A Rigidity Theorem for Composition Operators on Certain Bergman Spaces

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Let ϕ be an analytic self-map of the open unit disk $\mathbf{D}$ in the complex plane. We consider the composition operator C_ϕ, defined by $C_\phi f = f \circ \phi$, acting on a weighted Bergman space A^2_G. Here $G(r)$, $0 < r < 1$, is a positive continuous function and A^2_G consists of all f analytic on $\mathbf{D}$ with

$$\|f\|^2 \stackrel{\text{def}}{=} \int_{\mathbf{D}} |f(z)|^2 G(|z|)\, dA < \infty, \tag{1}$$

where dA is area measure on $\mathbf{D}$. We assume that G is non-increasing and that $G(|z|)$ is dA-integrable over $\mathbf{D}$. It is well known that the norm $\|\cdot\|$ defined by (1) makes A^2_G into a Hilbert space. The purpose of this note is to locate a family of "critical weights" G_* with the property that any A^2_G defined from a weight G which tends to zero more rapidly than G_* admits only compact and unitary composition operators.

It is known that if $G(r) = (1-r)^\alpha$ with $\alpha \geq 0$ (the standard weights), then every C_ϕ defines a bounded operator on A^2_G. Moreover, C_ϕ is a compact operator on these spaces exactly when ϕ has no finite angular derivative at any point on $\partial\mathbf{D}$. Recall that if ζ lies in the unit circle $\partial\mathbf{D}$, ϕ is said to have a (finite) angular derivative $\phi'(\zeta)$ at ζ if there exists w in $\partial\mathbf{D}$ such that

$$\phi'(\zeta) \stackrel{\text{def}}{=} \lim_{z\to\zeta} \frac{\phi(z)-w}{z-\zeta}$$

exists, where $z \to \zeta$ nontangentially. This happens exactly when the quantity

$$\liminf_{z\to\zeta} \frac{1-|\phi(z)|}{1-|z|} \tag{2}$$

is finite, where here $z \to \zeta$ unrestrictedly in $\mathbf{D}$; in this case expression (2) coincides with $|\phi'(\zeta)|$. Let us write $|\phi'(\zeta)|$ for (2) even when the lim inf is infinite. Note that when $\phi'(\zeta)$ exists as a finite limit, the nontangential limit of ϕ at ζ, call it $\phi(\zeta)$, exists and has modulus 1. Thus if the nontangential limit $\phi(\zeta)$ fails to exist, or if it exists but $|\phi(\zeta)| \neq 1$, then $|\phi'(\zeta)| = \infty$. If $\phi(\zeta) = \zeta$ and $\phi'(\zeta)$ exists, it is positive. For any ϕ and ζ, we have $0 < |\phi'(\zeta)| \leq \infty$. Thus compactness of C_ϕ on the standard weight spaces is characterized by: $|\phi'(\zeta)| = \infty$ for all ζ in $\partial\mathbf{D}$ (see [6]).

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The situation is different for fast weights G, those for which

$$\frac{G(r)}{(1-r)^{\alpha}} \tag{3}$$

tends to zero as $r \to 1$ for all $\alpha > 0$. We call a fast weight *regular* if the ratio (3) is actually decreasing for r near 1 for all $\alpha > 0$. The following facts are known [5].

(i) If G is a fast weight and C_ϕ is bounded on A_G^2, then $|\phi'(\zeta)| \geq 1$ for all ζ in $\partial\mathbf{D}$.

(ii) If G is fast and regular, then C_ϕ is compact on A_G^2 if and only if $|\phi'(\zeta)| > 1$ for all ζ in $\partial\mathbf{D}$.

Given a fast regular G, it is reasonable to ask whether there exists any bounded C_ϕ on A_G^2 that is not compact. The answer is always "yes": The rotations $\phi(z) = cz$, with $|c| = 1$, induce the entire collection of unitary composition operators on any A_G^2; the case $c = 1$ of course yields the identity operator. Here we show that, for weights that decay to zero more rapidly than the critical weights

$$G_*(r) = \exp\left\{-B\frac{1}{(1-r)^2}\right\}, \quad B > 0,$$

there are no other possibilities. In what follows we write G in the form $G(r) = e^{-h(r)}$ and assume that $h(r)$ is continuously differentiable and $h'(r)$ is nondecreasing on $(0,1)$.

THEOREM 1. *If $(1-r)^3 h'(r) \to \infty$ as $r \to 1$, then the only C_ϕ that act boundedly but not compactly on A_G^2 are those induced by the rotations $\phi(z) = cz$, $|c| = 1$.*

THEOREM 2. *If $(1-r)^3 h'(r)$ remains bounded as $r \to 1$, there exists a ϕ, not a rotation, such that C_ϕ is bounded but not compact on A_G^2.*

These results depend on the following boundedness criteria, which can be found in [5] and [4], respectively. We write $M(r)$ for the maximum modulus function $\max_\theta |\phi(re^{i\theta})|$.

(1) If $G(r)/G(M(r))$ remains bounded as $r \to 1$, then C_ϕ is a bounded operator on A_G^2.

(2) If $G(r)/G(M(r)) \to \infty$ as $r \to 1$, then C_ϕ is not bounded on A_G^2.

We also need the following variant of a result of Burns and Krantz [1; 3]. It will be seen in the proof that the function u defined in the statement satisfies $\operatorname{Re} u(r) \geq 0$, $0 < r < 1$.

PROPOSITION 1. *Let $\phi: \mathbf{D} \to \mathbf{D}$ be analytic with radial limit $\phi(1) = 1$ and angular derivative $\phi'(1) = 1$. Define u by $\phi(z) = z + u(z)$. If*

$$\liminf_{r \to 1^-} \frac{\operatorname{Re} u(r)}{(1-r)^3} = 0,$$

then $u \equiv 0$ and $\phi(z) = z$.

Proof. The function $(1+\phi)(1-\phi)^{-1}$ has positive real part in $\mathbf{D}$, and so has a Herglotz integral representation. In particular,

$$\frac{1+\phi(r)}{1-\phi(r)}=\int\frac{e^{i\theta}+r}{e^{i\theta}-r}\,d\mu(\theta)+ic,\quad 0<r<1,$$

where μ is a positive measure on $\partial\mathbf{D}$ and c is a real constant. This formula, of course, holds on all of $\mathbf{D}$, but we need it only for r in the unit interval. If we multiply this equation by $1-r$ and let $r\to 1$, we see that the left side tends to 2 because $\phi(1)=\phi'(1)=1$. The right side, on the other hand, tends to $2\mu(\{1\})$ by the dominated convergence theorem. It follows that $\mu(\{1\})=1$, and so $\mu=\delta_1+\nu$ where δ_1 is a unit point mass at 1 and ν is a positive measure with $\nu(\{1\})=0$. The preceding equation can thus be rewritten as

$$\frac{1+\phi(r)}{1-\phi(r)}-\frac{1+r}{1-r}=\int\frac{e^{i\theta}+r}{e^{i\theta}-r}\,d\nu(\theta)+ic,\quad 0<r<1. \tag{4}$$

Since $\phi(1)=1=\phi'(1)$, we see that

$$\frac{u(r)}{1-r}=1-\frac{\phi(r)-1}{r-1}\to 0$$

as $r\to 1$. Let us write s and t for the real and imaginary parts of $u(r)/(1-r)$. Note that $s\to 0$ and $t\to 0$ as $r\to 1$. According to Julia's lemma [2], ϕ maps every disk in $\mathbf{D}$ that is internally tangent to $\partial\mathbf{D}$ at 1 (a Julia disk at 1) into itself. Taking the disk whose boundary passes through r, we see that $\operatorname{Re}u(r)\geq 0$ for $0\leq r<1$, that is, $s\geq 0$. On taking real parts of both sides of (4), we calculate

$$\frac{1}{1-r}\left\{\frac{2s(1-s)-2t^2}{(1-s)^2+t^2}\right\}=\int P_r\,d\nu,$$

where P_r is the Poisson kernel at r. Note that the right side is bounded below by $(1-r)(1+r)^{-1}\nu(\partial\mathbf{D})$. Dividing the resulting inequality by $1-r$ and discarding the $-2t^2$ term yields

$$\frac{s}{(1-r)^2}\cdot\frac{2(1-s)}{(1-s)^2+t^2}\geq\frac{1}{1+r}\nu(\partial\mathbf{D}).$$

Since

$$\liminf_{r\to 1^-}\frac{s}{(1-r)^2}=\liminf_{r\to 1^-}\frac{\operatorname{Re}u(r)}{(1-r)^3}=0,$$

we have $\nu=0$, and thus

$$\frac{1+\phi(z)}{1-\phi(z)}=\frac{1+z}{1-z}+ic.$$

Let us consider the possibility $c\neq 0$. In this case we can solve the last equation for $\phi(z)$ to find

$$\phi(z)=\frac{\bar\alpha}{\alpha}\frac{\alpha-z}{1-\bar\alpha z},$$

where $\alpha=ic/(ic-2)$, a point in $\mathbf{D}$ satisfying $\operatorname{Re}\alpha=|\alpha|^2>0$. From this we compute

$$\operatorname{Re} u(r) = \frac{(1-r)^3 \operatorname{Re} \alpha}{|1-\bar{\alpha} r|^2}, \quad 0 < r < 1.$$

It follows that $(1-r)^{-3} \operatorname{Re} u(r)$ is bounded below, contradicting our hypothsis on $u(r)$. The only remaining possibility is $c = 0$, that is, $\phi(z) = z$. □

Note. In our original proof of Proposition 1 we had too hastily concluded that $c = 0$. We are grateful to the referee for pointing out that one really does need to rule out the automorphisms of $\mathbf{D}$ described in the last paragraph.

Now suppose ϕ maps $\mathbf{D}$ to $\mathbf{D}$ and at some ζ_0 in $\partial\mathbf{D}$ has angular derivative satisfying $|\phi'(\zeta_0)| = 1$. By Julia's lemma, $M(r) \geq r$. In fact, $M(r) > r$ for all $r < 1$ unless ϕ is a rotation. To see this, suppose $M(r_0) = r_0$ for some r_0. Choose c with $|c| = 1$ so that $c\phi(\zeta_0) = \zeta_0$ and note that $c\phi'(\zeta_0) = 1$. Since $c\phi$ maps the closed Julia disk at ζ_0 whose boundary contains $r_0\zeta_0$ into itself, the assumption $M(r_0) = r_0$ says that $r_0\zeta_0$ is a fixed point for $c\phi$. Moreover, if $r_0 < s < 1$, then $c\phi(s\zeta_0)$ must lie in the closed Julia disk J whose boundary passes through ζ_0 and $s\zeta_0$. On the other hand, consider the pseudohyperbolic metric on $\mathbf{D}$ defined by

$$d(z, w) = \left| \frac{z-w}{1-\bar{w}z} \right|.$$

Let $a = d(r_0\zeta_0, s\zeta_0)$ and put $B = \{z \in \mathbf{D}: d(r_0\zeta_0, z) \leq a\}$. Then B is a closed Euclidean disk containing $r_0\zeta_0$ with Euclidean center on the radius from 0 to ζ_0 and with $s\zeta_0$ in its boundary. By the Schwartz-Pick lemma, $c\phi$ maps B into B. Since $J \cap B = \{s\zeta_0\}$, we see that $c\phi$ fixes $s\zeta_0$ as well. Since s is arbitrary in the range $r_0 < s < 1$, we have $c\phi(z) \equiv z$ and ϕ is a rotation. This raises the question: Since $M(r) - r$ can never be zero for nonrotations, are there also restrictions on how fast it can tend to zero? The following consequence of Proposition 1 provides the answer; we will see below that it is sharp.

COROLLARY. *Suppose $\phi: \mathbf{D} \to \mathbf{D}$ is analytic with angular derivative satisfying $|\phi'(\zeta_0)| = 1$ at some ζ_0 in $\partial\mathbf{D}$. If the maximum modulus function $M(r)$ of ϕ satisfies*

$$\liminf_{r \to 1^-} \frac{M(r) - r}{(1-r)^3} = 0,$$

then $M(r) \equiv r$ and ϕ is a rotation.

Proof. By pre- and post-composing with rotations, we can assume that $\zeta_0 = 1$, $\phi(1) = 1$, and therefore $\phi'(1) = 1$. Then write $\phi(z) = z + u(z)$. We have

$$M(r) - r \geq |\phi(r)| - r \geq \operatorname{Re} \phi(r) - r = \operatorname{Re} u(r),$$

and so the corollary follows from Proposition 1. □

Proof of Theorem 1. Let G satisfy the hypothesis of Theorem 1 and suppose that C_ϕ is bounded but not compact on A^2_G. Then there exists ζ_0 in

$\partial\mathbf{D}$ with $|\phi'(\zeta_0)|=1$. Moreover, there exists a positive sequence $r_n\to 1$ with $G(r_n)/G(M(r_n))$ bounded. But

$$\frac{G(r)}{G(M(r))}=e^{h(M(r))-h(r)}=e^{h'(s)(M(r)-r)},$$

where $r\le s\le M(r)$. Since h' is nondecreasing we have $h'(s)\ge h'(r)$, and so for some constant $B>0$,

$$B\ge h'(r_n)(M(r_n)-r_n)=(1-r_n)^3h'(r_n)\frac{M(r_n)-r_n}{(1-r_n)^3}.$$

Since $(1-r_n)^3h'(r_n)\to\infty$, we may apply the corollary to conclude that ϕ is a rotation. □

To prove Theorem 2 and to provide some interesting examples, we consider the mappings

$$\phi(z)=z+t(1-z)^\beta,\quad t>0,\ \beta>1. \tag{5}$$

According to Proposition 1, such a ϕ cannot map $\mathbf{D}$ into $\mathbf{D}$ if $\beta>3$. Burns and Krantz [1; 3] observed that when $\beta=3$ and t is small enough, $\phi(\mathbf{D})\subset\mathbf{D}$, thus showing that their version of Proposition 1 (and ours as well) is sharp. These maps also appear in [5] to provide examples of composition operators on fast weight spaces, a theme that we pursue in what follows. Because none of these papers verifies $\phi(\mathbf{D})\subset\mathbf{D}$, we do so here, plus a bit more.

Proposition 2. *Suppose ϕ is given by* (5). *If $1<\beta\le 3$ and $0<t<2^{1-\beta}$, then ϕ maps $\mathbf{D}$ to $\mathbf{D}$ and $|\phi(e^{i\theta})|<1$ for all $e^{i\theta}\ne 1$. Moreover* (*and clearly*), *$\phi(1)=1$ and $\phi'(1)=1$.*

Proof. Consider a point $e^{i\theta}\ne 1$ on the unit circle, and write $1-e^{i\theta}$ in polar form as ρe^{ix} with $\rho>0$ and $-\pi/2<x<\pi/2$. A calculation shows that

$$|\phi(e^{i\theta})|^2=1+t^2\rho^{2\beta}+2t\rho^\beta[\cos\beta x-\rho\cos(\beta-1)x]. \tag{6}$$

Further, the condition $|1-\rho e^{ix}|=1$ translates to $\rho=2\cos x$; on substituting this into (6) and using the identity

$$\cos\beta x-2\cos(\beta-1)x\cos x=-\cos(\beta-2)x,$$

we find

$$|\phi(e^{i\theta})|^2=1+t2^\beta[t2^\beta\cos^\beta x-2\cos(\beta-2)x]\cos^\beta x.$$

Clearly, we want the second term on the right to be negative. But this follows since $t<2^{1-\beta}$ and because the inequality $1<\beta\le 3$ implies that $\cos(\beta-2)x\ge\cos x\ge\cos^\beta x$ for $-\pi/2<x<\pi/2$. □

Lemma. *Let ϕ be given by* (5) *with $1<\beta\le 3$ and $0<t<2^{1-\beta}$. If $\{z_n\}$ is a sequence in $\mathbf{D}$ with $z_n\to 1$ tangentially, then $|\phi(z_n)|<|z_n|$ for n large.*

Proof. First consider the case $\beta=3$. Let $z_n\to 1$ tangentially. Without loss of generality we can take z_n in the first quadrant of $\mathbf{D}$. Let us abbreviate

$z_n = z$ and introduce the polar forms $z = re^{i\theta}$ and $1-z = \rho e^{ix}$. We have $0 < \theta < \pi/2$ and $-\pi/2 < x < 0$. Then

$$\phi(re^{i\theta}) = re^{i\theta} + t\rho^3 e^{3ix},$$

and we compute that

$$|\phi(re^{i\theta})|^2 - r^2 = t\rho^3[t\rho^3 + 2r\cos(\theta - 3x)]. \tag{7}$$

As $z \to 1$, $r \to 1$ and $\theta \to 0$. Tangential approach means that $x \to -\pi/2$ and $\rho/\theta \to 1$ as well. We see from (7) that it suffices to show $t\theta^3 + \cos(\theta - 3x) < 0$ when z is tangentially close to 1.

Consider now two triangles. The first has corners at 0, 1, and $re^{i\theta}$. The interior angle at the vertex 1 is clearly $-x$. The second triangle, isoceles, has corners at 0, 1, and $e^{i\theta}$, and interior angles α at 1 and $e^{i\theta}$. Clearly $-x < \alpha$ and $2\alpha + \theta = \pi$. Thus

$$\theta - 3x < \theta + 3\alpha = \frac{3\pi}{2} - \frac{\theta}{2},$$

so $\theta - 3x$ is tending to $3\pi/2$ from below as $z \to 1$ tangentially. But just to the left of $3\pi/2$ the cosine satisfies

$$\cos s < \frac{1}{2}\left(s - \frac{3\pi}{2}\right),$$

so that

$$\cos(\theta - 3x) < \cos\left(\frac{3\pi}{2} - \frac{\theta}{2}\right) < -\frac{\theta}{4}.$$

Since $t\theta^3 < \theta/4$ for small positive θ, we are done.

The case $1 < \beta < 3$ is more obvious. With ρ and x as before,

$$\phi(re^{i\theta}) = re^{i\theta} + t\rho^\beta e^{i\beta x}.$$

In the limit we have $\beta x \to -\beta\pi/2$, so that eventually the small complex number $t\rho^\beta e^{i\beta x}$ lies inside a sector opening from 0 into the left half-plane, symmetric about the negative real axis and with angle less than π. Clearly, then, $\phi(z)$ is closer to zero than z for z near 1. □

Now consider the weights

$$G(r) = \exp\left\{-B\frac{1}{(1-r)^\alpha}\right\}, \quad B > 0, \ \alpha > 0.$$

Theorem 1 implies that if $\alpha > 2$, the only bounded composition operators C_ϕ on A^2_G are those with $|\phi'(\zeta)| > 1$ for all ζ in $\partial\mathbf{D}$, and those with $\phi(z) = cz$, $|c| = 1$. Accordingly, we concentrate on the range $0 < \alpha \le 2$.

THEOREM 3. *Consider the exponential weights G as just defined with $0 < \alpha \le 2$. Let ϕ be one of the maps* (5) *with $1 < \beta \le 3$ and $t < 2^{1-\beta}$. Then C_ϕ is bounded on A^2_G if and only if $\beta \ge \alpha + 1$.*

Note. Since $\phi'(1)=1$, none of the operators C_ϕ in Theorem 3 can be compact.

Proof. Let $0<\alpha\le 2$ and $1<\beta\le 3$, and write $M(r)$ for the maximum modulus function of ϕ. As in the proof of Theorem 1, we have

$$\frac{G(r)}{G(M(r))}=\exp\left\{\alpha B\frac{M(r)-r}{(1-s)^{\alpha+1}}\right\},$$

where $r\le s\le M(r)$. Since the minimum value of $|\phi'(\zeta)|$ over $\partial\mathbf{D}$ is exactly 1, $1-M(r)\sim 1-r$ as $r\to 1$ (see [5]), and so $G(r)/G(M(r))$ remains bounded or tends to ∞ with the quantity

$$\frac{M(r)-r}{(1-r)^{\alpha+1}}. \tag{8}$$

Thus, if C_ϕ is bounded, the expression (8) remains bounded on some sequence $r_n\to 1$, and since $t(1-r)^\beta=|\phi(r)|-r\le M(r)-r$ we see that $\beta\ge\alpha+1$.

On the other hand, if C_ϕ is unbounded, then (8) is unbounded on a sequence $r_n\to 1$. Select z_n in $\mathbf{D}$ with $|z_n|=r_n$ and $|\phi(z_n)|=M(r_n)$. Since $M(r_n)\to 1$, it follows from Proposition 2 that $z_n\to 1$. By the lemma, z_n must tend to 1 nontangentially, that is, $|1-z_n|\le C(1-r_n)$ for some $C>0$ and all n. Thus

$$\begin{aligned}\frac{M(r_n)-r_n}{(1-r_n)^{\alpha+1}}&=\frac{|\phi(z_n)|-r_n}{(1-r_n)^{\alpha+1}}\\&\le\frac{t|1-z_n|^\beta}{(1-r_n)^{\alpha+1}}\\&\le(\text{constant})(1-r_n)^{\beta-\alpha-1}.\end{aligned}$$

Since the left side tends to ∞, we see that $\beta<\alpha+1$. □

Remark. The last half of the proof of Theorem 3, when applied in the case $\alpha=2$, shows that for the functions

$$\phi(z)=z+t(1-z)^3,\quad t<\tfrac{1}{4},$$

one has

$$M(r)-r\le C(1-r)^3,\quad 0<r<1,$$

and thus that our corollary is indeed sharp.

Proof of Theorem 2. The work has mostly been done. We let ϕ be the cubic in the last remark and invoke the hypothesis on h to find

$$\begin{aligned}h(M(r))-h(r)&=\int_r^{M(r)}h'(x)\,dx\\&\le C\int_r^{M(r)}\frac{1}{(1-x)^3}\,dx\end{aligned}$$

$$\leq C \frac{M(r)-r}{(1-M(r))^3}$$
$$\leq C' \frac{M(r)-r}{(1-r)^3}.$$

By our remark the right side is bounded, so $G(r)/G(M(r))$ is bounded as well, and C_ϕ acts boundedly in A_G^2. □

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