A Characterization of the Finite Multiplicity of a CR Mapping

YIFEI PAN

1. Introduction

In this paper we give a characterization of the finite multiplicity of a CR mapping between real analytic hypersurfaces. The finite multiplicity of a CR mapping was defined algebraically by Baouendi and Rothschild in [BR2] (see the definition below). We will prove that, under certain conditions on hypersurfaces, the finite multiplicity of a CR mapping is equivalent to the finiteness of the map's preimage. More precisely, we prove the following.

THEOREM 1. *Suppose that M_1, M_2 are real analytic hypersurfaces of essential finite type in $\mathbf{C}^n$, and that M_2 contains no complex variety of positive dimension. Then a smooth CR mapping $f: M_1 \to M_2$ is of finite multiplicity at $z_0 \in M_1$ if and only if $f^{-1}(f(z_0))$ is finite.*

The proof of Theorem 1 relies on the real analyticity result of [BR2] and the following Theorem 2, which we shall prove. In [BR2], Baouendi and Rothschild proved that a smooth CR mapping of finite multiplicity from a real analytic hypersurface to a real analytic hypersurface of essential finite type is real analytic. This result with the proof of Theorem 1 implies the following.

COROLLARY 1. *A smooth CR mapping of finite multiplicity between real analytic hypersurfaces of essential finite type is the restriction of a locally proper holomorphic mapping in $\mathbf{C}^n$.*

THEOREM 2. *Suppose that $f: M_1 \to M_2$ is a smooth CR mapping between real analytic hypersurfaces in $\mathbf{C}^n$. Suppose further that M_1 is essentially finite and that M_2 contains no complex variety of positive dimension. If $f^{-1}(f(z_0)) \setminus \{z_0\}$ is discrete for a point $z_0 \in M_1$, then f extends holomorphically to a neighborhood of z_0 in $\mathbf{C}^n$.*

A simple example shows that the condition that M_2 contains no complex variety of positive dimension is necessary in Theorems 1 and 2.

Received September 26, 1995. Final revision received March 18, 1996.
Supported in part by a grant from Purdue Research Foundation.
Michigan Math. J. 43 (1996).

COROLLARY 2. *Suppose that $f: M_1 \to M_2$ is a smooth CR mapping between real analytic hypersurfaces of finite type of D'Angelo in $\mathbf{C}^n$. If $f^{-1}(f(z_0)) \setminus \{z_0\}$ is discrete for a point $z_0 \in M_1$, then f extends holomorphically to a neighborhood of z_0 in $\mathbf{C}^n$.*

A well-known problem in the study of real analyticity of CR mappings is whether every smooth CR mapping between real analytic hypersurfaces of finite type of D'Angelo in $\mathbf{C}^n$ is real analytic.

COROLLARY 3. *Let $f: M_1 \to M_2$ be a smooth CR mapping between real analytic hypersurfaces of finite type of D'Angelo in $\mathbf{C}^n$. If f is real analytic on $M_1 \setminus \{p\}$, then f is also real analytic at p.*

This can be viewed as a "removable singularity theorem" for real analyticity of CR mappings. As another corollary of the proof of Theorem 2, one has the following.

COROLLARY 4. *A finite-to-one smooth CR mapping from a real analytic hypersurface of essential finite type to another real analytic hypersurface is real analytic.*

Here a map $f: M_1 \to M_2$ is said to be *finite-to-one* if $f^{-1}(q)$ is finite for any $q \in M_2$. The proofs of these results depend on the work of Baouendi and Rothschild [BR2] and Diederich and Fornaess [DF] on real analyticity; the Hopf lemma of [BR4]; and the work of Tumonov [T] on holomorphic extension of CR functions. However, we will directly prove the holomorphic extension of CR mappings whenever their works do not apply. For earlier results, see [BBR; BJT; BB; B; DW; L; Pi1; Pi2]. Theorem 1 will be proved in Section 2; Theorem 2, along with its corollaries, will be proved in Section 3. The work of this paper is in part inspired by a paper of Pinchuk [Pi2].

ACKNOWLEDGMENT. The author thanks the referee for very helpful suggestions.

2. Proof of Theorem 1

To prove Theorem 1, we first recall some basic definitions. Let M be a real analytic hypersurface in $\mathbf{C}^n$ containing the origin and defined locally by $\rho(z, \bar{z}) = 0$, $\nabla\rho \neq 0$, $z \in \mathbf{C}^n$, where ρ is a real-valued analytic function, $\rho(0) = 0$. As introduced in [BJT], M is said to be *essentially finite* at 0 if for any sufficiently small $z \in \mathbf{C}^n \setminus \{0\}$ there exists an arbitrarily small $\zeta \in \mathbf{C}^n$ satisfying $\rho(z, \zeta) \neq 0$ and $\rho(0, \zeta) = 0$. We point out that if M does not contain any complex variety of positive dimension through 0, then M is essentially finite at 0. Consequently, a real analytic hypersurface of finite type of D'Angelo is essentially finite. The finite multiplicity of a CR mapping is introduced by Baouendi and Rothschild in [BR2] as follows. If $f: M_1 \to M_2$ is a smooth CR mapping between two smooth real analytic hypersurfaces in $\mathbf{C}^n$, then there exist n CR functions $f_1, \dots, f_n$ defined on M_1 such that $f = (f_1, \dots, f_n)$. On the other hand, if j is a smooth CR function defined on M_1 near 0, and if there exists a formal holomorphic power series

$J(Z) = \sum a_\alpha Z^\alpha$ in n indeterminates such that $U \in u \to Z(u) \in \mathbf{C}^n$ (U an open neighborhood of 0 in $\mathbf{R}^{2n-1}$, $Z(0) = 0$) is a parameterization of M_1, then the Taylor series of $j(Z(u))$ at 0 is given by $J(Z(u))$. We can choose holomorphic coordinates Z such that $\rho(Z, 0) = \alpha(Z)Z_n$, $\alpha(0) \neq 0$. With $Z = (z', z_n)$ and $z' = (z_1, \ldots, z_{n-1})$, the mapping f is said to be of *finite multiplicity* at 0 if

$$\dim_{\mathbf{C}} O[[z']]/(F(z', 0)) < \infty, \tag{1}$$

where $F(z', 0)$ is the ideal generated by $F_1(z', 0), \ldots, F_n(z', 0)$, the power series associated to the CR functions $f_1, \ldots, f_n$; $O[[z']]$ is the ring of formal power series in $n - 1$ indeterminates; and the dimension is taken in the sense of vector spaces.

After another holomorphic change of coordinates near 0, we may further assume that M_1 is given by the equations

$$\Im z_n = \psi(z', \bar{z}', \Re z_n), \qquad \psi(0) = d\psi(0) = \psi(z, 0, 0) = \psi(0, \zeta, 0) = 0,$$

with $(z', z_n) \in \mathbf{C}^{n-1} \times \mathbf{C}$. We assume that M_2 is another real analytic hypersurface defined by

$$\Im z_n = \phi(z', \bar{z}', \Re z_n), \qquad \phi(0) = d\phi(0) = \phi(z, 0, 0) = \phi(0, \zeta, 0) = 0,$$

with $(z', z_n) \in \mathbf{C}^{n-1} \times \mathbf{C}$. Let $f = (f', f_n)$ be a CR map from M_1 to M_2 with $f(0) = 0$. We say that f_n is the *normal component* of f, and call z_n the *normal direction* at 0.

Proof of Theorem 1. We actually prove that if $f: M_1 \to M_2$ is a smooth CR mapping of finite multiplicity between real analytic hypersurfaces of essential finite type, then $f^{-1}(0)$ is finite. By Theorem 1 of [BR1], f is real analytic at 0. Let $F = (F_1(z), \ldots, F_n(z))$ be the holomorphic extension of f to $\mathbf{C}^n$ near 0. If $f^{-1}(0)$ is not finite, then $S = F^{-1}(0)$ must be a complex variety of positive dimension. By Theorem 4 of [BR3], we have

$$\frac{\partial F_n}{\partial z_n}(0) \neq 0. \tag{2}$$

We claim that S lies in M_1. Indeed, by (2),

$$\Im F_n(z) - \phi(F'(z), \bar{F}'(z), \Re F_n(z)) = 0$$

defines a real analytic hypersurface in $\mathbf{C}^n$ that clearly coincides with M_1 near the origin where $F' = (F_1, \ldots, F_{n-1})$. This proves the claim.

Now we let S' be any complex curve in S parameterized by

$$z(\zeta) = (z_1(\zeta), \ldots, z_n(\zeta))$$

passing through 0. We claim that $z_n(\zeta) \equiv 0$. Indeed, in the coordinates chosen above, by Lemma 3.7 of [BR2] we have

$$F_n(z) = z_n G(z).$$

By (2), we see $G(0) \neq 0$. On S', it follows that $F_n(z(\zeta)) = z_n(\zeta)G(z(\zeta)) = 0$, which implies $z_n(\zeta) = 0$. Therefore, $F_1(z', 0), \ldots, F_n(z', 0)$ have common zeros near 0 and hence the dimension

$$\dim_{\mathbf{C}} O[[z']]/(F_1(z',0),\ldots,F_n(z',0))$$

is infinite, a contradiction to the finite multiplicity of f at 0.

As proved above, S lies in M_1 and hence $f^{-1}(0) = F^{-1}(0)$. This means that F is a locally proper holomorphic mapping, which gives a proof of Corollary 1.

Now we prove that, under the conditions in Theorem 1, if $f^{-1}(0)$ is finite then f is of finite multiplicity. Indeed, by Theorem 2 (whose proof will not depend on Theorem 1), f is real analytic at 0. As before, let F be the holomorphic extension of f. We notice that $F_n(z) \not\equiv 0$ since M_2 contains no complex variety of positive dimension and since, by Theorem 4 of [BR3], f is of finite multiplicity at 0. This could also be proved directly. Indeed, by Theorem 4 of [BR3], (2) holds. As above, this implies that $F^{-1}(0)$ is finite; therefore F is locally proper, which implies the finite multiplicity of f. $\square$

We close this section with an example. Let $M_1 = \{\Im z_3 = |z_1|^2 + |z_2|^2\}$ and $M_2 = \{\Im z_3 = |z_1|^2 - |z_2|^2\}$. Consider $f = (g, g, 0)$ where g is the restriction of $e^{-1/z_3^{1/3}}$, which is holomorphic in $\Im z_3 > 0$ and smooth up to the boundary. It is easy to see that $f^{-1}(0) = 0$ but f is not of finite multiplicity. Note that M_2 contains a complex curve and that M_1, M_2 are both of essential finite type.

3. Proof of Theorem 2

Following Tumanov [T], we say that a real hypersurface M_1 is *minimal* at z_0 if there is no germ of complex holomorphic hypersurface contained in M_1 and passing through z_0. By a theorem of Trepreau [Tr], f extends holomorphically to one side of M_1. The main result of [BBR; BR2; DF] can be stated as follows.

THEOREM. *Let M_1 be a real analytic hypersurface that is essentially finite at* $0 \in M_1$. *If M_2 is another real analytic hypersurface and $f\colon M_1 \to M_2$ is a smooth CR mapping with $f(0) = 0$ and $\frac{\partial f_n}{\partial z_n}(0) \neq 0$, then f extends holomorphically to a neighborhood of* 0 *in* $\mathbf{C}^n$.

This theorem has many important applications to global proper holomorphic mappings. For example, it was proved in [BR2; DF] that every proper holomorphic mapping between bounded pseudoconvex domains with real analytic boundaries extends holomorphically across the boundary. In [BR3] it was shown that, if the normal component of f is not flat (i.e., if there exists a number $k > 0$ such that $\frac{\partial^k f_n}{\partial z_n^k}(0) \neq 0$) in the normal direction at 0, then the condition $\frac{\partial f_n}{\partial z_n}(0) \neq 0$ holds automatically. As an application of this result, it was proved in [HP] that the unique continuation property holds for proper holomorphic mappings between bounded domains with real analytic boundaries. This result in turn proves that every proper holomorphic mapping between bounded real analytic domains that is smooth up to the boundary extends holomorphically across the boundary.

In order to prove Theorem 2, we need the following lemmas. First we recall the definition of a correspondence. Let Ω be a domain in $\mathbf{C}^n$, and let $f\colon \Omega \to \mathbf{C}^n$ be a holomorphic mapping. Denote the graph of f by

$$\Gamma_f = \{ (z, w) : w = f(z), z \in \Omega \}.$$

Let

$$B((z_0, w_0), \varepsilon) = \{ (z, w) \in \mathbf{C}^n \times \mathbf{C}^n : |z - z_0| < \varepsilon, |w - w_0| < \varepsilon \}.$$

We say that f extends as a *correspondence* to a neighborhood of (z_0, w_0) if there exists $\varepsilon > 0$ and a pure n-dimensional subvariety $V \subset B((z_0, w_0), \varepsilon)$ such that

$$\Gamma_f \cap B((z_0, w_0), \varepsilon) \subset V \cap B((z_0, w_0), \varepsilon).$$

Now we state a lemma due to Bedford and Bell [BB].

LEMMA 1. *Let Ω be a bounded domain in $\mathbf{C}^n$ with smooth boundary near $z_0 \in \partial\Omega$, and let $f: \Omega \to \mathbf{C}^n$ be a holomorphic mapping that is C^∞ smooth up to the boundary of Ω near z_0. If f extends as a correspondence at z_0,, then f extends holomorphically to a neighborhood of z_0 in $\mathbf{C}^n$.*

Let M_1 and M_2 be smooth real hypersurfaces in $\mathbf{C}^n$, and let Ω_1, Ω_2 be two domains in $\mathbf{C}^n$ with defining functions r_i for $i = 1, 2$ such that $\nabla r_i \neq 0$ on $\bar{\Omega}_i$ for $i = 1, 2$. Set $\Omega_i^+ = \{ z \in \Omega_i : r_i(z) > 0 \}$ and $\Omega_i^- = \{ z \in \Omega_i : r_i(z) < 0 \}$ for $i = 1, 2$. If $F: \Omega_1^- \to \mathbf{C}^n$ is a holomorphic mapping, we denote by Jac F the determinant of the Jacobian matrix of F.

As will become clear, in order to prove Theorem 2 one need only consider the case when w_0 is a minimal, but not minimally convex, point in the sense of [BR4]. For this matter, we prove the following result.

LEMMA 2. *Let $f: M_1 \to M_2$ be a smooth CR mapping between smooth real hypersurfaces M_1, M_2 in $\mathbf{C}^n$. Suppose that f extends holomorphically to an one-sided neighborhood of M_1, say F and Ω_1^-. Given a point $z_0 \in M_1$, if M_2 contains no nontrivial complex variety through $f(z_0)$ and if $f^{-1}(f(z_0)) \setminus \{z_0\}$ is discrete, then* Jac $F \not\equiv 0$. *Furthermore, if $f(z_0)$ is not minimally convex then f extends holomorphically to a neighborhood of z_0 in $\mathbf{C}^n$.*

We remark that no real analyticity on hypersurfaces has been assumed.

Proof. Let $F(z)$: $\Omega_1^- \to \mathbf{C}^n$ be the extension of f. First we prove two facts to be used later.

We notice that $F(\Omega_1^-) \not\subset M_2$, since M_2 contains no complex variety of positive dimension. Now we claim that Jac $F(z) \not\equiv 0$. Indeed, if Jac $F(z) \equiv 0$ in Ω_1^- then we let μ be the maximal rank of the Jacobian matrix of F in Ω_1^-. We have $0 < \mu < n$, and the set

$$\{ z \in \Omega_1^- : \text{Rank}\, F = \mu \}$$

is an open dense subset of Ω_1^-. By the rank theorem and the fact $F(\Omega_1^-) \not\subset M_2$, we may find a sequence of points $z_k \in \Omega_1^-$ converging to z_0 such that $F(z_k) \notin M_2$ and, for each k, the analytic set

$$\{ z \in \Omega_1^- : F(z) = F(z_k) \}$$

has an irreducible component $V_k \subset \Omega_1^-$ of dimension $n - \mu > 0$ passing through z_k. Since $F(z_k) \notin M_2$ it follows that, for each k, V_k does not have limit points on M_1. Therefore $\bar{V}_k$ is a closed analytic variety in Ω_1. Now let $z' \in \overline{\bigcup \bar{V}_k} \setminus \bigcup \bar{V}_k$; we see that

$$f(z') = F(z') = \lim F(z_k) = F(z_0) = w_0.$$

This implies that $z' \in f^{-1}(w_0)$. But $f^{-1}(w_0) \setminus \{z_0\}$ is discrete, and we see that the sequence of the sets $\bar{V}_k$ clusters on M_1 only at discrete points near z_0. Thus, by the generalized continuity principle [S], we conclude that $F(z)$ extends holomorphically to a neighborhood of z_0 in $\mathbf{C}^n$. As before, this implies that $\operatorname{Jac} F(z) \not\equiv 0$ since F is locally proper.

Using these facts, we will prove that F extends holomorphically to a neighborhood of z_0 in $\mathbf{C}^n$.

When $w_0 \in M_2$ is not a minimally convex point, an important fact is that every holomorphic function defined on one side of M_2 that admits a distribution limit up to M_2 extends holomorphically to a small open neighborhood of w_0. (See [BR1, Thm. 7; BR4, Thm. 1; T]; this fact has been used in [HP; P].)

To prove that F is a holomorphic extension when w_0 is not minimally convex, we will construct pieces of proper holomorphic mappings near z_0. Since $f^{-1}(w_0) \setminus \{z_0\}$ is discrete and $f^{-1}(w_0)$ is closed, we may choose an open neighborhood Ω_1 of z_0 such that

$$\partial\Omega_1 \cap \{f^{-1}(w_0)\} = \emptyset.$$

Thus we have that $\operatorname{dist}(\partial\Omega_1, \{f^{-1}(w_0)\}) = \delta > 0$.

Now consider

$$V = \{z \in \Omega_1^-, F(z) = w_0\}.$$

Then V is an analytic variety in Ω_1^-. If $\dim V \geq 1$, let V' be an irreducible component of V. Since V only has limit points $f^{-1}(w_0)$ on M_1, we see that V is also an analytic variety in $\Omega_1 \setminus f^{-1}(w_0)$, and by a theorem of Remmert and Stein [RS], $\bar{V}'$ is an analytic variety in Ω_1. The continuity principle implies that F extends holomorphically to a neighborhood of z_0. There is nothing to prove in this case.

Now we may assume that $\dim V = 0$. This means V is a discrete set in Ω_1^-. We may shrink Ω_1 slightly so that $\partial\Omega_1 \cap V = \emptyset$. We therefore have

$$\operatorname{dist}(w_0, F(\partial\Omega_1^- \setminus M_1)) > 0.$$

We can then choose a very small open neighborhood Ω_2 of w_0 such that

$$\operatorname{dist}(\partial\Omega_2, F(\partial\Omega_1^- \setminus M_1)) > 0. \tag{3}$$

Since $F(\Omega_1^-) \not\subset M_2$, $F(\Omega_1^-)$ intersects at least one side of M_2; there are two possibilities as follows.

(I) For any small neighborhood Ω_2 of w_0, we have

$$F(\Omega_1^-) \cap \Omega_2^- \neq \emptyset \quad \text{and} \quad F(\Omega_1^-) \cap \Omega_2^+ \neq \emptyset.$$

(II) There is an arbitrarily small neighborhood Ω_2 of w_0 such that

$$F(\Omega_1^-) \subset \overline{\Omega_2^-} \quad \text{or} \quad F(\Omega_1^-) \subset \overline{\Omega_2^+}.$$

We consider case (I); case (II) can be dealt with similarly.

Consider two nonempty open sets in Ω_1^-,

$$U^+ = F^{-1}(\Omega_2^+) \quad \text{and} \quad U^- = F^{-1}(\Omega_2^-).$$

We claim that the restriction of F to U^+ (resp. U^-) is a proper map from U^+ to Ω_2^+ (resp. U^- to Ω_2^-). Indeed, let $F^+ = F|_{U^+}$ and let $K \subset\subset \Omega_2^+$ be a compact subset; we want to prove that $(F^+)^{-1}(K)$ is a compact subset in U^+. If $(F^+)^{-1}(K)$ is not compact in U^+, then there exists a point $p \in \partial U^+$ such that $F^+(p) \in K$. Since $K \cap M_2 = \emptyset$ we have $p \notin M_1$ and, by (3), $p \in \Omega_1^-$. Therefore, there exists a neighborhood O of p such that $F(O) \subset \Omega_2^+$. Hence p cannot be a boundary point of U^+, a contradiction.

Now we observe that the open set $U^+ \cup U^-$ is, in general, not connected. We make some simple observations that are crucial to what follows in the proof of Lemma 2.

Claim 1: The set $U^+ \cup U^-$ is an open dense set near z_0 in Ω_1^- along M_1.

Indeed, if this is not the case then, for any small neighborhood of z_0, there exists a point $p \in \Omega_1^-$ in that neighborhood, and there exists a small neighborhood O of p contained in Ω_1^- so that $F(O) \subset \partial\Omega_2 \cup M_2$ (since by continuity $F(O) \subset \bar{\Omega}_2$). This is impossible since Jac $F(z) \not\equiv 0$ in Ω_1^-.

Claim 2: The open set $U^+ \cup U^-$ has finitely many connected components.

Indeed, if this is not the case then we may assume that U^+ has infinitely many components. We let U_j be connected components of U^+ for $j = 1, 2, \ldots$. Therefore $F: U_j \to \Omega_2^+$ is proper for each j. Let $p_0 \in \Omega_2^+$ fixed. By the properness of $F|_{U_j}$, there exists a point $z_j \in U_j$ such that $F(z_j) = p_0$ for $j = 1, 2, \ldots$. We notice that z_j are different points. On the other hand, F is proper from U^+ to Ω_2^+ and therefore $F^{-1}(p_0)$ is finite, a contradiction to $F^{-1}(p_0) = \{z_j\}$.

Now let $\{U_j^+\}_{j=1}^k$ be connected components of U^+, and similarly $\{U_j^-\}_{j=1}^l$ for U^-. Let g_j be the restriction of F on U_j^+, and h_j on U_j^-. It follows that $g_j: U_j^+ \to \Omega_2^+$ and $h_j: U_j^- \to \Omega_2^-$ are proper holomorphic mappings.

We then consider a proper mapping g from D to G, where the pairing (D, G) is either (U_j^+, Ω_2^+) or (U_j^-, Ω_2^-) and g is either g_j or h_j. The graph of g is defined to be

$$\Gamma_g = \{(z, w) \in D \times G, w = g(z)\}.$$

By the proper mapping theorem, g is a covering from $D \setminus g^{-1}(g(V_g))$ to $G \setminus g(V_g)$ of multiplicity m, where

$$V_g = \{ z \in D : \operatorname{Jac} g = 0 \}.$$

Let $G_1, G_2, \ldots, G_m$ be the local inverses defined on $G \setminus g(V_g)$. On $D \times G \setminus g(V_g)$ define

$$H_i(z, w) = \prod_{j=1}^{m} (z_i - (G_j(w))_i).$$

By the removable singularity result of bounded holomorphic functions, H_i extends to be holomorphic on $D \times G$. Let

$$A_g = \{(z, w) \in D \times G : H_1 = H_2 = \cdots = H_n = 0\}.$$

It is easy to check that $\Gamma_g = A_g$.

Let Γ_{g_j}, Γ_{h_j} be the graphs of g_j, h_j (respectively), and let A_{g_j}, A_{h_j} be associated with g_j, h_j as defined above. We see that the graph of F over $U^+ \cup U^-$ is given by

$$\bigcup_{j=1}^{k} \Gamma_{g_j} \cup \bigcup_{j=1}^{l} \Gamma_{h_j},$$

which is equal to

$$\bigcup_{j=1}^{k} A_{g_j} \cup \bigcup_{j=1}^{l} A_{h_j}.$$

As we have observed, the open set $U^+ \cup U^-$ is an open dense set along M_1 near z_0 (Claim 1). By continuity, we conclude that the graph of F over a small one-sided neighborhood of M_1 near z_0 is contained in

$$\bigcup_{j=1}^{k} A_{g_j} \cup \bigcup_{j=1}^{l} A_{h_j}.$$

Now we want to show that

$$\bigcup_{j=1}^{k} A_{g_j} \cup \bigcup_{j=1}^{l} A_{h_j}$$

extends to be an analytic variety of pure dimension n in $\mathbf{C}^n \times \mathbf{C}^n$ near (z_0, w_0). Indeed, we notice that for each g (either g_i or h_j),

$$H_i(z, w) = z_i^m + S_{m-1}(w) z_i^{m-1} + \cdots + S_0(w),$$

where $S_j(w)$ is the jth symmetric function of $(G_j(w))i$ for $j = 1, \ldots, m$. Since $S_j(w)$ are bounded, and since w_0 is not minimally convex, it follows that $S_j(w)$ extends to be holomorphic in a neighborhood of w_0 in $\mathbf{C}^n$ from either side wherever applicable. Therefore $H_i(z, w)$ extends to be holomorphic in a neighborhood of (z_0, w_0) in $\mathbf{C}^n \times \mathbf{C}^n$, and this in turn implies that

$$\bigcup_{j=1}^{k} A_{g_j} \cup \bigcup_{j=1}^{l} A_{h_j}$$

is an analytic variety of pure dimension n in a neighborhood of (z_0, w_0), which implies that F extends to be a correspondence to a neighborhood of z_0. Lemma 1 then gives the holomorphic extension of F at z_0. This completes the proof of Lemma 2 for case (I); case (II) can be proved similarly. □

Proof of Theorem 2. Let $z_0 \in M_1$ and $w_0 = f(z_0) \in M_2$. Since M_1 is minimal at z_0, by Trepreau's theorem f extends holomorphically to a one-sided neighborhood of M_1, say Ω_1^-, whose extension is denoted by $F(z)$. Therefore $F(z)\colon \Omega_1^- \to \mathbf{C}^n$ is a holomorphic mapping such that $F = f$ on M_1. If w_0 is minimally convex, then the complex Hopf lemma of [BR4] (since Jac $F \not\equiv 0$ by Lemma 2) and [DF] imply that f extends holomorphically to a neighborhood of z_0. If w_0 is not minimally convex then Lemma 2 applies again. This completes the proof. □

Corollary 2 is a special case of Theorem 2, since a real analytic hypersurface of finite type of D'Angelo is essentially finite and contains no nontrivial complex varieties.

Proof of Corollary 3. It suffices to prove that $f^{-1}(f(p)) \setminus \{p\}$ is discrete. Let $q \in f^{-1}(f(p)) \setminus \{p\}$ but $q \neq p$. We want to prove that q is an isolated point. Since f is real analytic at q by assumption, f extends holomorphically to a neighborhood of q with extension (say) F. By a result of [BR3], the Hopf lemma holds at q for the normal component of F. Let ρ be a real analytic defining function of M_2 near w_0. By the Hopf lemma just mentioned, at q it is easy to see, by changes of coordinates at both q and $f(q)$, that $\rho \circ F$ is again a defining function of M_1 near q. Therefore the equation

$$\{z \in \mathbf{C}^n : \rho \circ F(z) = 0\}$$

defines a real analytic hypersurface near q that is identical to M_1 near q. This implies that $F^{-1}(f(q))$ is contained in M_1. Since M_1 is of finite type of D'Angelo and $F^{-1}(f(q))$ is a complex analytic variety, we conclude that q is an isolated point in M_1. Theorem 2 then applies at p since $f^{-1}(f(p)) \setminus \{p\}$ is discrete. $\square$

In order to prove Corollary 4, we need to prove a version of Lemma 2 under the conditions of Corollary 4.

LEMMA 3. *Let $f: M_1 \to M_2$ be a finite-to-one smooth CR mapping between smooth real hypersurfaces that extends holomorphically to Ω_1^- as F. Let $z_0 \in M_1$ and $f(z_0) = w_0 \in M_2$. If M_2 is minimal but not minimally convex at w_0, then f extends holomorphically to a neighborhood of z_0 in $\mathbf{C}^n$.*

Proof. By the proof of Lemma 2, it suffices to prove the following: (a) $F(\Omega_1^-) \not\subset M_2$, and (b) $\operatorname{Jac} f(z) \not\equiv 0$.

Indeed, if $F(\Omega_1^-) \subset M_2$ then $\operatorname{Jac} F(z) \equiv 0$ in Ω_1^-. This implies that the Jacobian matrix of the map $f: M_1 \to M_2$ considered as a real map of the real manifolds is of maximal rank μ such that $0 < \mu < 2n - 1$. Therefore, by the rank theorem there exists a point w' near w_0 such that $f^{-1}(w')$ is a manifold of dimension $n - \mu$, a contradiction to f being finite-to-one; (b) follows as well.

Proof of Corollary 4. First we observe that $F(\Omega_1^-) \not\subset M_2$ by the proof of Lemma 3. If w_0 is not minimal then, by a unique continuation result for holomorphic mappings [P, Thm. 2], F does not vanish to infinite order at z_0 in the normal component. Therefore [BR3] implies that the normal derivative of F is nonzero at 0, and hence F extends holomorphically to a neighborhood of z_0. When w_0 is minimal, the proof follows as in that of Theorem 2 by using Lemma 3 instead of Lemma 2. $\square$

References

[BBR] M. S. Baouendi, S. Bell, and L. P. Rothschild, *Mappings of three-dimensional CR manifolds and their holomorphic extension,* Duke Math. J. 56 (1988), 503–530.

[BJT] M. S. Baouendi, H. Jacobwitz, and F. Treves, *On the analyticity of CR mappings,* Ann. of Math. (2) 122 (1985), 365–400.

[BR1] M. S. Baouendi and L. P. Rothschild, *Normal forms for generic manifolds and holomorphic extension of CR functions,* J. Differential Geom. 25 (1987), 431–467.

[BR2] ———, *Germs of CR maps between real analytic hypersurfaces,* Invent. Math. 93 (1988), 481–500.

[BR3] ———, *Geometric properties of mappings between hypersurfaces in complex space,* J. Differential Geom. 31 (1990), 473–499.

[BR4] ———, *A generalized complex Hopf lemma and its applications to CR mappings,* Invent. Math. 111 (1993), 331–348.

[BB] E. Bedford and S. Bell, *Extension of proper holomorphic mappings past the boundary,* Manuscripta Math. 50 (1985), 1–10.

[B] S. Bell, *Analytic hypoellipticity of the $\bar{\partial}$-Neumann problem and extendability of holomorphic mappings,* Acta. Math. 147 (1981), 109–116.

[DF] K. Diederich and J. Fornaess, *Proper holomorphic mappings between real-analytic pseudoconvex domains in* $\mathbf{C}^n$, Math. Ann. 282 (1988), 681–700.

[DW] K. Diederich and S. Webster, *A reflection principle for degenerate real hypersurfaces,* Duke Math. J. 47 (1980), 835–843.

[HP] X. Huang and Y. Pan, *Proper holomorphic mappings between real analytic domains in* $\mathbf{C}^n$, Duke Math. J. 82 (1996), 437–446.

[L] H. Lewy, *On the boundary behavior of holomorphic mappings,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 35 (1977), 1–8.

[P] Y. Pan, *Real analyticity of homeomorphic CR mappings between real analytic hypersurfaces in* $\mathbf{C}^2$, Proc. Amer. Math. Soc. 123 (1995), 373–380.

[Pi1] S. Pinchuk, *On the analytic continuation of biholomorphic mappings,* Math. USSR-Sb. 27 (1975), 375–392.

[Pi2] ———, *CR transformations of real manifolds in* $\mathbf{C}^n$, Indiana Univ. Math. J. 41 (1992), 1–16.

[RS] R. Remmert and K. Stein, *Über die wesentlichen Singularitäten analytischer Mengen,* Math. Ann. 126 (1953), 263–306.

[S] B. V. Shabat, *Introduction to complex analysis. Part II. Functions of several complex variables,* Amer. Math. Soc., Providence, RI, 1992.

[Tr] J. Trepreau, *Sur le prolongement holomorphe des fonctions CR défines sur une hypersurface réelle de classe* C^2 *dans* $\mathbf{C}^n$, Invent. Math. 83 (1986), 583–592.

[T] A. Tumanov, *Extension of CR-functions into a wedge from a manifold of finite type* (Russian), Mat. Sb. (N.S.) 136 (1988), 128–139. Translation in Math. USSR-Sb. 64 (1989), 129–140.

Department of Mathematics
Indiana University – Purdue University
Fort Wayne, IN 46805