The Best Approximation and Composition with Inner Functions

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0. Introduction

Let $f \in L^p(T)$, $0 < p \leq +\infty$, where T is the unit circle in the complex plane C and $\mathcal{P}_n$ the set of complex polynomials whose degree is not greater than n. The best approximation of f by the polynomials of degree n in L^p-metric is defined by

$$E_n(f) = E_n(f)_p = \inf\{\|f-g\|_p : g \in \mathcal{P}_n\}.$$

An application of the Hahn-Banach theorem to the quotient space $L^p/\mathcal{P}_n$, $1 \leq p < +\infty$, enables us to get a convenient expression of $E_n(f)_p$ for further application (see the equality (1.1) below). Using (1.1), we prove that if $F \in H^p$ $(1 < p < +\infty)$, w is an inner function, and $w(0) = 0$, then $E_n(F \circ w)_p \geq E_n(F)_p$.

In Sections 2 and 3, we consider applications of this result. In Section 2, we give a characterization of the Lipschitz-Besov space $X = A(\alpha, p, q)$ by means of the best approximation. Using this result, we prove that if $F \circ w \in X$ then $F \in X$, where X is a Lipschitz-Besov space and w is an inner function.

If φ is an inner function and if φ has no zeros in the unit disk then $\varphi = A \circ w$, where w is an inner function and A is the atomic function. In Section 3, combining this fact with results obtained in previous sections, we prove that if φ is an inner function with a nonconstant singular factor and $1 \leq p < +\infty$, then there exists a positive constant C such that

$$E_k(\varphi)_p \geq C k^{-1/2p}.$$

This result is sharp. The case $p = 2$ of this result is due to Newman and Shapiro [6].

Finally, applications to the growth of integral means of inner functions are given. Using the lower bound on the rate at which $E_n(f)_p$ may go to zero, together with the characterization of $A_0(p)$, we prove that if φ is an inner function with nonconstant singular factor and $0 < p < +\infty$ then

$$\limsup_{r \to 1_-} (1-r)^{1-1/2p} M_p(r, \varphi') > 0.$$

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An immediate corollary of this is the following result, due to Ahern [2]: If the atomic function does not belong to X then no nonconstant singular function belongs to X, where X is a Lipschitz-Besov space.

1. Theorem 1.1

Let $U = \{z : |z| < 1\}$ be the open unit disk in the complex plane. The class of all holomorphic (analytic) functions in U will be denoted by $H(U)$. If $f \in H(U)$, $0 \le r < 1$, we define the integral means of f by

$$M_p(r,f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p}, \quad I_p(r,f) = M_p^p(r,f), \quad 0 < p < +\infty;$$

$$M_\infty(r,f) = \sup_\theta |f(re^{i\theta})|.$$

For $0 < p < \infty$, the class H^p is defined to consist of all $f \in H(U)$ for which $M_p(r,f)$ remains bounded as $r \to 1_-$.

A function $f \in H(U)$ is said to be *subordinate* to a function $F \in H(U)$ (written $f \prec F$) if $f(z) = F(w(z))$ for some function $w(z)$ analytic in U, satisfying $|w(z)| \le |z|$.

Throughout the paper let the letters n, k, and v denote natural numbers. From now on, all sums without limits are taken from 0 to ∞. In the theorems that follow, C denotes a positive constant though not always the same one. In order to give a background of our investigation, we first sketch the proof of Theorem 1.1 in the case $p = 2$. First we need the following.

THEOREM L (Littlewood's subordination theorem). *Let $f, F \in H(U)$ and $f \prec F$ in U. Then*

$$M_p(r,f) \le M_p(r,F), \quad 0 < p \le \infty.$$

Let $f(z) = \sum a_n z^n$, $F(z) = \sum A_n z^n$, $S_n(z) = \sum_{k=0}^n A_k z^k$, and $f \prec F$. Since $S_n(w(z)) = \sum_{k=0}^n a_k z^k + \sum_{k=n+1}^\infty b_k z^k$, an application of Theorem L gives the next theorem.

THEOREM A. *Under the conditions of Theorem L, we have*

$$\sum_{k=0}^n |a_k|^2 \le \sum_{k=0}^n |A_k|^2, \quad n \ge 1.$$

Let $T = \{z : |z| = 1\}$. For the following result, see [8].

THEOREM B. *Let w be an inner function and $w(0) = 0$, and let $h \in L^1(T)$. Then $h \circ w \in L^1(T)$ and*

$$\int_0^{2\pi} h(w(e^{i\theta}))\, d\theta = \int_0^{2\pi} h(e^{i\theta})\, d\theta.$$

Combining Theorem A with Theorem B, we find the following.

PROPOSITION A. *Let $f \in H^2$ and $f = F \circ w$, where w is an inner function and $w(0) = 0$. Then*

$$\sum_{k=n}^{\infty} |a_k|^2 \geq \sum_{k=n}^{\infty} |A_k|^2.$$

Since $E_n(f)_2 = \sum_{k=n}^{\infty} |\hat{f}(k)|^2$ with $n \geq 0$ and $f \in H^2$, the following result is a generalization of Proposition A.

THEOREM 1.1. *Suppose that $F \in H^p$ $(1 < p < +\infty)$ and $f = F \circ w$, where w is an inner function with $w(0) = 0$. Then*

$$E_n(f)_p \geq E_n(F)_p.$$

Before we give a proof, for the convenience of the reader we emphasize some similarities between the proofs of Proposition A and Theorem 1.1. Instead of a simple expression for the best approximation in L^2-metric and Littlewood's subordination theorem, we shall use (respectively) the equality (1.1) and Lemma 1.1. Theorem B has the same role in both proofs.

The following result, which plays an essential role in the theory of extremal problems, will be needed in the proof of Theorem 1.1.

THEOREM C. *Let X be a Banach space with dual space X^*, and let S be a closed subspace of X. Then, for each fixed $x \in X$,*

$$\inf_{y \in S} \|x - y\| = \max_{\psi \in S^{\perp}, \|\psi\| = 1} |\psi(x)|,$$

where $S^{\perp}$ is the annihilator of the subspace S, that is, the set of all linear functionals $\psi \in X^*$ such that $\psi(x) = 0$ for all $x \in S$.

Proof of Theorem 1.1. It is well known that the dual space of $L^p = L^p(T)$ $(1 \leq p < \infty)$ is $L^{p'} = L^{p'}(T)$, where the pairing is given by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta, \quad f \in L^p, \ g \in L^{p'}$$

and $1/p + 1/p' = 1$. Therefore, it is easily verified that

$$\mathcal{P}_n^{\perp} = L_n^{p'} = \{g : g \in L^{p'}, \hat{g}(v) = 0, v = 0, 1, 2, \ldots, n\}.$$

Now, an application of Theorem C gives

$$E_n(f)_p = \sup\{|\langle f, g \rangle| : g \in L_n^{p'}, \|g\|_{p'} \leq 1\}. \tag{1.1}$$

To complete the proof we need the following result.

LEMMA 1.1. *Let w be an inner function and let $w(0) = 0$. If $g \in L_n^p$, $1 < p \leq \infty$, then $g \circ w \in L_n^p$ and*

$$\|g \circ w\|_p = \|g\|_p. \tag{1.2}$$

Now, by this lemma and (1.1) (recall that $f = F \circ w$),

$$E_n(f)_p \geq \sup\{|\langle F \circ w, g \circ w\rangle| : g \in L_n^{p'}, \|g\|_{p'} \leq 1\}.$$

Hence, by (1.1) and Theorem B, we obtain Theorem 1.1. □

Proof of Lemma 1.1. First, suppose that $1 < p < \infty$. Next, let

$$G(z) = \sum_{v=n+1}^{\infty} \hat{g}(v) z^v \quad \text{and} \quad g_1 = g - G.$$

A routine application of known results from H^p theory gives that there exists a function $g_2 \in H^p$ such that $g_2(e^{i\theta}) = \overline{g_1(e^{i\theta})}$ a.e. on T. Hence

$$\overline{(g_1 \circ w)\hat{}\,(v)} = \frac{1}{2\pi} \int_0^{2\pi} \overline{g_1(w(e^{i\theta}))e^{-iv\theta}}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} g_2(w(e^{i\theta}))e^{iv\theta}\, d\theta, \quad v \geq 0.$$

Thus $g_1 \circ w \in L_n^p$. Next it is easily seen, from

$$G(w(z)) = \sum_{v=n+1}^{\infty} a_v [w(z)]^v$$

and the hypothesis, that $(G \circ w)\hat{}\,(v) = 0$, $v = 0, 1, \ldots, n$. Thus $G \circ w \in L_n^p$ and consequently $g \circ w \in L_n^p$. To finish the proof of this case, we need only observe that the equality (1.2) follows from Theorem B.

In the case $p = \infty$, we can use Theorem B to conclude that $\|g \circ w\|_\infty = \|g\|_\infty$. Now, it follows from the previous consideration that $(g \circ w)\hat{}\,(v) = 0$, $v = 0, 1, \ldots, n$. Thus $g \circ w \in L_n^\infty$. □

The following result will be needed in Section 3.

PROPOSITION 1.1. *Let $f = IF$, where I is an inner function and $F \in H^p$. Then*

$$E_n(f)_p \geq C_p E_n(F)_p, \quad 1 < p < +\infty.$$

Proof. Let $H_n^{p'} = L_n^{p'} \cap H^{p'}$ where $1/p' + 1/p = 1$, and let g be an arbitrary function in $L_n^{p'}$. Since $1 < p' < \infty$, the M. Riesz theorem (see e.g. [3]) guarantees that the "analytic projection" $Pg(z) = \sum_{n=0}^{+\infty} c_n z^n$ of the original $L^{p'}$ function $g(e^{i\theta}) \sim \sum_{n=-\infty}^{+\infty} c_n e^{in\theta}$ is in $H^{p'}$. Furthermore, there exists a constant A_p such that

$$\|Pg\|_{p'} \leq A_{p'} \|g\|_{p'}, \quad g \in L^{p'}. \tag{1.3}$$

From (1.1) and (1.3) and since $\langle f, g\rangle = \langle f, Pg\rangle$ for every $f \in H^p$ and $g \in L^{p'}$, it follows that

$$\begin{aligned} &\sup\{|\langle f, G\rangle| : G \in H_n^{p'}, \|G\|_{p'} \leq 1\} \\ &\qquad \leq E_n(f)_p \leq A_{p'} \sup\{|\langle f, G\rangle| : G \in H_n^{p'}, \|G\|_{p'} \leq 1\}. \end{aligned} \tag{1.4}$$

Thus (recall that $f = IF$) we have

$$E_n(f)_p \geq \sup\left\{\left|\int_0^{2\pi} F(e^{i\theta}) I(e^{i\theta}) \overline{G(e^{i\theta})}\, d\theta\right| : G \in H_n^{p'}, \|G\|_{p'} \leq 1\right\}.$$

Since $GI \in H_n^{p'}$ for every $G \in H_n^{p'}$ and $\|GI\|_{p'} = \|G\|_{p'}$, we find

$$E_n(f)_p \geq \sup\left\{\left|\int_0^{2\pi} F(e^{i\theta})I(e^{i\theta})\overline{G(e^{i\theta})I(e^{i\theta})}\,d\theta\right| : G\in H_n^{p'}, \|G\|_{p'}\leq 1\right\}.$$

Now, the second inequality in (1.4) gives

$$E_n(f)_p \geq C_p E_n(F)_p,$$

where $C_p = 1/A_{p'}$. □

2. Membership in $A(\alpha, p, q)$

Let

$$K_n(x) = \sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)e^{ikx} \quad (n\geq 1), \quad K_0(x)=0,$$

be Fejer's kernel and let V_n be de la Vallee Poussin's kernel

$$V_{2m}(x) = 2K_{2m}(x) - K_m(x) \quad (m\geq 1), \quad V_0(x) = 0,$$
$$V_{2m+1}(x) = V_{2m}(x) \quad (m\geq 1).$$

If $f\in L^1(0,2\pi]$ we write $V_n(f)(x)$ for

$$V_n * f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})V_n(x-t)\,dt,$$

the convolution of V_n with f. Let $W_{2^n} = V_{2^n} - V_{2^{n-1}}$ for $n\geq 1$.

LEMMA 2.1. *If $1\leq p\leq\infty$, $f\in L^p(0,2\pi)$, and $n\geq 0$, then*

(a) $\|V_n(f)\|_p \leq 3\|f\|_p$;
(b) $\|f - V_n(f)\|_p \leq 4\|f\|_p$;
(c) $\|f * W_{2^n}\|_p \leq 6\|f\|_p$, $n\geq 1$.

Proof. From the definition of Fejer's kernel K_n, it follows that $K_n(x)\geq 0$ and $\int_0^{2\pi} K_n(x)\,dx = 2\pi$. Hence, by combining this with Minkowski's inequality we obtain (a). Using (a), it is easy to prove (b) and (c). □

An immediate corollary of the Riesz projection theorem is: if $1<p<\infty$, $f\in L^p[0,2\pi]$, and $S_nf(e^{ix}) = \sum_{k=0}^n \hat{f}(k)e^{ikx}$, then there exists a constant $C_p > 0$ such that

$$C_p\|f - S_nf\|_p \leq E_n(f)_p \leq \|f - S_nf\|_p.$$

This inequality does not hold for $p = 1, \infty$.

We need the following result, which can be viewed as a generalization of Lemma 2.1 to the case $1\leq p\leq\infty$.

LEMMA 2.2. *Let $1\leq p\leq\infty$, $f\in L^p(0,2\pi)$, and $n\geq 0$. Then*

$$\tfrac{1}{4}\|f - V_{2n}(f)\|_p \leq E_n(f)_p \leq \|f - V_n(f)\|_p. \tag{2.1}$$

Proof. Let $P_n = \sum_{k=0}^{n} a_k e^{ikx}$ be an arbitrary polynomial. Because

$$V_{2n}(k) = \begin{cases} 1 & \text{if } |k| \le n, \\ (2n-|k|)/n & \text{if } n \le k \le 2n, \\ 0 & \text{if } |k| \ge 2n, \end{cases}$$

it is easy to check that $V_{2n}(P_n) = P_n$.

Therefore, by part (a) of Lemma 2.1,

$$\|f-P_n\|_p \ge \tfrac{1}{4}\|(f-P_n)-V_{2n}(f-P_n)\| = \tfrac{1}{4}\|f-V_{2n}(f)\|_p.$$

Thus, we have the left-hand inequality of (2.1); the right-hand inequality follows immediately from the definition of $E_n(f)_p$. □

Let n be a positive integer with $0<\alpha<n$. We say that a function $f \in H(U)$ belongs to the Lipschitz–Besov space $X = A(\alpha, p, q)$, $0 < p \le +\infty$, if

$$\|f\|_X = \left\{ \int_0^1 (1-r)^{(n-\alpha)q-1} M_p^q(r, f^{(n)})\, dr \right\} < +\infty, \quad 0<q<\infty,$$

and

$$\|f\|_X = \sup_{0 \le r < 1} (1-r)^{n-\alpha} M_p(r, f^{(n)}) < +\infty, \quad q = \infty.$$

In the case $q = +\infty$, we will use the notation $A(\alpha, p)$ instead of $A(\alpha, p, \infty)$. By $A_0(\alpha, p)$ we denote the subspace of $A(\alpha, p)$ consisting of $f \in H(U)$ for which

$$(1-r)^{n-\alpha} M_p(r, f^{(n)}) \to 0$$

when $r \to 1$.

It is well known that $A(\alpha_1, p_1, q_1) \subset A(\alpha_2, p_2, q_2)$ if either

(a) $\alpha_1 > \alpha_2$ (q_1 and q_2 need not be related) or
(b) $\alpha_1 = \alpha_2$ and $q_1 \le q_2$.

Let $f = \sum a_k z^k$ be an analytic function on the unit disk U, and let $\beta \ge 0$. The fractional derivative $f^{[\beta]}$ of f is defined by

$$f^{[\beta]}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+\beta+1)}{k!} a_k z^k, \quad z \in U.$$

It is known that the definition of a Lipschitz–Besov space is independent of n, and that we define the same space if we write $f^{[n]}$ instead of $f^{(n)}$.

The following result is probably known (at least partially) to specialists in the field. Our proof is based on the ideas developed in [5].

THEOREM 2.1. *Let $1 \le p \le \infty$ and $0 < q \le +\infty$. Then the following conditions are equivalent:*

(a) $f \in A(\alpha, p, q)$;
(b) $(2^{k\alpha}\|f * W_{2^k}\|_p)_{k=1}^{\infty} \in l^q$;
(c) $(2^{k\alpha} E_{2^k}(f)_p)_{k=1}^{\infty} \in l^q$.

We will make a few definitions and comments before giving the proof of this result.

For $f \in H(U)$, we define $S_n f(z) = \sum_{k=0}^{n} \hat{f}(k) z^k$, $\Delta_n f(z) = S_{2^n} f(z) - S_{2^{n-1}} f(z)$ $(n \geq 1)$, and $\Delta_0(z) = 0$ for $z \in U$. In [5], we proved the following result.

THEOREM MP. *If* $1 < p < \infty$, *then* $f \in A(\alpha, p, q)$ *if and only if*

$$\{2^{k\alpha} \|\Delta_k f\|\}_{k=1}^{\infty} \in l^q. \tag{2.2}$$

The proof of the equivalence of Theorem 2.1(a) and (b) is similar to the one given in [5], where to prove that $f \in A(\alpha, p, q)$ implies (2.2) we used a corollary of the Riesz projection theorem: If $1 < p < \infty$ and if $f \in H^p$, then there exists a constant $C_p > 0$ such that

$$\|\Delta_n f\|_p \leq C_p \|f\|_p.$$

Here we shall instead use part (c) of Lemma 2.1.

Combining the equality $f(z) - S_{2^n} f(z) = \sum_{v=n+1}^{\infty} \Delta_v f(z)$, $z \in U$, with the Riesz projection theorem and Minkowski's inequality, we can prove the following result: If $1 < p < \infty$ and if $f \in H^p$ then there exists a constant $C_p > 0$ such that

$$C_p \|\Delta_n f\|_p \leq E_n(f)_p \leq \sum_{v=n+1}^{\infty} \|\Delta_v f\|_p.$$

Using this inequality and Theorem MP, we can prove that for the case $1 < p < +\infty$, Theorem 2.1(a) holds if and only if (c) holds. To get an extension of this to the case $1 \leq p \leq \infty$, we shall use the inequalities (2.5) and (2.6).

Proof of Theorem 2.1. Let $\beta_k = \|f * W_{2^k}\|_p$ and $\gamma_k = E_{2^k}(f)_p$, $k \geq 1$. By Lemma 2.1(c) and [5, Lemma 3.1, inequality (3.4)], we have

$$\sup_{0 \leq r < 1} C^{-1} 2^{kn} r^{2^k} \beta_k \leq \|f_r^{[n]}\|_p. \tag{2.3}$$

On the other hand, using Minkowski's inequality and again [5, (3.4)], we obtain

$$\|f_r^{[n]}\|_p \leq \sum_{k=1}^{\infty} \|f_r^{[n]} * W_{2^k}\|_p \leq C\left(\sum_{k=2}^{\infty} 2^{kn} \beta_k r^{2^{k-2}} + 2^n \beta_1\right). \tag{2.4}$$

Now, [5, Prop. 4.1], together with (2.3) and (2.4), shows the equivalence of (a) and (b).

Let P_{2^k} denote the polynomial of order 2^k, $k \geq 1$. By Lemma 2.1(c),

$$6\|f - P_{2^k}\|_p \geq \|(f - P_{2^k}) * W_{2^{k+2}}\|_p = \|f * W_{2^{k+2}}\|_p,$$

so that

$$6\gamma_k \geq \beta_{k+2}. \tag{2.5}$$

Thus, (c) implies (b).

Combining Lemma 2.2, the equality $f - V_{2^k} f = \sum_{v=k+1}^{\infty} f * W_{2^v}$, and Minkowski's inequality, we have

$$\gamma_k \leq \sum_{v=k+1}^{\infty} \beta_v. \tag{2.6}$$

Let us prove that (b) implies (c). By (2.6), it suffices to prove that $K_1 = \Sigma(2^{k\alpha}\beta k)^q < +\infty$ implies that $K_2 = \Sigma(2^{k\alpha}R_k)^q < \infty$, where $R_k = \Sigma_{v=k}^{\infty}\beta_k$.

First, let $q \leq 1$. Since $R_k = \beta_k + R_{k+1}$, we have

$$K_2 \leq \sum_{k=1}^{\infty}(2^{k\alpha}\beta_k)^q + 2^{-\alpha q}\sum_{k=1}^{\infty}[2^{(k+1)\alpha}R_{k+1}]^q.$$

This gives the desired result.

If $q \geq 1$, we use the Minkowski inequality to obtain

$$K_2^{1/q} \leq K_1^{1/q} + \left\{\sum_{k=1}^{\infty}(2^{k\alpha}R_{k+1})^q\right\}^{1/q}.$$

The rest is similar to the case $q \leq 1$. □

THEOREM 2.2. *For $1 \leq p \leq +\infty$, the following conditions are equivalent:*

(a) $f \in A_0(\alpha, p)$;
(b) $2^{k\alpha}\|f * W_{2^k}\|_p \to 0$, $k \to +\infty$;
(c) $2^{k\alpha}E_{2^k}(f)_p \to 0$, $k \to +\infty$.

Proof. Let $f \in A_0(\alpha, p)$. For given $\epsilon > 0$, there is an $r_0 \in (0,1)$ such that

$$(1-r)^{n-\alpha}\|f_r^{(n)}\|_p \leq \epsilon, \quad r \geq r_0. \tag{2.7}$$

From (2.3) and (2.7), it follows that there exists a k_0 such that

$$C^{-1}2^{-k(n-\alpha)}2^{kn}\beta_k < \epsilon, \quad k \geq k_0,$$

so that (a) implies (b).

In order to prove the converse, suppose that (b) holds. This means that, for given $\epsilon > 0$, there exists a natural number k_0 such that

$$2^{k\alpha}\beta_k < \epsilon, \quad k \geq k_0. \tag{2.8}$$

Therefore, this inequality and (2.4) show that

$$(1-r)^{n-\alpha}\|f_r^{(n)}\|_p \leq K(1-r)^{n-\alpha}\sum_{k=1}^{k_0}2^{kn}\beta_k + K\epsilon(1-r)^{n-\alpha}\sum_{k=k_0}^{\infty}2^{k(n-\alpha)}r^{2^{k-2}}.$$

Because the first term on the right-hand side is arbitrarily small for r close enough to 1, and since

$$\sum 2^{k(n-\alpha)}r^{2^k} \leq K(1-r)^{\alpha-n},$$

we obtain (a).

As (2.5) shows that (c) implies (b), it remains only to prove that (b) implies (c). Suppose that we have (b), so that (2.8) holds for given $\epsilon > 0$. Combining (2.6) and (2.8) yields

$$\gamma_k \leq \sum_{v=k+1}^{\infty} 2^{-v\alpha} \leq M\epsilon 2^{-k\alpha}$$

so that we have (c). □

THEOREM 2.3. *Let F be an analytic function in U and let w be an inner nonconstant function. If $f = F \circ w$, $1 \le p \le \infty$, and $0 < q \le \infty$, then*

(a) *$f \in A(\alpha, p, q)$ implies $F \in A(\alpha, p, q)$ and*
(b) *$f \in A_0(\alpha, p)$ implies $F \in A_0(\alpha, p)$.*

Proof. Suppose that $f \in A(\alpha, p, q)$. Hence $f \in A(\alpha, p)$ and since $A(\alpha, p) \subset H^p$, we conclude that $f \in H^p$. A result of Stephenson [9] (see also Rudin [7]) states that $F \in H^p$, so that $E_n(f)_p \ge E_n(F)_p$, by Theorem 1.1. Therefore (a) follows from Theorem 2.1.

In a similar manner, by using Theorem 2.2 instead of Theorem 2.1 we can prove (b). □

3. Growth of Means of a Singular Inner Function

LEMMA 3.1. *Let*

$$A(z) = A_s(z) = \exp\left(s\frac{z+1}{z-1}\right), \quad s > 0$$

(the atomic function), and let $1 \le p \le \infty$. Then there are two constants C_1 and C_2 such that

(a) *$E_n(A)_p \le C_1 n^{-1/2p}$, $n \ge 1$, and*
(b) *$E_n(A)_p \ge C_2 n^{-1/2p}$, $n \ge 1$.*

Proof. Mateljević and Pavlović [4] proved, in the case $k = 1$, that if $p > 1/2$ then there exists a constant $C_p > 0$ such that

$$M_p(r, A_s') \le C_p(1-r)^{1/2p-1}, \quad 0 \le r < 1.$$

Using the argument given in [4], it is easy to get a proof of this inequality in the general case $s > 0$. This means that $A \in A(1/2p, p)$. Hence, by Theorem 2.1, we have part (a) of Lemma 3.1.

The case $p = 2$ of part (b) is due to Newman and Shapiro [6]. It is interesting that we can induce the general case to this one. Let $T_n A = A - V_n(A)$. By Hölder's inequality,

$$\|T_n A_s\|_2^2 \le \|T_n A_s\|_p \|T_n A_s\|_{p'},$$

where $1 \le p \le +\infty$ and $1/p + 1/p' = 1$. This inequality, part (a), and Lemma 2.1 show that

$$Cn^{-1/2} \le \|T_n A_s\|_p n^{-1/2p'},$$

so we have (b). □

THEOREM 3.1. *Let φ be an inner function with a nonconstant singular factor, and let $1 \le p \le +\infty$. Then there exists a constant C such that*

$$E_k(\varphi)_p \ge Ck^{-1/2p}.$$

Proof. Consider first the case $1 < p < +\infty$. If φ has no zero in U, it is known (see e.g. [6, p. 254]) that

$$\varphi(z) = e^{i\gamma} A_s[w(z)], \quad z \in U,$$

where γ is a real number, w is an inner function with $w(0) = 0$, $s > 0$, and

$$A_s(z) = \exp\left(s\frac{z+1}{z-1}\right)$$

is the atomic function. Hence, applying Theorem 1.1, we find

$$E_n(\varphi)_p \geq E_n(A_s)_p, \quad n \geq 1.$$

The result then follows from this inequality and Lemma 3.1(b). If φ has zeros, we factor out the Blaschke product $B(z)$ to obtain a nonvanishing inner function $S(z) = \varphi(z)/B(z)$. Therefore, by Proposition 1.1,

$$E_n(\varphi)_p \geq C_p E_n(S)_p) \geq Cn^{-1/2p}, \quad n \geq 1.$$

Consider now the case $p = 1$. By Lemma 2.2,

$$\begin{aligned} E_n(\varphi)_1 &\geq \frac{1}{4}\|\varphi - V_{2n}(\varphi)\|_1 \\ &= \frac{1}{4}\int_0^{2\pi} |\varphi(e^{i\theta}) - V_{2n}(\varphi)(e^{i\theta})|^{-1} |\varphi(e^{i\theta}) - V_{2n}(\varphi)(e^{i\theta})|^2 \, d\theta. \end{aligned}$$

Since $\|\varphi - V_{2n}(\varphi)\|_\infty \leq 4$ by Lemma 2.1(b), we have

$$E_n(\varphi)_1 \geq 4^{-1}\|\varphi - V_{2n}(\varphi)\|_2^2.$$

Hence, by Lemma 2.2 again,

$$E_n(\varphi)_1 \geq 4^{-1} E_{2n}(\varphi)_2^2.$$

Combining the last inequality with $E_{2n}(\varphi)_2 \geq Cn^{-1/4}$ yields the result. □

For $0 < p < \infty$, it is convenient to use the notation $A_0(p) = A_0(1/2p, p)$.

COROLLARY 3.1. *Let φ be an inner function with nonconstant singular factor. Then $\varphi \notin A_0(p)$, $0 < p < +\infty$.*

Proof. Let $0 < p < q < +\infty$, and let n be a natural number such that $n > 1/2p$. Next let $z \in U$. By the Cauchy integral formula we have

$$\varphi^{(n)}(z) = \frac{n!}{2\pi i}\int_{K(z,\rho)} \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta,$$

where $K(z, \rho)$ is the circle defined by $\gamma(\theta) = z + \rho e^{i\theta}$, $0 \leq \theta \leq 2\pi$, and $\rho = \rho(z) = (1 - |z|)/2$. Since $\varphi \in H^\infty$, it follows from the last equality that there exists a positive constant $C > 0$ such that

$$|\varphi^{(n)}(z)| \leq C(1 - |z|)^{-n}, \quad z \in U.$$

Using this inequality and

$$I_q(r, \varphi^{(n)}) = \int_0^{2\pi} |\varphi^{(n)}(re^{it})|^{q-p} |\varphi^{(n)}(re^{it})|^p \, dt, \quad 0 \leq r < 1,$$

we conclude that

$$I_q(r, \varphi^{(n)}) \le C_1(1-r)^{np-nq} I_p(r, \varphi^{(n)}), \quad 0 < r < 1.$$

Thus, if $0 < p < q < \infty$ and $\varphi \in A_0(p)$ then $\varphi \in A_0(q)$. This shows that it is enough to prove the theorem for $p > 1$. Now the result follows from Theorem 2.2 and Theorem 3.1. □

In order to give a geometric interpretation of Corollary 3.1, we introduce the notation

$$l(r) = l(r, \varphi) = \int_0^{2\pi} |\varphi'(re^{it})|\, dt.$$

Note that $l(r)$ is the length of the curve γ_r: $w = \varphi(re^{it})$, $0 \le t \le 2\pi$, for $0 \le r < 1$. If $p = 1$, Corollary 3.1 states that

$$\lim_{r \to 1_-} \sup (1-r)^{1/2} l(r) > 0.$$

In this connection, we conjecture that there exists a constant $C > 0$ such that

$$l(r) = l(r, \varphi) \ge C(1-r)^{-1/2}$$

for r close to 1.

The following result, which is due to P. Ahern [2], is an immediate consequence of Corollary 3.1.

THEOREM Ah. *If S is a singular inner function, $p > 0$, $q > 0$, $0 < \alpha < 1$, and $1/2p \le \alpha$, then*

$$\int_0^1 (1-r)^{(1-\alpha)q-1} M_p^q(r, S')\, dr = +\infty. \tag{3.1}$$

Proof. Conversely, suppose that the integral (3.1) converges. Then

$$(1-r)^{1-\alpha} M_p(r, S') \to 0$$

when $r \to 1$. Hence, since $\alpha \ge 1/2p$, we find $S \in A_0(p)$. But this is not true, by Corollary 3.1. □

The interested reader can find further references related to this result, as well as contributions of other authors, in [1], [2], and [4]. In a forthcoming paper we will give a new approach to Theorem Ah and some related results using Littlewood's subordination principle and duality.

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