^{p'}$ are also zero sets for B_α^p and that the converse is not true, so in particular $B_\alpha^p \not\subset B_\alpha^{p'}$.

Among many other results, Coifman and Rochberg [CR, Prop. 4.2, Prop. 4.4] prove that if $p \le p' \le 1$, $\alpha', \alpha \ge 0$, and $\alpha + n/p = \alpha' + n/p'$, then $B_\alpha^p \subset B_{\alpha'}^{p'}$. Their proof is valid for a whole class of symmetric domains.

Proof of Lemma 2.1. (a) Assume $\alpha + n/p \le \alpha' + n/p'$ and $f \in B_\alpha^p$. The case $p' = \infty$ was settled by Lemma 1.3. For $p' < \infty$, one has

$$\int_{\mathbb{B}^n} |f(z)|^{p'}(1-|z|^2)^{\alpha' p' - 1}\, dm(z) \\ \le c^{p'-p} \int_{\mathbb{B}^n} |f(z)|^p (1-|z|^2)^{(n/p+\alpha)(p-p')+\alpha' p' - 1}\, dm(z).$$

This integral is controlled by $\|f\|_{\alpha,p}^p$ whenever $(n/p+\alpha)(p-p') + \alpha' p' \ge \alpha p$, that is, when $\alpha + n/p \le \alpha' + n/p'$.

Conversely, assume $\alpha + n/p > \alpha' + n/p'$. As in Section 1.2, let $f_{\gamma,a}(z) = (1 - z\cdot\bar{a})^{-\gamma}$. Whenever $\gamma > \alpha' + n/p'$, $\|f_{\gamma,a}\|_{\alpha',p'} \simeq (1-|a|^2)^{\alpha'+n/p'-\gamma}$. Choosing $\gamma > \alpha + n/p$, we see that $\|f_{\gamma,a}\|_{\alpha',p'}/\|f_{\gamma,a}\|_{\alpha,p}$ cannot be bounded as $|a|$ tends to 1.

(b) Assume $\alpha < \alpha'$ and $f \in B_\alpha^p$. Hölder's inequality with exponents $p/p' \ge 1$ and $p/(p-p')$ yields

$$\int_{\mathbb{B}^n} |f(z)|^{p'}(1-|z|^2)^{\alpha' p' - 1}\, dm(z) \\ \le \left(\int_{\mathbb{B}^n} |f(z)|^p (1-|z|^2)^{\alpha p - 1}\, dm(z)\right)^{p'/p} \\ \times \left(\int_{\mathbb{B}^n} (1-|z|^2)^{(\alpha'-\alpha)p'p/(p-p')-1}\, dm(z)\right)^{(p-p')/p},$$

and by hypothesis both integrals are finite. If p' or p is infinite, the analogous proof goes through even more easily.

When $\alpha = 0$, the hypothesis becomes $f \in H^p$. Using $\int_S |f|^{p'} d\sigma \le \int_S |f|^p\, d\sigma$ and integration in polar coordinates, we see that

$$\int_{\mathbb{B}^n} |f(z)|^{p'}(1-|z|^2)^{\alpha' p' - 1}\, dm(z) \preceq \|f\|_{H^p}^p \int_0^1 (1-r^2)^{\alpha' p' - 1}\, dr < \infty.$$

When $\alpha = \alpha' = 0$, $B_0^p = H^p \subset B_0^{p'} = H^{p'}$ if and only if $p \ge p'$.

Assume now $\alpha \ge \alpha' > 0$ and $p > p'$. In order to construct a function $F \in B_\alpha^p(\mathbb{B}^n) \setminus B_{\alpha'}^{p'}(\mathbb{B}^n)$, we use the Ryll–Wojtaszczyk polynomials: there exists a family of homogeneous polynomials W_k of degree k such that $c^{-1} \le \|W_k\|_{H^2} \le \|W_k\|_{H^\infty} \le 1$ for some constant c independent of k (see [Ru2, Thm. 2.1, p. 4]).

It clearly suffices to prove the following.

Claim: The gap series $F(z) = \sum_k a_k W_{2^k}(z)$ satisfies the estimate

$$\|F\|_{p,\alpha}^p \simeq \sum_{k=0}^{\infty} \frac{|a_k|^p}{2^{kp\alpha}}.$$

Proof. The homogeneity of W_k and [Ru1, 1.4.7] yield:

$$\begin{aligned}
\|F\|_{p,\alpha}^p &= \int_{\mathbb{B}^n} |F(z)|^p (1-|z|^2)^{\alpha p-1}\, dm(z) \\
&= 2n \int_S \int_0^1 \Big|\sum_k a_k W_{2^k}(r\zeta)\Big|^p (1-r^2)^{\alpha p-1} r^{2n-1}\, dr\, d\sigma(\zeta) \\
&= 2n \int_S \int_0^1 \frac{1}{2\pi} \int_0^{2\pi} \Big|\sum_k a_k (re^{i\theta})^{2^k} W_{2^k}(\zeta)\Big|^p d\theta \\
&\quad \times (1-r^2)^{\alpha p-1} r^{2n-1}\, dr\, d\sigma(\zeta).
\end{aligned}$$

Applying now [Zy, Chap. V, Thm. 8.20], it follows that

$$\begin{aligned}
\|F\|_{p,\alpha}^p &\simeq \int_S \int_0^1 \frac{1}{2\pi} \left(\int_0^{2\pi} \Big|\sum_k a_k (re^{i\theta})^{2^k} W_{2^k}(\zeta)\Big|^2 d\theta\right)^{p/2} \\
&\quad \times (1-r^2)^{\alpha p-1} r^{2n-1}\, dr\, d\sigma(\zeta) \\
&\simeq \int_S \int_0^1 \left(\sum_k |a_k|^2 |W_{2^k}(\zeta)|^2 r^{2^{k+1}}\right)^{p/2} (1-r^2)^{\alpha p-1} r^{2n-1}\, dr\, d\sigma(\zeta),
\end{aligned}$$

which by [MP, Thm. 1] finishes the proof. □

The sequence spaces we are considering verify similar inclusions.

LEMMA 2.2. *Suppose $\{a_k\}_k$ is separated. Then:*

(a) *if $p \le p'$, then $l^p_{n/p+\alpha} \subset l^{p'}_{n/p'+\alpha'}$ if and only if $n/p + \alpha \le n/p' + \alpha'$;*

(b) *if $p \ge p'$ and $\alpha < \alpha'$, then $l^p_{n/p+\alpha} \subset l^{p'}_{n/p'+\alpha}$.*

Conversely to (b),

(c) *if $p > p'$ and $\alpha \ge \alpha'$, then there exists a separated sequence $\{a_k\}_k$ such that $l^p_{n/p+\alpha} \not\subset l^{p'}_{n/p'+\alpha'}$.*

Proof. (a) Left to the reader (similar to Lemma 2.1).

(b) First observe that the separation of the sequence implies that, for any $\varepsilon > 0$, $\sum_k (1-|a_k|^2)^{n+\varepsilon} < \infty$ (see Lemma 4.1). For $p < \infty$, applying Hölder's inequality yields

$$\begin{aligned}
&\sum_k (1-|a_k|^2)^{n+\alpha' p'} |v_k|^{p'} \\
&\le \left(\sum_k \left((1-|a_k|^2)^{\alpha p'} |v_k|^{p'}\right)^{p/p'} (1-|a_k|^2)^n\right)^{p'/p} \\
&\quad \times \left(\sum_k \left((1-|a_k|^2)^{(\alpha'-\alpha)p'}\right)^{1/(1-p'/p)} (1-|a_k|^2)^n\right)^{1-p'/p}.
\end{aligned}$$

Now the exponent of $1-|a_k|^2$ in the last sum is $(\alpha'-\alpha)/(1/p'-1/p)+n>n$, so we are done. Notice that if the sequence $\{a_k\}_k$ were sparse then we could allow smaller values of α', so there is no hope of a general converse statement analogous to case (a). The reasoning in the case $p=\infty$ is similar and simpler.

(c) Our example will be a separated sequence that is as crowded as possible (a net in the sense of [Ro] or [Lu]), that is, having the property that all points in the ball are less than a constant invariant distance away from a point in the sequence.

Let $r\in(0,1)$. We pick a sequence

$$\{a_{m,l},\ m\in\mathbb{Z}_+^*,\ 0\le l\le L_m\}$$

such that, for any l, $a_{m,l}=1-r^m$ and for each m the set $\{a_{m,l},\ 0\le l\le L_m\}$ is maximal in the sphere of radius $1-r^m$ for the property that the invariant balls $E(a_{m,l},r)$ be disjoint. It is easy to check that this sequence is separated and that $L_m\simeq r^{-nm}$.

We choose $v_{m,l}$ so that $u_m:=|v_{m,l}|$ depends only on m. Now, for $0<p<\infty$,

$$\sum_{m,l}(1-|a_k|^2)^{n+\alpha p}|v_{m,l}|^p\simeq\sum_m r^{-nm+m(n+\alpha p)}u_m^p$$

and

$$\sum_{m,l}(1-|a_k|^2)^{n+\alpha'p'}|v_{m,l}|^{p'}\simeq\sum_m r^{m\alpha'p'}u_m^{p'}\ge\sum_m\left(r^{m\alpha p}u_m^p\right)^{p'/p}.$$

By choosing u_m appropriately, we can then make the last sum diverge, while $\sum_m r^{m\alpha p}u_m^p<\infty$.

In the case $p=\infty$, simply taking $u_m=r^{-m\alpha}$, we see that $\sum_m r^{m\alpha'p'}u_m^{p'}\ge\sum_m 1=\infty$. □

3. Inclusions for $\mathrm{Int}(B_\alpha^p)$

In this section we show that the inclusions given in Lemma 2.1 and Lemma 2.2 are also verified by the corresponding spaces of interpolating sequences. First we recall some known facts about Carleson measures, which will also be used in Section 4.

For any $t>0$ and $\zeta_0\in S$, consider the *Carleson window* with center ζ_0 and radius t defined by $C_t(\zeta_0)=\{z\in\mathbb{B}^n:|1-\bar\zeta_0 z|<t\}$. A Borel measure ν in $\mathbb{B}^n$ is a *q-Carleson measure* if

$$\nu(C_t(\zeta_0))=O(t^q)\quad\forall t>0,\ \forall\zeta_0\in S.$$

An n-Carleson measure is simply called a *Carleson measure.* What we call q-Carleson measures were studied in [AB] under the name "Carleson measures of order q/n".

One of the main features of Carleson measures is the following.

LEMMA 3.1 [Hr; CW]. *Let $q\ge p>0$ and $\alpha\ge0$. Let μ be a positive measure. Then the following are equivalent:*

(a) $\left(\int_{\mathbb{B}^n} |f(z)|^q \, d\mu(z)\right)^{1/q} \le c\|f\|_{p,\alpha}$ *for all* $f \in B^p_\alpha$;
(b) $\mu(C_t(\zeta)) = O(t^{(n/p+\alpha)q})$ *for all* $t > 0$ *and for all* $\zeta \in S$.

In the sequel, we will be mainly interested in Carleson conditions for the measures $\sum_k (1-|a_k|^2)^q \delta_{a_k}$, which have important relationships with the values

$$K(\{a_k\}, p, q) := \sup_{k \in \mathbb{Z}_+} \sum_{j : j \ne k} \frac{(1-|a_k|^2)^p (1-|a_j|^2)^q}{|1-\bar{a}_k a_j|^{p+q}}, \qquad p, q > 0.$$

LEMMA 3.2.
(a) *If* $\sum_k (1-|a_k|^2)^n \delta_k$ *is a Carleson measure then, for any* $p > 0$,

$$K(\{a_k\}, p, n) < \infty.$$

(b) *A positive measure* μ *on* $\mathbb{B}^n$ *is a Carleson measure if and only if there exists some* $\beta > n/2$ *such that*

$$\sup_{b \in \mathbb{B}^n} \int_{\mathbb{B}^n} \frac{(1-|b|^2)^{2\beta-n}}{|1-\bar{b}z|^{2\beta}} \, d\mu(z) < \infty.$$

Part (a) is an immediate consequence of [Ma, Lemma 1.4]. Part (b) is a well-known result that can be found in [Ma, Lemma 1.2] and ultimately goes back to [Ga, p. 239].

We now come to the main result of this section.

THEOREM 3.3. *Let* $p, p' > 0$ *and* $\alpha, \alpha' \ge 0$ *satisfy one of the following conditions*:
(a) $p \le p'$ *and* $n/p + \alpha < n/p' + \alpha'$;
(b) $p \ge p'$ *and* $\alpha < \alpha'$.
Then $\operatorname{Int}(B^p_\alpha) \subset \operatorname{Int}(B^{p'}_{\alpha'})$.

In the special case where $n = 1$, Seip [Se1] has proved a stronger result than Theorem 3.3(b), namely, that $\operatorname{Int}(B^\infty_\alpha) = \operatorname{Int}(B^p_\alpha)$ (see Section 5). This suggests that the inequality $\alpha < \alpha'$ in (b) is critical.

In fact, if we take a sequence $a \subset \mathbb{B}^1 \times \{0\} \subset \mathbb{B}^n$, then we see by Lemma 1.8 that $a \in \operatorname{Int}(B^p_\alpha(\mathbb{B}^n))$ iff $a \in \operatorname{Int}(B^p_{\alpha+(n-1)/p}(\mathbb{B}^1))$, which, by Seip's result, is the same as $\operatorname{Int}(B^\infty_{\alpha+(n-1)/p}(\mathbb{B}^1))$. Since we know from [Se1] that $\operatorname{Int}(B^\infty_\beta(\mathbb{B}^1)) \subsetneq \operatorname{Int}(B^\infty_{\beta'}(\mathbb{B}^1))$ when $\beta < \beta'$, this shows that $\operatorname{Int}(B^p_\alpha(\mathbb{B}^n)) \ne \operatorname{Int}(B^{p'}_{\alpha'}(\mathbb{B}^n))$ when $\alpha + (n-1)/p \ne \alpha' + (n-1)/p'$, and that $\operatorname{Int}(B^p_\alpha(\mathbb{B}^n)) \not\subset \operatorname{Int}(B^{p'}_{\alpha'}(\mathbb{B}^n))$ when $\alpha + (n-1)/p > \alpha' + (n-1)/p'$. This shows that the inclusions in parts (a) and (b) of Theorem 3.3 are strict.

Proof of Theorem 3.3. Let $\{a_k\}_k$ be B^p_α-interpolating, and take $f_k \in B^p_\alpha$ with $(1-|a_j|^2)^{n/p+\alpha} f_k(a_j) = \delta_{jk}$ and $\|f_k\|_{p,\alpha} \le c$ for some constant $c > 0$ independent of k. Given $m > 0$, define $G_k(z) = g_k(z) \cdot f_k(z)$, where

$$g_k(z) = \frac{(1-|a_k|^2)^{n/p+\alpha+m}}{(1-\bar{a}_k z)^{n/p'+\alpha'+m}}.$$

For a given $\{\lambda_k\}_k \in l^{p'}$, let $G := \sum \lambda_k G_k$. From this definition it follows immediately that $(1-|a_k|^2)^{n/p'+\alpha'} G(a_k) = \lambda_k$, and we need to prove that $G \in B^{p'}_{\alpha'}$. Assume first that $\alpha > 0$.

Case $0 < p' \leq 1$. Since $\|\cdot\|^{p'}_{p',\alpha'}$ satisfies the triangle inequality, it is enough to show the following.

Claim: There exists $c > 0$ independent of k such that $\|G_k\|_{p',\alpha'} \leq c$ for all k.

Proof. By definition of G_k,

$$\|G_k\|^{p'}_{p',\alpha'} = \int_{\mathbb{B}^n} \frac{(1-|a_k|^2)^{(n/p+\alpha+m)p'}}{|1-\bar{a}_k z|^{(n/p'+\alpha'+m)p'}} (1-|z|^2)^{\alpha' p'-1} |f_k(z)|^{p'}\, dm(z).$$

(a) Since $p \leq p'$, estimate $|f_k(z)|^{p'-p}$ by Lemma 1.3. We see that

$$\begin{aligned}\|G_k\|^{p'}_{p',\alpha'} &\preceq \|f_k\|^{p'-p}_{p,\alpha} \\ &\quad\times \int_{\mathbb{B}^n} \frac{(1-|a_k|^2)^{(n/p+\alpha+m)p'}(1-|z|^2)^{\alpha' p'+(p-p')(n/p+\alpha)-\alpha p}}{|1-\bar{a}_k z|^{(n/p'+\alpha'+m)p'}} \\ &\quad\times (1-|z|^2)^{\alpha p-1} |f_k(z)|^p\, dm(z),\end{aligned}$$

which, since $(n/p+\alpha+m)p' + [\alpha' p' + (p-p')(n/p+\alpha) - \alpha p] = (n/p' + \alpha' + m)p'$, shows that $\|G_k\|^{p'}_{p',\alpha'} \preceq \|f_k\|^{p'}_{p,\alpha}$.

(b) We may assume $p > p'$, and apply Hölder's inequality with exponents p/p' and $p/(p-p')$. Since $\alpha' - \alpha > 0$, one has, for m large enough,

$$\begin{aligned}&\|G_k\|^{p'}_{p',\alpha'} \\ &\preceq \left[\int_{\mathbb{B}^n} \frac{(1-|a_k|^2)^{(\frac{n}{p}+\alpha+m)\frac{p'p}{p-p'}}(1-|z|^2)^{(\alpha'-\alpha)\frac{p'p}{p-p'}-1}}{|1-\bar{a}_k z|^{(\frac{n}{p'}+\alpha'+m)\frac{p'p}{p-p'}}}\, dm(z)\right]^{\frac{p-p'}{p}} \|f_k\|^{p'}_{p,\alpha} \\ &\preceq (1-|a_k|^2)^{[(\frac{n}{p}+\alpha+m)\frac{p'p}{p-p'}-(\frac{n}{p'}+\alpha'+m)\frac{p'p}{p-p'}+(\alpha'-\alpha)\frac{p'p}{p-p'}-1+n+1]\frac{p-p'}{p}} \|f_k\|^{p'}_{p,\alpha} \\ &\simeq \|f_k\|^{p'}_{p,\alpha}.\end{aligned}$$

This concludes the case $p' \leq 1$. $\square$

Case $p' > 1$. First we give a useful estimate.

LEMMA 3.4. *Let $\{a_k\}_k$ and $\{g_k\}_k$ be as above, and let A be such that*

$$(n/p+\alpha+m)A > n.$$

Then

$$\sum_k |g_k(z)|^A \preceq (1-|z|^2)^{A[(n/p+\alpha)-(n/p'+\alpha')]}.$$

This is a consequence of [Ma]; more precisely, it follows from Lemma 4.1(d) below with exponents $P := A[(n/p'+\alpha') - (n/p+\alpha)]$ and $Q := A(n/p+\alpha+m)$.

(a) Using in succession Hölder's inequality (with $1/p' + 1/q' = 1$), Lemma 3.4, and Lemma 1.3 for $|f_k(z)|$, we obtain

$$\begin{aligned}\|G\|_{p',\alpha'}^{p'} &= \int_{\mathbb{B}^n} \left|\sum_k \lambda_k g_k(z) f_k(z)\right|^{p'} (1-|z|^2)^{\alpha' p'-1}\, dm(z) \\ &\preceq \int_{\mathbb{B}^n} \left[\sum_k |g_k(z)|^{q'}\right]^{p'/q'} \left[\sum_k |\lambda_k|^{p'} |f_k(z)|^{p'}\right] (1-|z|^2)^{\alpha' p'-1}\, dm(z) \\ &\preceq \int_{\mathbb{B}^n} \sum_k |\lambda_k|^{p'} |f_k(z)|^p (1-|z|^2)^{\alpha p-1} \\ &\quad \times (1-|z|^2)^{p'(n/p+\alpha-n/p'-\alpha')+(p-p')(n/p+\alpha)+\alpha' p'-\alpha p}\, dm(z),\end{aligned}$$

which is controlled by $\sum_k |\lambda_k|^{p'}$, since

$$p'[(n/p+\alpha)-(n/p'+\alpha')]+(p-p')(n/p+\alpha)+\alpha' p'-\alpha p = 0.$$

(b) Let $1/p + 1/q = 1$. We first estimate

$$\begin{aligned}|G(z)| &= \left|\sum_k \lambda_k g_k(z) f_k(z)\right| \\ &\le \left(\sum_k |\lambda_k|^{(1-p'/p)q} |g_k(z)|^q\right)^{1/q} \left(\sum_k |\lambda_k|^{p'} |f_k(z)|^p\right)^{1/p}.\end{aligned}$$

Then, applying again Hölder's inequality with exponents p/p' and $p/(p-p')$, it follows that $\|G\|_{p',\alpha'}^{p'}$ is bounded by

$$\begin{aligned}&\int_{\mathbb{B}^n} \left[\sum_k |\lambda_k|^{(1-\frac{p'}{p})q} |g_k(z)|^q (1-|z|^2)^{q(\alpha'-\alpha+\frac{1}{p}-\frac{1}{p'})}\right]^{\frac{p'}{q}} \\ &\quad \times \left[\sum_k |\lambda_k|^{p'} |f_k(z)|^p (1-|z|^2)^{\alpha p-1}\right]^{\frac{p'}{p}} dm(z) \\ &\preceq \left[\int_{\mathbb{B}^n} \left[\sum_k |\lambda_k|^{\frac{p-p'}{p}q} |g_k(z)|^q (1-|z|^2)^{q(\alpha'-\alpha+\frac{1}{p}-\frac{1}{p'})}\right]^{\frac{p'p}{q(p-p')}} dm(z)\right]^{\frac{p-p'}{p}} \\ &\quad \times \left[\int_{\mathbb{B}^n} \sum_k |\lambda_k|^{p'} |f_k(z)|^p (1-|z|^2)^{\alpha p-1}\, dm(z)\right]^{\frac{p'}{p}}.\end{aligned}$$

The second factor is controlled by $\sum_k |\lambda_k|^{p'} \|f_k\|_{p,\alpha}^p \preceq \sum_k |\lambda_k|^{p'}$. Taking $\gamma, \delta \ge 1$ with $1/\gamma + 1/\delta = 1$ and applying Hölder's inequality with exponents $a = pp'/q(p-p')$ and $b = p'(p-1)/p(p'-1)$, using Lemma 3.4 we can bound the integral appearing in the first factor by

$$\int_{\mathbb{B}^n}\left[\sum_k |g_k(z)|^{\frac{qb}{\delta}}\right]^{\frac{a}{b}}\left[\sum_k |\lambda_k|^{p'}|g_k(z)|^{\frac{qa}{\gamma}}(1-|z|^2)^{qa(\alpha'-\alpha+\frac{p'-p}{pp'})}\right] dm(z)$$
$$\preceq \int_{\mathbb{B}^n}\sum_k |\lambda_k|^{p'}(1-|a_k|^2)^{(\frac{n}{p}+\alpha+m)\frac{qa}{\gamma}}$$
$$\times \frac{(1-|z|^2)^{qa(\alpha'-\alpha)-1+\frac{qa}{\delta}(\frac{n}{p}+\alpha-\frac{n}{p'}-\alpha')}}{|1-\bar{a}_k z|^{(\frac{n}{p'}+\alpha'+m)\frac{qa}{\gamma}}}\, dm(z).$$

We can choose $\delta > 1$ so that $qa(\alpha'-\alpha)-1+(qa/\delta)(n/p+\alpha-n/p'-\alpha') > -1$ and the integral is finite. Then, once more by Lemma 1.1, we see that the integral is bounded by $\sum_k |\lambda_k|^{p'}$. This concludes the case $\alpha > 0$.

We now turn to the case $\alpha = 0$. First we handle the special situation $p' = p$.

LEMMA 3.5. *For any* $\alpha' > 0$, $\mathrm{Int}(H^p) \subset \mathrm{Int}(B^p_{\alpha'})$.

Accepting this, suppose (p', α') satisfy (a) in Theorem 3.3; then there exists $\alpha_1 > 0$ such that $n/p+\alpha_1 < n/p'+\alpha'$. Likewise, if (p', α') satisfy (b) then there exists $\alpha_1 > 0$ such that $\alpha_1 < \alpha'$. In either case, applying Lemma 3.5 followed by the case $\alpha > 0$ of Theorem 3.3, we get $\mathrm{Int}(H^p) \subset \mathrm{Int}(B^p_{\alpha_1}) \subset \mathrm{Int}(B^{p'}_{\alpha'})$. □

Proof of Lemma 3.5. Define f_k, g_k, G_k, and G as before for $\{\lambda_k\}_k \in l^p$.

Case $p \le 1$. It is enough to prove that $\|G_k\|^p_{p,\alpha'} \le c$ for all k, that is,

$$\int_{\mathbb{B}^n}(1-|z|^2)^{\alpha' p-1}|g_k(z)|^p|f_k(z)|^p\, dm(z) \le c \quad \text{for all } k.$$

Applying Lemma 3.1 with $p = q$ and $\alpha = 0$, we see that it suffices to prove that, for an appropriate choice of the parameter m in g_k,

$$(1-|z|^2)^{\alpha' p-1}|g_k(z)|^p\, dm(z)$$

is a Carleson measure with Carleson norm independent of a_k. To see this, we apply Lemma 3.2(b) with $2\beta = n+\alpha' p + mp$. By the hypothesis on α' and m, we have $2\beta > n+mp > n$. Then, by Lemma 1.1(c),

$$\sup_{a,b\in\mathbb{B}^n}\int_{\mathbb{B}^n}\frac{(1-|b|^2)^{2\beta-n}(1-|z|^2)^{\alpha' p-1}(1-|a|^2)^{mp+n}}{|1-\bar{b}z|^{2\beta}|1-\bar{a}z|^{n+\alpha' p+mp}}\, dm(z)$$
$$\preceq \sup_{a,b\in\mathbb{B}^n}\frac{(1-|b|^2)^{\alpha' p+mp}(1-|a|^2)^{mp+n}}{|1-\bar{a}b|^{n+\alpha' p+mp}}\{\min(1-|a|^2, 1-|b|^2)\}^{-mp},$$

which is finite since $\max(1-|a|^2, 1-|b|^2) \preceq |1-\bar{a}b|$.

Case $1 < p \le 2$. By Hölder's inequality,

$$\int_{\mathbb{B}^n}(1-|z|^2)^{\alpha' p-1}|G(z)|^p\, dm(z)$$
$$\le \sum_j |\lambda_j|^p\int_{\mathbb{B}^n}(1-|z|^2)^{\alpha' p-1}\left(\sum_k |g_k(z)|^q\right)^{p/q}|f_j(z)|^p\, dm(z),$$

where $1/p + 1/q = 1$. Again by Lemmas 3.1 and 3.2(b), it is enough to consider

$$\sup_{b\in\mathbb{B}^n} \int_{\mathbb{B}^n} \frac{(1-|b|^2)^{2\beta-n}}{|1-\bar{b}z|^{2\beta}} (1-|z|^2)^{\alpha' p-1} \left(\sum_k |g_k(z)|^q\right)^{p/q} dm(z). \tag{1}$$

Since $p/q = p-1 \le 1$, this integral is bounded by

$$S := \sum_k \int_{\mathbb{B}^n} \frac{(1-|b|^2)^{2\beta-n}(1-|z|^2)^{\alpha' p-1}(1-|a_k|^2)^{mp+n}}{|1-\bar{b}z|^{2\beta}|1-\bar{a}_k z|^{n+\alpha' p+mp}}\, dm(z).$$

Choosing again $2\beta = n+\alpha' p + mp > n + mp > n$ and applying Lemma 1.1(c), we get $S \le S_1 + S_2$, where

$$S_1 := \sum_{k:1-|a_k|^2\le 1-|b|^2} \frac{(1-|b|^2)^{\alpha' p+mp}(1-|a_k|^2)^n}{|1-\bar{a}_k b|^{n+\alpha' p+mp}},$$

$$S_2 := \sum_{k:1-|a_k|^2> 1-|b|^2} \frac{(1-|b|^2)^{\alpha' p}(1-|a_k|^2)^{mp+n}}{|1-\bar{a}_k b|^{n+\alpha' p+mp}},$$

so $\sup_{b\in\mathbb{B}^n} S_1 \le K(\{a_k\}, \alpha' p + mp, mp+n)$ and $\sup_{b\in\mathbb{B}^n} S_2 \preceq K(\{a_k\}, \alpha' p, n)$. Because $\{a_k\} \in \operatorname{Int}(H^p)$, $p > 1$, we know from Theorem 2.2 of [Th1] that $\sum_k (1-|a_k|^2)^n \delta_{a_k}$ is an n-Carleson measure, so Lemma 3.2(a) allows us to conclude that both quantities are finite.

Case $2 < p < \infty$. As before, it is enough to consider (1). We first apply Hölder's inequality with exponents $(p-1)/(p-2)$ and $p-1$ and then apply Lemma 3.4 to yield

$$\begin{aligned}\left(\sum_k |g_k(z)|^q\right)^{p-1} &\le \left(\sum_k |g_k(z)|^{p/2(p-2)}\right)^{p-2} \sum_k |g_k(z)|^{p/2} \\ &\preceq (1-|z|^2)^{-\alpha' p/2} \sum_k |g_k(z)|^{p/2},\end{aligned}$$

so that in this case the integral in (1) is controlled by

$$S := \sup_{b\in\mathbb{B}^n} \sum_k \int_{\mathbb{B}^n} \frac{(1-|b|^2)^{2\beta-n}(1-|z|^2)^{\frac{\alpha' p}{2}-1}(1-|a_k|^2)^{\frac{1}{2}(mp+n)}}{|1-\bar{b}z|^{2\beta}|1-\bar{a}_k z|^{\frac{1}{2}(n+\alpha' p+mp)}}\, dm(z).$$

This time choose $2\beta = \frac{1}{2}(n+\alpha' p+mp) > \max(\frac{1}{2}(mp+n), n)$, which requires $m > n/p - \alpha'$. As above, $S \le S_1 + S_2$, where

$$S_1 := \sum_{k:1-|a_k|^2\le 1-|b|^2} \frac{(1-|b|^2)^{\frac{1}{2}(-n+\alpha' p+mp)}(1-|a_k|^2)^n}{|1-\bar{a}_k b|^{\frac{1}{2}(n+\alpha' p+mp)}},$$

$$S_2 := \sum_{k:1-|a_k|^2> 1-|b|^2} \frac{(1-|b|^2)^{\frac{\alpha' p}{2}}(1-|a_k|^2)^{\frac{1}{2}(mp+n)}}{|1-\bar{a}_k b|^{\frac{1}{2}(n+\alpha' p+mp)}},$$

and requiring finally $mp \ge n$, we conclude as before.

Case $p = \infty$. We can estimate $\sup_{z\in\mathbb{B}^n}(1-|z|^2)^{\alpha'}|G(z)|$ by a straightforward application of Lemma 3.4. □

REMARK. We have actually shown a slightly stronger result—namely, that if functions f_k exist with the properties mentioned at the very beginning of the proof (i.e., interpolating values that are zero at all but one of the points), then $\{a_k\} \in \operatorname{Int}(B^{p'}_{\alpha'})$ for (α', p') verifying (a) or (b).

Notice that the proof of Theorem 3.3 cannot be used to show the intuitive conjecture that $\operatorname{Int}(H^p) \subset \operatorname{Int}(H^{p'})$ for $p' < p$. It is also interesting to note that the proof uses only that $\sum_k (1-|a_k|^2)^n \delta_{a_k}$ is an n-Carleson measure in the case $\{a_k\} \in \operatorname{Int}(H^p)$, $p > 1$, which follows the arguments of [CG] and is much easier to prove than for $p = 1$ (see [Th2, Sec. 2.2]), while the case $p < 1$ is not known to us for $n \geq 2$.

4. Sufficient Conditions

In this section we give sufficient conditions for a sequence $\{a_k\}_k$ to be B^p_α-interpolating in terms of the values $K(\{a_k\}, p, q)$ defined in the previous section.

LEMMA 4.1 [Ma]. *The following conditions are equivalent*:

(a) *$\{a_k\}_k$ is the union of a finite number of separated sequences*;
(b) *$K(\{a_k\}, p, q) < +\infty$ for all $q > n$ and all $p \leq q$*;
(c) *there exist $q \geq n$, p such that $K(\{a_k\}, p, q) < +\infty$*;
(d) *for all $p > 0$ and all $q > n$,*

$$\sup_{z\in\mathbb{B}^n} \sum_{k=1}^{\infty} \frac{(1-|z|^2)^p (1-|a_k|^2)^q}{|1-\bar{a}_k z|^{p+q}} < +\infty;$$

(e) *$\sum_k (1-|a_k|^2)^q \delta_{a_k}$ is a q-Carleson measure for $q > n$.*

We will consider, given a sequence $\{a_k\}_k$, the restriction map $T(f) = \{f(a_k)\}_k$. Notice that

$$\|T(f)\|^p_{l_{n/p+\alpha}} = \sum_k |f(a_k)|^p (1-|a_k|^2)^{n+\alpha p} = \int_{\mathbb{B}^n} |f(z)|^p \, d\mu(z),$$

where $\mu = \sum_k (1-|a_k|^2)^{n+\alpha p} \delta_{a_k}$. From Lemma 3.1 and Lemma 4.1 we thus deduce that T maps B^p_α boundedly on $l_{n/p+\alpha}(\{a_k\})$ if and only if $\{a_k\}_k$ is a finite union of separated sequences. This gives a partial converse to Proposition 1.6.

The first result we give in this section deals with the case $p = 1$.

THEOREM 4.2. *Let $\{a_k\}_k$ be a separated sequence in $\mathbb{B}^n$. If there exists an $m > 0$ such that $K(\{a_k\}, m, n+\alpha) < 1$, then $\{a_k\}_k$ is B^1_α-interpolating.*

Observe that, by Theorem 3.3, this implies that $\{a_k\}_k \in \operatorname{Int}(B^{p'}_{\alpha'})$ whenever $p' > 1$ and $n + \alpha < n/p' + \alpha'$.

Proof. For the case $\alpha = 0$, see [Th1]. Consider $T\colon B^1_\alpha \to l^1_{n+\alpha}(\{a_k\})$ defined above. In order to show that T is onto, we define, given $v = \{v_k\} \in l^1_{n+\alpha}(\{a_k\})$, the *approximate extension*

$$E(v)(z) = \sum_{k=1}^{\infty} v_k \left(\frac{1-|a_k|^2}{1-\bar{a}_k z}\right)^{n+\alpha+m}.$$

Using Lemma 1.1, it is immediately verified that $E(v)$ is in B^1_α:

$$\begin{aligned}
\|E(v)\|_{1,\alpha} &\le \int_{\mathbb{B}^n} (1-|z|^2)^{\alpha-1} \sum_{k=1}^{\infty} |v_k| \frac{(1-|a_k|^2)^{n+\alpha+m}}{|1-\bar{a}_k z|^{n+\alpha+m}}\, dm(z) \\
&\le \sum_{k=1}^{\infty} |v_k| (1-|a_k|^2)^{n+\alpha+m} \int_{\mathbb{B}^n} \frac{(1-|z|^2)^{\alpha-1}}{|1-\bar{a}_k z|^{n+\alpha+m}}\, dm(z) \simeq \|v\|.
\end{aligned}$$

On the other hand, $TE - \mathrm{Id}$, regarded as an operator on $l^1_{n+\alpha}(\{a_k\})$, has norm strictly smaller than 1:

$$\begin{aligned}
\|TE(v) - v\| &= \sum_{k=1}^{\infty} (1-|a_k|^2)^{n+\alpha} |(TE(v))_k - v_k| \\
&\le \sum_{k=1}^{\infty} (1-|a_k|^2)^{n+\alpha} \sum_{j:j\neq k} |v_j| \left(\frac{1-|a_j|^2}{|1-\bar{a}_k a_j|}\right)^{n+\alpha+m} \\
&= \sum_{j=1}^{\infty} (1-|a_j|^2)^{n+\alpha} |v_j| \left(\sum_{k:k\neq j} \frac{(1-|a_j|^2)^m (1-|a_k|^2)^{n+\alpha}}{|1-\bar{a}_k a_j|^{n+\alpha+m}} \right) \\
&< \|v\|.
\end{aligned}$$

Hence the series

$$\mathrm{Id} + \sum_{k=1}^{\infty} (TE - \mathrm{Id})^k$$

converges and defines an inverse to TE. The operator $\tilde{E} = E(TE)^{-1}$ provides, finally, the inverse of T. □

Notice that, by the invariance under automorphisms of the B^p_α-interpolating sequences, the hypothesis in Theorem 4.2 can be replaced by the seemingly weaker assumption of the existence of an automorphism φ_ζ such that

$$K(\{\varphi_\zeta(a_k)\}, m, n+\alpha) < 1.$$

Corollary 4.3.

(a) *Let $\{a_k\}_k$ be a separated sequence. There exists $\alpha > 0$ such that $\{a_k\}_k$ is B^1_α-interpolating.*
(b) *Let $\alpha > 0$. There exists a $\delta \in (0,1)$ such that any sequence $\{a_k\}_k$ verifying $d_G(a_j, a_k) \ge \delta$ for any $j \neq k$ is B^1_α-interpolating.*

Proof. Lemma 4.1 shows that

$$K(\{a_k\}, n+\alpha/2, n+\alpha/2) = \sup_k \sum_{j:j\neq k} \left(1 - d_G^2(a_j, a_k)\right)^{n+\alpha/2} < +\infty.$$

If δ is such that $d_G(a_j, a_k) \geq \delta$, we have

$$\sup_{k\in\mathbb{Z}_+} \sum_{j:j\neq k} \left(1 - d_G^2(a_j, a_k)\right)^{n+\alpha} \leq (1-\delta^2)^{\alpha/2} K(\{a_k\}, n+\alpha/2, n+\alpha/2).$$

In both cases (a) and (b), we can finish the proof by choosing (respectively) α or δ so that $(1-\delta^2)^{\alpha/2} K(\{a_k\}, n+\alpha/2, n+\alpha/2) < 1$ and then applying Theorem 4.2. □

Theorem 4.2, along with the following lemma, provides another characterization of the sequences appearing in Lemma 4.1.

MILLS'S LEMMA (cf. [Ga] or [Th1]). *Let A_{jk}, $j, k \in \mathbb{Z}_+$, be nonnegative real numbers such that $A_{jk} = A_{kj}$ and $A_{jj} = 0$ for any j and k. If $\sup_{k\in\mathbb{Z}_+} \sum_{j\in\mathbb{Z}_+} A_{jk} = M < +\infty$, then there exists a partition $\mathbb{Z}_+ = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$, such that*

$$\sup_{k\in S_i} \sum_{j\in S_i} A_{jk} \leq \frac{M}{2}, \quad i = 1, 2.$$

COROLLARY 4.4. *A sequence $\{a_k\}_k$ is the union of a finite number of separated sequences if and only if it is the union of a finite number of B_α^1-interpolating sequences.*

Proof. The reverse implication is given directly by Proposition 1.6. To see the direct one, we apply (b) of Lemma 4.1 with $p = q = n+\alpha$ and Mills's lemma with

$$A_{jk} = \frac{(1-|a_k|^2)^{n+\alpha}(1-|a_j|^2)^{n+\alpha}}{|1-\bar{a}_k a_j|^{2(n+\alpha)}}.$$

For any $N \in \mathbb{Z}_+$ one can split $\{a_k\}_k$ into 2^N sequences $\{b_k^l\}_k$, $l = 1, \dots, 2^N$, such that $K(\{b_k^l\}, n+\alpha, n+\alpha) < (1/2^N) K(\{a_k\}, n+\alpha, n+\alpha)$. Taking N sufficiently large, this term becomes smaller than 1, which (by Theorem 4.2) yields the stated result. □

With the same methods it is also possible to obtain sufficient conditions for a sequence to be B_α^p-interpolating, $p > 1$. However, these conditions are not so well adapted to the nature of the B_α^p spaces, in the sense that they are symmetrical in p and the conjugated exponent q. In the proof, which follows [Th1, Prop. 3.2], we will use the duality between B_α^p spaces. Consider the product given by

$$\langle f, g\rangle =: \int_{\mathbb{B}^n} f(z)\overline{g(z)}(1-|z|^2)^{\alpha p-1}\, dm(z).$$

Using Lemma 1.1 and some standard results for classical Bergman spaces (see [Am, Lemme 1.2.3] and [Ru1, Chap. 7]), it is easy to prove that when $1 < p < \infty$

the dual space of B^p_α with respect to this product is B^q_β, where $1/p + 1/q = 1$ and $\beta q = \alpha p$. Furthermore, there is a reproducing kernel for B^p_α functions, namely,

$$K_z(\zeta) = \frac{\Gamma(n+\alpha p)}{\Gamma(n+1)\Gamma(\alpha p)} \frac{1}{(1-\zeta \bar{z})^{n+\alpha p}}.$$

Theorem 4.5. *Let $1 < p < \infty$ and let q be its conjugated exponent. Let $\beta = \alpha p/q$. If there exist $c_1, c_2 > 0$ such that $c_1 c_2 < 1$ and*

$$K\left(\{a_k\}, \frac{n}{p}+\alpha, \frac{n}{q}+\beta\right) \le c_1^q, \quad K\left(\{a_k\}, \frac{n}{q}+\beta, \frac{n}{p}+\alpha\right) \le c_2^p,$$

then $\{a_k\}_k$ is B^p_α-interpolating.

Notice that $n/p + \alpha \le n$ yields results that do not follow from Theorem 4.2.

Proof. Given $\{v_k\} \in l^p_{n/p+\alpha}(\{a_k\})$, take the approximate extension

$$E(v)(z) = \sum_{k=1}^{\infty} v_k \left(\frac{1-|a_k|^2}{1-\bar{a}_k z}\right)^{n+\alpha p}.$$

Using the duality described above, the reproducing kernel for B^p_α, and Lemma 4.1 with $\mu = \sum_k (1-|a_k|^2)^{n+\beta q}\delta_{a_k}$, one has that $\|E(v)\|_{p,\alpha} \le c\|v\|_{p,n/p+\alpha}$.

On the other hand, if T denotes the operator on B^p_α associated to $\{a_k\}_k$, we have $\|TE - \mathrm{Id}\| < 1$, since

$$\begin{aligned}
&\sum_{k=1}^{\infty}(1-|a_k|^2)^{n+\alpha p}|(TE(v)-v)_k|^p \\
&\quad \le \sum_{k=1}^{\infty}\left(\sum_{j:j\ne k}\frac{(1-|a_k|^2)^{n/p+\alpha}(1-|a_j|^2)^{n/q+\beta}}{|1-\bar{a}_k a_j|^{n+\alpha p}}\right)^{p/q} \\
&\quad\quad \times \sum_{j:j\ne k}\frac{(1-|a_k|^2)^{n/p+\alpha}(1-|a_j|^2)^{n/q+\beta}}{|1-\bar{a}_k a_j|^{n+\alpha p}}(1-|a_j|^2)^{n+\alpha p}|v_j|^p \\
&\quad \le c_1^p \sum_{j=1}^{\infty}(1-|a_j|^2)^{n+\alpha p}|v_j|^p \sum_{k:k\ne j}\frac{(1-|a_j|^2)^{n/q+\beta}(1-|a_k|^2)^{n/p+\alpha}}{|1-\bar{a}_k a_j|^{n+\alpha p}} \\
&\quad \le (c_1 c_2)^p \|v\|_{p,n/p+\alpha}.
\end{aligned}$$

This shows that TE is invertible and, as in Theorem 4.2, that T is onto. □

Similar results to Corollaries 4.3 and 4.4 can be derived from Theorem 4.5, with some restrictions on the values p and α.

Corollary 4.6. *Let $p > 1$, let q be the conjugated exponent, and denote $A(\alpha, p) = (n+\alpha p)\min(1/p, 1/q)$.*

(a) *If there exists $c_0 < 1$ such that*

$$K(\{a_k\}, A(\alpha,p), A(\alpha,p)) \le 2^{-(n+\alpha p)|1-2/p|}c_0,$$

then $\{a_k\}_k$ is B^p_α-interpolating.

(b) *Let $\{a_k\}_k$ be separated. Then there exists an $\alpha > 0$ such that $\{a_k\}$ is B^p_α-interpolating.*

(c) *Let $A(\alpha, p) > n$. Then there exists a $\delta > 0$ such that any sequence $\{a_k\}_k$ verifying $d_G(a_j, a_k) \geq \delta$ for all $j \neq k$ is B^p_α-interpolating.*

(d) *Let $A(\alpha, p) > n$. Then $\{a_k\}_k$ is a finite union of separated sequences if and only if it is a finite union of B^p_α-interpolating sequences.*

Part (c), like (b) of Corollary 4.3, is a particular case of a theorem of Rochberg which actually shows that the result holds for any $p > 0$ and $\alpha > 0$ (see [Ro, p. 231]).

Proof. Part (a) is the analog of [Th1, Cor. 3.3]. Parts (b), (c), and (d) are derived from (a), as in Corollaries 4.3 and 4.4. □

In view of Theorem 1.10, we see that the conclusions of Theorems 4.2 and 4.5 (and their corollaries) still hold if their hypotheses are verified only for the original sequence deprived of a finite number of points. This, however, is still far from giving us a sufficient condition in terms of density.

5. Appendix

Given a sequence $\{a_k\}_k \subset \mathbb{B}^1$, consider its upper uniform density

$$D(\{a_k\}) = \limsup_{r\to 1} \sup_{z\in\mathbb{B}^1} \frac{\sum_{1/2<|\varphi_z(a_k)|<r} \log(1/|\varphi_z(a_k)|)}{\log(1/(1-r))}.$$

LEMMA 5.1. *Let $\{a_k\}_k \subset \mathbb{B}^1$. Then $\{a_k\} \in \operatorname{Int}(B^p_\alpha)$ if and only if $D(\{a_k\}) < \alpha$.*

Proof. Assume first that $D(\{a_k\}) < \alpha$ and define $\epsilon = 1/2(\alpha - D(\{a_k\}))$. There exist functions $f_k \in B^\infty_{\alpha-\epsilon}$ and $C > 0$ with $\|f_k\|_{\infty,\alpha-\epsilon} \leq C$ for all k and $f_k(a_j) = \delta_{jk}(1 - |a_k|^2)^{-\alpha+\epsilon}$ (see [Se1, p. 34]). As in Theorem 3.3 (see the remark at the end of the proof), this implies that $\{a_k\}_k$ is $B^{p'}_{\alpha'}$-interpolating for any (p', α') such that $\alpha' > \alpha - \epsilon$, and in particular for α and p.

Assume now that $\{a_k\}_k$ is B^p_α-interpolating.

LEMMA 5.2. *Let $z \in \mathbb{B}^1$ such that $d(z, \{a_k\}) \geq \delta_0$. Then there exists an $f \in B^p_\alpha$ with $f(a_k) = 0$ for all k, $f(z) = 1$, and $\|f\|_{p,\alpha} \leq 1 + M\delta_0^{-1}$, where M denotes the constant of interpolation of $\{a_k\}$.*

Proof. By invariance under automorphisms we can suppose that $z = 0$. There exists an $f_0 \in B^p_\alpha$ such that $f_0(a_1) = 1/a_1$, $f_0(a_j) = 0$ for all $j \geq 2$, and $\|f_0\|_{p,\alpha} \leq M\delta_0^{-1}$. Then the function $f(z) = 1 - zf(z)$ satisfies all the requirements. □

We now resume the proof of Lemma 5.1. Since $M^p_p(f, r) = \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$ is an increasing function of r,

$$\|f\|_{p,\alpha}^p = \int_0^1 2\pi M_p^p(f,r)(1-r^2)^{\alpha p-1} r\,dr \geq M_p^p(f,r_0)\frac{\pi}{\alpha p}(1-r_0^2)^{\alpha p}.$$

Hence, $M_p^p(f,r) \preceq \|f\|_{p,\alpha}^p(1-r^2)^{-\alpha p}$. By Jensen's inequality,

$$\exp\left(p\int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi}\right) \leq M_p^p(f,r) \leq C(1-r^2)^{-\alpha p},$$

and thus

$$\int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} \leq \frac{1}{p}\log C + \alpha\log\left(\frac{1}{1-r^2}\right).$$

From Jensen's formula it now follows that

$$\sum_{1/2<|\varphi_z(a_k)|<r} \log\frac{r}{|\varphi_z(a_k)|} \leq \frac{1}{p}\log C + \alpha\log\left(\frac{1}{1-r^2}\right),$$

and therefore $D(\{a_k\}) \leq \alpha$. To see that the inequality is strict, take a sequence $\{a_k'\}_k$ and δ_0 such that $d(a_k, a_k') < \delta_0$ for all k and $(1+\delta_0)D(\{a_k\}) \leq D(\{a_k'\})$ (see the proof of [Se1, Lemma 6.6]). An application of the argument above to the sequence $\{a_k'\}_k$ shows finally that $D(\{a_k\}) < \alpha$. □

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M. Jevtić
Matematički Fakultet
Studentski Trg 16
11000 Beograd
Yugoslavia

XPMFM39@yubgss21.rzs.bg.ac.yu

X. Massaneda
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra
Spain

xavier@manwe.mat.uab.es

P. J. Thomas
UFR MIG
Université Paul Sabatier
31062 Toulouse
France

pthomas@cict.fr