# Functional Calculus for Noncommuting Operators

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## 1. Notation and Preliminaries

Throughout this paper, $\Lambda$ stands for the set $\{1, 2, \dots, n\}$ where $n$ is a fixed natural number. For every $k \in \mathbb{N}^* = \{1, 2, \dots\}$ let $F(k, \Lambda)$ be the set of all functions from the set $\{1, 2, \dots, k\}$ to $\Lambda$, and let

$$\mathfrak{F} = \bigcup_{k=0}^{\infty} F(k, \Lambda), \tag{1.1}$$

where $F(0, \Lambda)$ stands for the set $\{0\}$.

A sequence $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ of unilateral shifts on a Hilbert space $\mathcal{H}$ with orthogonal final spaces is called a *$\Lambda$-orthogonal shift* if the operator matrix $[S_1, S_2, \dots, S_n]$ is nonunitary, that is, $\mathcal{L} := \mathcal{H} \ominus (\bigoplus_{\lambda \in \Lambda} S_\lambda \mathcal{H}) \neq \{0\}$. This definition is essentially the same as that from [4]. The dimension of $\mathcal{L}$ is called the *multiplicity* of the $\Lambda$-orthogonal shift. Two $\Lambda$-orthogonal shifts are *unitarily equivalent* if and only if they have the same multiplicity (see [6, Thm. 1.2]).

Let us consider a model $\Lambda$-orthogonal shift of multiplicity 1, acting on the full Fock space [3]

$$F^2(H_n) = \mathbf{C}1 \oplus \bigoplus_{m \geq 1} H_n^{\otimes m}, \tag{1.2}$$

where $H_n$ is an $n$-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, \dots, e_n\}$.

For each $\lambda \in \Lambda$ we define the isometry $S_\lambda$ by

$$S_\lambda h = e_\lambda \otimes h \quad \text{for } h \in F(H_n). \tag{1.3}$$

It is easy to see that $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ is a $\Lambda$-orthogonal shift of multiplicity 1. This model will play an important role in our investigation. We shall denote by $\mathcal{P}$ the set of all $p \in F^2(H_n)$ of the form

$$p = a_0 + \sum_{\substack{1 \leq i_1, \dots, i_k \leq n \\ 1 \leq k \leq m}} a_{i_1 \dots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}, \quad m \in \mathbf{N},$$

where $a_0, a_{i_1 \dots i_k} \in \mathbf{C}$. The set $\mathcal{P}$ may be viewed as the algebra of the polynomials in $n$ noncommuting indeterminates, with $p \otimes q$, $p, q \in \mathcal{P}$, as multi-

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plication. For any sequence $\{T_1, T_2, \ldots, T_n\}$ of bounded operators on a Hilbert space $\mathcal{H}$, let $p(T_1, \ldots, T_n)$ denote the operator acting on $\mathcal{H}$, given by

$$p(T_1, \ldots, T_n) = a_0 I_{\mathcal{H}} + \sum a_{i_1 \ldots i_k} T_{i_1} \cdots T_{i_k}. \tag{1.4}$$

The von Neumann inequality [8] for $(B(\mathcal{H})^n)_1$ (see [6]) asserts that if

$$(T_1, \ldots, T_n) \in (B(\mathcal{H})^n)_1 := \left\{(T_1, \ldots, T_n) \in B(\mathcal{H})^n : \sum_{i=1}^{n} T_i T_i^* \le I_{\mathcal{H}}\right\}$$

and $p \in \mathcal{P}$, then

$$\|p(T_1, \ldots, T_n)\| \le \|p(S_1, \ldots, S_n)\| = \sup_{q \in (\mathcal{P})_1} \|p \otimes q\|_{F^2(H_n)}, \tag{1.5}$$

where

$$(\mathcal{P})_1 := \{q \in \mathcal{P} : \|q\|_{F^2(H_n)} \le 1\}.$$

Now we define $F^\infty$ as being the set of all $g \in F^2(H_n)$ for which

$$\|g\|_\infty := \sup_{q \in (\mathcal{P})_1} \|g \otimes q\|_2 < \infty, \quad \text{where } \|\ \|_2 := \|\ \|_{F^2(H_n)}. \tag{1.6}$$

Let us recall from [6] that, if $f \in F^\infty$ and $g \in F^2(H_n)$, then the multiplication defined by

$$f \otimes g := \lim_{n \to \infty} f \otimes p_n \tag{1.7}$$

(the convergence being in $F^2(H_n)$), where $p_n \in \mathcal{P}$ and $\|p_n - g\|_2 \to 0$, is well-defined and $f \otimes g \in F^2(H_n)$. Notice also that $\|f \otimes g\|_2 \le \|f\|_\infty \|g\|_2$ and, according to [6], $(F^\infty, \|\ \|_\infty)$ is a noncommutative Banach algebra, which can be viewed as a noncommutative analog of the Hardy space $H^\infty$.

Now let us recall the Wold decomposition theorem for sequences of isometries [4]. Let $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ be a sequence of isometries on a Hilbert space $\mathcal{K}$ such that $\sum_{\lambda \in \Lambda} V_\lambda V_\lambda^* \le I_{\mathcal{K}}$. Then $\mathcal{K}$ decomposes into an orthogonal sum $\mathcal{K} = \mathcal{K}_u \oplus \mathcal{K}_s$ such that $\mathcal{K}_u$ and $\mathcal{K}_s$ reduce each operator $V_\lambda$ ($\lambda \in \Lambda$), and we have

$$\left(I_{\mathcal{K}} - \sum_{\lambda \in \Lambda} V_\lambda V_\lambda^*\right)\Bigg|_{\mathcal{K}_u} = 0 \quad \text{and} \quad \{V_\lambda|_{\mathcal{K}_s}\}_{\lambda \in \Lambda}$$

is a $\Lambda$-orthogonal shift acting on $\mathcal{K}_s$. This decomposition is uniquely determined; indeed, we have

$$\mathcal{K}_u = \bigcap_{k=0}^{\infty} \left(\bigoplus_{f \in F(k, \Lambda)} V_f \mathcal{K}\right) \quad \text{and} \quad \mathcal{K}_s = \bigoplus_{f \in \mathcal{F}} V_f \mathcal{L},$$

where $\mathcal{L} = \mathcal{K} \ominus (\bigoplus_{\lambda \in \Lambda} V_\lambda \mathcal{K})$.

We recall from [4] that, for any sequence $\mathcal{T} = \{T_\lambda\}_{\lambda \in \Lambda}$ of operators on a Hilbert space $\mathcal{H}$ such that $\sum_{\lambda \in \Lambda} T_\lambda T_\lambda^* \le I_{\mathcal{H}}$, there exists a minimal isometric dilation $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ on a Hilbert Space $\mathcal{K} \supset \mathcal{H}$ that is uniquely determined up to an isomorphism; that is, the following conditions hold:

(i) $V_\lambda^* V_\lambda = I_{\mathcal{K}}$ for any $\lambda \in \Lambda$;
(ii) $\sum_{\lambda \in \Lambda} V_\lambda V_\lambda^* \le I_{\mathcal{K}}$;
(iii) $V_\lambda^* \mathcal{H} \subset \mathcal{H}$ and $V_\lambda^*|_{\mathcal{H}} = T_\lambda^*$ for any $\lambda \in \Lambda$; and
(iv) $\mathcal{K} = \bigvee_{f \in \mathcal{F}} V_f \mathcal{H}$.

## 2. $C^*(V_1, \dots, V_n)$ and Similarity

It is well known that if two isometries $V_1$ and $V_2$ are similar then they are unitarily equivalent. As a consequence, the $C^*$-algebra generated by $V_1$ is $*$-isometric to the $C^*$-algebra generated by $V_2$.

In this section we extend this result to sequences of isometries with orthogonal final spaces.

THEOREM 2.1. *Let $\{V_i\}_{i=1}^n \subset B(\mathcal{K})$ and $\{W_i\}_{i=1}^n \subset B(\mathcal{K}')$ be two sequences of isometries such that $\sum_{i=1}^n V_i V_i^* \le I_{\mathcal{K}}$ and $\sum_{i=1}^n W_i W_i^* \le I_{\mathcal{K}'}$. If there exists an invertible operator $X \in B(\mathcal{K}, \mathcal{K}')$ such that*

$$XV_i = W_i X, \quad i = 1, 2, \dots, n,$$

*then there exists a unitary operator $U \in B(\mathcal{K}, \mathcal{K}')$ such that*

$$UV_i = W_i U, \quad i = 1, 2, \dots, n.$$

*Proof.* According to the Wold decomposition, $\mathcal{K}$ decomposes into an orthogonal sum $\mathcal{K} = \mathcal{K}_u \oplus \mathcal{K}_s$ such that $\mathcal{K}_u$ and $\mathcal{K}_s$ reduce each operator $V_i$ $(i = 1, 2, \dots, n)$, and we have

$$\left(I_{\mathcal{K}} - \sum_{i=1}^n V_i V_i^*\right)\bigg|_{\mathcal{K}_u} = 0 \quad \text{and} \quad \{V_i|_{\mathcal{K}_s}\}_{i=1}^n$$

is a $\Lambda$-orthogonal shift acting on $\mathcal{K}_s$. Moreover, we have

$$\mathcal{K}_u = \bigcap_{k=0}^{\infty}\left(\bigoplus_{f \in F(k,\Lambda)} V_f \mathcal{K}\right) \quad \text{and} \quad \mathcal{K}_s = \bigoplus_{f \in \mathcal{F}} V_f \mathcal{L}, \tag{2.1}$$

where $\mathcal{L} = \mathcal{K} \ominus (\bigoplus_{i=1}^n V_i \mathcal{K})$.

The Wold decomposition for $\{W_i\}_{i=1}^n$ provides the corresponding spaces $\mathcal{K}_u'$, $\mathcal{K}_s'$, and $\mathcal{L}'$.

Let us denote

$$V_i|_{\mathcal{K}_u} := A_i, \quad V_i|_{\mathcal{K}_s} := B_i, \quad W_i|_{\mathcal{K}_u'} := C_i, \quad \text{and} \quad W_i|_{\mathcal{K}_s'} = D_i. \tag{2.2}$$

Since $V_i^* X^* = X^* W_i^*$ and $X$ is invertible, it is easy to see that

$$X^*(\operatorname{Ker} W_i^*) = \operatorname{Ker} V_i^*, \quad i = 1, 2, \dots, n.$$

Hence, we get

$$X^*\left(\bigcap_{i=1}^n \operatorname{Ker} W_i^*\right) = \bigcap_{i=1}^n \operatorname{Ker} V_i^*.$$

Since $\mathcal{L}' = \bigcap_{i=1}^n \operatorname{Ker} W_i^*$ and $\mathcal{L} = \bigcap_{i=1}^n \operatorname{Ker} V_i^*$, we infer that $\dim \mathcal{L}' = \dim \mathcal{L}$. According to [6, Thm. 1.2] there is a unitary operator $M \in B(\mathcal{K}_s, \mathcal{K}_s')$ such that

$$MB_i = D_i M, \quad i = 1, 2, \dots, n. \tag{2.3}$$

On the other hand, since $XV_i = W_i X$ $(i = 1, 2, \dots, n)$, the relation (2.1) implies $X(\mathcal{K}_u) = \mathcal{K}_u'$. Therefore

$$(X|_{\mathcal{K}_u})(V_i|_{\mathcal{K}_u}) = (W_i|_{\mathcal{K}_u'})(X|_{\mathcal{K}_u}), \quad i = 1, 2, \dots, n. \tag{2.4}$$

Let us denote $X|_{\mathcal{K}_u} := X_0$. Obviously $X_0$ is an invertible operator from $\mathcal{K}_u$ to $\mathcal{K}'_u$. By (2.2) the relation (2.4) becomes

$$X_0 A_i = C_i X_0, \quad i = 1, 2, \ldots . \tag{2.5}$$

Notice that $\sum_{i=1}^n A_i A_i^* = I_{\mathcal{K}_u}$ and $\sum_{i=1}^n C_i C_i^* = I_{\mathcal{K}'_u}$. By (2.5), we have

$$X_0 A_i A_i^* = C_i X_0 A_i^*, \quad i = 1, 2, \ldots, n.$$

Hence, it follows that

$$X_0 = C_1 X_0 A_1^* + C_2 X_0 A_2^* + \cdots + C_n X_0 A_n^*.$$

Since $C_i^* C_j = 0$ $(i \neq j)$ and $C_i^* C_i = I_{\mathcal{K}'_u}$, we deduce that

$$C_i^* X_0 = X_0 A_i^*, \quad i = 1, 2, \ldots, n.$$

Taking the adjoint of these relations, we obtain

$$A_i X_0^* = X_0^* C_i, \quad i = 1, 2, \ldots, n,$$

which implies that

$$A_i X_0^* X_0 = X_0^* C_i X_0 = X_0^* X_0 A_i, \quad i = 1, 2, \ldots, n.$$

Hence,

$$A_i (X_0^* X_0)^{1/2} = (X_0^* X_0)^{1/2} A_i, \quad i = 1, 2, \ldots, n. \tag{2.6}$$

The polar decomposition of $X_0$ gives $X_0 = \Omega R$, where $\Omega$ is a unitary operator and $R = (X^* X_0)^{1/2}$ is an invertible operator. The relation (2.6) becomes

$$A_i R = R A_i, \quad i = 1, 2, \ldots, n.$$

Now, for each $i = 1, 2, \ldots, n$ we have

$$\begin{aligned} C_i \Omega &= C_i \Omega R R^{-1} = C_i X_0 R^{-1} \\ &= X_0 A_i R^{-1} = \Omega R A_i R^{-1} = \Omega A_i. \end{aligned}$$

Therefore

$$C_i \Omega = \Omega A_i \quad \text{for any } i = 1, 2, \ldots, n, \tag{2.7}$$

and $\Omega$ is a unitary operator from $\mathcal{K}_u$ onto $\mathcal{K}'_u$. Let us define the unitary operator $U = M \oplus \Omega$. According to the relations (2.2), (2.3), and (2.7), it follows that

$$U V_i = W_i U, \quad i = 1, 2, \ldots, n.$$

The proof is complete. □

Now let us denote by $C^*(V_1, \ldots, V_n)$ the $C^*$-algebra generated by $\{V_1, \ldots, V_n\}$ (see [1; 2]).

COROLLARY 2.2. *Under the hypothesis of Theorem 2.1, the mapping*

$$\phi : C^*(V_1, \ldots, V_n) \to C^*(W_1, \ldots, W_n)$$

*defined by*

$$\phi(X) = U X U^* \quad \textit{for } X \in C^*(V_1, \ldots, V_n)$$

*is an isometric $*$-isomorphism.*

## 3. Von Neumann Inequality

Let us recall some facts concerning the Cuntz algebra $\mathcal{O}_n$ and a certain extension of $\mathcal{O}_n$. In [2] the $C^*$-algebra $\mathcal{O}_n$ ($n \geq 2$) was defined as the $C^*$-algebra generated by $n$ isometries $V_1, V_2, \ldots, V_n$ such that $\sum_{i=1}^n V_i V_i^* = I$. It was shown that $\mathcal{O}_n$ does not depend, up to canonical isomorphism, on the choice of the generators $V_1, \ldots, V_n$. In other words, if $\hat{V}_1, \ldots, \hat{V}_n$ is a second family of isometries satisfying $\sum_{i=1}^n \hat{V}_i \hat{V}_i^* = I$, then $C^*(\hat{V}_1, \ldots, \hat{V}_n)$ is canonically isomorphic to $C^*(V_1, \ldots, V_n)$; that is, the map $\hat{V}_i \to V_i$ extends to an isomorphism from $C^*(\hat{V}_1, \ldots, \hat{V}_n)$ onto $C^*(V_1, \ldots, V_n)$. In what follows we need the following result due to Cuntz [2, Prop. 3.1].

LEMMA 3.1. *Let $V_1, \ldots, V_n$ be isometries on a Hilbert space $\mathcal{K}$ such that $\sum_{i=1}^n V_i V_i^* \leq I_{\mathcal{K}}$ ($n$ finite). Then the projection $P = I_{\mathcal{K}} - \sum_{i=1}^n V_i V_i^*$ generates a closed two-sided ideal $\mathcal{J}$ in $C^*(V_1, \ldots, V_n)$ which is isomorphic to the $C^*$-algebra of all compact operators on an infinite-dimensional separable Hilbert space, and contains $P$ as a minimal projection. The short sequence*

$$0 \to \mathcal{J} \to C^*(V_1, \ldots, V_n) \to \mathcal{O}_n \to 0 \tag{3.1}$$

*is exact.*

The main result of this section is the following.

THEOREM 3.2. *Let $\{V_i\}_{i=1}^n$ ($n \geq 2$) be a sequence of isometries on a Hilbert space $\mathcal{K}$ such that*

$$\sum_{i=1}^n V_i V_i^* \leq I_{\mathcal{K}}.$$

*Then the $C^*$-algebra $C^*(V_1, \ldots, V_n)$ is $*$-isomorphic either to $C^*(S_1, \ldots, S_n)$ or to $\mathcal{O}_n$.*

*Proof.* According to the Wold decomposition for the sequence $\{V_i\}_{i=1}^n$, the Hilbert space $\mathcal{K}$ decomposes into an orthogonal sum

$$\mathcal{K} = \mathcal{K}_u \oplus \mathcal{K}_s \tag{3.2}$$

such that $\mathcal{K}_u$ and $\mathcal{K}_s$ reduce each operator $V_i$ ($i = 1, 2, \ldots, n$), and we have

$$\sum_{i=1}^n W_i W_i^* = I_{\mathcal{K}_u}. \tag{3.3}$$

Moreover, $\{U_i\}_{i=1}^n$ is a $\Lambda$-orthogonal shift on $\mathcal{K}_s$, where

$$V_i = W_i \oplus U_i, \quad i = 1, 2, \ldots, n,$$

is the decomposition of the operator $V_i$ with respect to (3.2).

Now if $\mathcal{K}_s = \{0\}$ then $\sum_{i=1}^n V_i V_i^* = I_{\mathcal{K}}$ and, according to the result of Cuntz [2], $C^*(V_1, \ldots, V_n)$ is $*$-isomorphic to $\mathcal{O}_n$. Let us consider the case when $\mathcal{K}_s \neq \{0\}$. For any polynomial $p(X_1, \ldots, X_n; Y_1, \ldots, Y_n)$ in $2n$ noncommuting indeterminates we have

$$
\begin{aligned}
&p(V_1, \dots, V_n; V_1^*, \dots, V_n^*) \\
&\qquad = p(W_1, \dots, W_n; W_1^*, \dots, W_n^*) \oplus p(U_1, \dots, U_n; U_1^*, \dots, U_n^*),
\end{aligned}
$$

whence

$$
\begin{aligned}
&\|p(V_1, \dots, V_n; V_1^*, \dots, V_n^*)\| \\
&\quad = \max\{\|p(W_1, \dots, W_n; W_1^*, \dots, W_n^*)\|, \|p(U_1, \dots, U_n; U_1^*, \dots, U_n^*)\|\}. \qquad (3.4)
\end{aligned}
$$

If the multiplicity of the $\Lambda$-orthogonal shift $\{U_1, \dots, U_n\}$ is $\alpha$, then the operator $p(U_1, \dots, U_n; U_1^*, \dots, U_n^*)$ is unitarily equivalent to the direct sum of $\alpha$ copies of $p(S_1, \dots, S_n; S_1^*, \dots, S_n^*)$, where $\{S_1, \dots, S_n\}$ is the model $\Lambda$-orthogonal shift of multiplicity 1 acting on the Fock space $F^2(H_n)$. Therefore,

$$
\|p(U_1, \dots, U_n; U_1^*, \dots, U_n^*)\| = \|p(S_1, \dots, S_n; S_1^*, \dots, S_n^*)\|. \qquad (3.5)
$$

Since $\sum_{i=1}^n W_i W_i^* = I_{\mathcal{K}_u}$, according to [2] we have

$$
\|p(W_1, \dots, W_n; W_1^*, \dots, W_n^*)\| = \|p(\sigma_1, \dots, \sigma_n; \sigma_1^*, \dots, \sigma_n^*)\|, \qquad (3.6)
$$

where $\{\sigma_1, \dots, \sigma_n\}$ is a system of generators for the Cuntz algebra $\mathcal{O}_n$.

On the other hand (see Lemma 3.1), we have the following short exact sequence,

$$
0 \to \mathcal{I} \to C^*(S_1, \dots, S_n) \to \mathcal{O}_n \to 0,
$$

where $\mathcal{I}$ denotes the closed two-sided ideal in $C^*(S_1, \dots, S_n)$ generated by $P_{\mathbf{C}1}$, which is the orthogonal projection from $F^2(H_n)$ onto $\mathbf{C}1$.

Thus, if $\pi$ denotes the quotient map from $B(F^2(H_n))$ onto $B(F^2(H_n))/\mathcal{I}$, we then have

$$
\begin{aligned}
\|p(\sigma_1, \dots, \sigma_n; \sigma_1^*, \dots, \sigma_n^*)\| &= \|p(\pi(S_1), \dots, \pi(S_n); \pi(S_1^*), \dots, \pi(S_n^*))\| \\
&= \|\pi(p(S_1, \dots, S_n; S_1^*, \dots, S_n^*))\| \\
&\le \|p(S_1, \dots, S_n; S_1^*, \dots, S_n^*)\|.
\end{aligned}
$$

Hence, using the relations (3.4), (3.5), and (3.6), we infer that

$$
\|p(V_1, \dots, V_n; V_1^*, \dots, V_n^*)\| = \|p(S_1, \dots, S_n; S_1^*, \dots, S_n^*)\|.
$$

Therefore, the mapping $V_i \mapsto S_i$ $(i = 1, 2, \dots, n)$ extends to an isometry from $C^*(V_1, \dots, V_n)$ onto $C^*(S_1, \dots, S_n)$ which is also a $*$-isomorphism. The proof is complete. □

COROLLARY 3.3. *If* $\{V_1, \dots V_n\}$ *is a sequence of isometries on a Hilbert space* $\mathcal{K}$ *such that*

$$
\sum_{i=1}^n V_i V_i^* \le I_{\mathcal{K}}, \qquad (3.7)
$$

*then*

$$
\|p(V_1, \dots, V_n; V_1^*, \dots, V_n^*)\| \le \|p(S_1, \dots, S_n; S_1^*, \dots, S_n^*)\| \qquad (3.8)
$$

*for any polynomial* $p(X_1, \dots, X_n; Y_1, \dots, Y_n)$ *in* $2n$ *noncommuting indeterminates. Moreover, if* (3.7) *is not an equality then* (3.8) *is an equality.*

Let us now denote by $\mathcal{P}(X, Y)$ the set of all polynomials in $2n$ noncommuting indeterminates $p(X_1, \ldots, X_n; Y_1, \ldots, Y_n)$ of the following form,

$$p(X_1, \ldots, X_n; Y_1, \ldots, Y_n) = \sum a_{i_1, \ldots, j_m} X_{i_1} \cdots X_{i_k} Y_{j_1} \cdots Y_{j_m},$$

where $a_{i_1, \ldots, j_m} \in \mathbf{C}$, $i_1, \ldots, i_k, j_1, \ldots, j_m \in \{0, 1, 2, \ldots, n\}$, and $k, m \in \mathbb{N}^*$. Here we use the convention $X_0 = 1$ and $Y_0 = 1$.

An extension of the von Neumann inequality [6, Thm. 2.1] is as follows.

THEOREM 3.4. *Let $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$, $n \geq 2$. Then $\sum_{i=1}^n T_i T_i^* \leq I_{\mathcal{H}}$ if and only if*

$$\|p(T_1, \ldots, T_n; T_1^*, \ldots, T_n^*)\| \leq \|p(S_1, \ldots, S_n; S_1^*, \ldots, S_n^*)\| \tag{3.9}$$

*for any $p \in \mathcal{P}(X, Y)$.*

*Proof.* Since $(T_1, \ldots, T_n) \in (B(\mathcal{H})^n)_1$, there is a minimal isometric dilation $(V_1, \ldots, V_n) \in (B(\mathcal{K})^n)_1$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that

$$V_i^* V_i = I_{\mathcal{K}}, \quad i = 1, 2, \ldots, n;$$

$$\sum_{i=1}^n V_i V_i^* \leq I_{\mathcal{K}}, \tag{3.10}$$

$$V_i^* |_{\mathcal{H}} = T_i^*, \quad i = 1, 2, \ldots, n. \tag{3.11}$$

By (3.11), it follows that

$$p(T_1, \ldots, T_n; T_1^*, \ldots, T_n^*) = P_{\mathcal{H}} p(V_1, \ldots, V_n; V_1^*, \ldots, V_n^*)|_{\mathcal{H}}, \quad p \in \mathcal{P},$$

where $P_{\mathcal{H}}$ stands for the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Hence, we get

$$\|p(T_1, \ldots, T_n; T_1^*, \ldots, T_n^*)\| \leq \|p(V_1, \ldots, V_n; V_1^*, \ldots, V_n^*)\|.$$

According to Corollary 3.3 the result follows.

Conversely, by setting $p(X, Y) = X_1 Y_1 + \cdots + X_n Y_n$ in (3.9) we obtain

$$\left\|\sum_{i=1}^n T_i T_i^*\right\| \leq \left\|\sum_{i=1}^n S_i S_i^*\right\| \leq 1.$$

The proof is complete. □

Let us remark that this theorem is also true when the relation (3.9) holds for matrices of polynomials in $\mathcal{P}(X, Y)$.

Let $\mathcal{P}(X)$ denote the set of all polynomials in $n$ noncommuting indeterminates $\{X_1, \ldots, X_n\}$. It is clear that $\mathcal{P}(X) \subset \mathcal{P}(X, Y)$ and that $\mathcal{P}(X)$ can be identified with $\mathcal{P} \subset F^2(H_n)$ (see Section 1).

According to Theorem 3.4, for any $p \in \mathcal{P}(X)$ we have

$$\|p(T_1, \ldots, T_n)\| \leq \|p(S_1, \ldots, S_n)\|, \quad p \in \mathcal{P}(X). \tag{3.12}$$

Thus, we again find the von Neumann inequality (1.5) proved in [6, Thm. 2.1]. It would be interesting to know the answer to the following question.

If the relation (3.12) holds for polynomials in $\mathcal{P}(X)$, does this imply that $(T_1, \ldots, T_n)$ is in $(B(\mathcal{H})^n)_1$?

For any polynomial $p \in \mathcal{P}(X)$,

$$p = \sum a_{i_1, i_2, \ldots, i_k} X_{i_1} X_{i_2} \cdots X_{i_k},$$

let us define

$$\tilde{p} = \sum \bar{a}_{i_1, i_2, \ldots, i_k} X_{i_k} \cdots X_{i_2} X_{i_1}.$$

COROLLARY 3.5. *If $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$ such that $\sum_{i=1}^n T_i^* T_i \le I_{\mathcal{H}}$, then*

$$\|p(T_1, \ldots, T_n)\| \le \|\tilde{p}\|_\infty, \quad p \in \mathcal{P}(X).$$

*Proof.* According to (3.12), we have

$$\begin{aligned}\|p(T_1, \ldots, T_n)\| &= \|(p(T_1, \ldots, T_n))^*\| \\ &= \|\tilde{p}(T_1^*, \ldots, T_n^*)\| \le \|\tilde{p}(S_1, \ldots, S_n)\| = \|\tilde{p}\|_\infty. \qquad \square\end{aligned}$$

## 4. $F^\infty$-Functional Calculus

In this section we extend [6, Thm. 3.6] to a more general setting. More precisely, we will obtain a noncommutative analog of the Sz.-Nagy–Foiaş $H^\infty$-functional calculus for completely noncoisometric (c.n.c.) contractions. An important role is played by the functional model for a c.n.c. contraction $[T_1, \ldots, T_n]$ (see [5]) and the von Neumann inequality (1.5).

Let us note that any element $f \in F^2(H_n)$ can be written as follows:

$$f = \sum_{f \in \mathfrak{F}} a_f e_f \quad \text{with } a_f \in \mathbf{C}$$

and

$$\|f\|_2 = \sum_{f \in \mathfrak{F}} |a_f|^2 < \infty, \tag{4.1}$$

where $e_f$ stands for $e_{f(1)} \otimes \cdots \otimes e_{f(k)}$ if $f \in F(k, \Lambda)$, $k \ge 1$, and $e_0 = 1$. We make the natural identification of $e_f \otimes 1$ with $e_f$ for any $f \in \mathfrak{F}$.

For any $0 < r < 1$ and $f = \sum_{f \in \mathfrak{F}} a_f e_f$, define

$$f_r = \sum_{f \in \mathfrak{F}} r_f a_f e_f,$$

where $r_f = r_{f(1)} \cdots r_{f(k)}$, if $f \in F(k, \Lambda)$, $k \ge 1$, $r_0 = 1$, and $r_{f(k)} = r$ for any $k \in \{1, 2, \ldots\}$. Notice that $\|f_r\|_2 \le \|f\|_2$.

LEMMA 4.1. *If $f \in F^2(H_n)$, then*

$$\|f_r - f\|_2 \to 0 \quad \textit{as } r \to 1.$$

*Proof.* An easy computation shows that

$$\|f_r - f\|_2 = \sum_{f \in \mathfrak{F}} (r_f - 1)^2 |a_f|^2 = \sum_{k=1}^\infty (r^k - 1)^2 \sum_{f \in F(k, \Lambda)} |a_f|^2.$$

Because (4.1) holds, it follows that

$$\|f_r - f\|_2 \to 0 \quad \text{as } r \to 1. \qquad \square$$

PROPOSITION 4.2. *If* $f \in F^\infty$ *then* $f_r \in F^\infty$ $(0 < r < 1)$ *and*

$$f(S_1, \ldots, S_n) = \text{so-}\lim_{r \to 1} f_r(S_1, \ldots, S_n),$$

*where* so- *denotes the strong operator topology.*

*Proof.* Since $\|f_r - f\|_2 \to 0$ as $r \to 1$, it follows that, for any $p \in \mathcal{P}$,

$$\|f_r \otimes p - f \otimes p\|_2 \to 0 \quad \text{as } r \to 1. \tag{4.2}$$

On the other hand, since $r < 1$, the inequality (1.5) applied to $(rS_1, rS_2, \ldots, rS_n)$ implies that

$$\|f_r\|_\infty = \|f_r(S_1, \ldots, S_n)\| = \|f(rS_1, \ldots, rS_n)\| \le \|f\|_\infty.$$

Let $\epsilon > 0$ and $h \in F^2(H_n)$. There exists a polynomial $p \in \mathcal{P}$ such that

$$\|h - p\|_2 \le \epsilon / \|f\|_\infty.$$

On the other hand,

$$\begin{aligned} \|f_r \otimes h - f \otimes h\|_2 &\le \|f_r \otimes (h-p)\|_2 + \|(f_r - f) \otimes p\|_2 + \|f \otimes (h-p)\|_2 \\ &\le \|f_r\|_\infty \|h-p\|_2 + \|(f_r - f) \otimes p\|_2 + \|f\|_\infty \|h-p\|_2 \\ &\le \epsilon + \|(f_r - f) \otimes p\|_2. \end{aligned}$$

Taking into account (4.2), we infer that for any $\epsilon > 0$

$$\limsup_{r \to 1} \|f_r \otimes h - f \otimes h\|_2 \le \epsilon.$$

Therefore,

$$\lim_{r \to 1} \|f_r \otimes h - f \otimes h\|_2 = 0,$$

which is equivalent to

$$f(S_1, \ldots, S_n) = \text{so-}\lim_{r \to 1} f_r(S_1, \ldots, S_n).$$

The proof is complete. $\square$

Let us remark that the above proposition remains true if $\{S_1, \ldots, S_n\}$ is replaced by a $\Lambda$-orthogonal shift of arbitrary multiplicity.

Now we extend the functional calculus for $(T_1, \ldots, T_n) \in (B(\mathcal{H})^n)_1$ (see [6, Thm. 3.6]) to a more general setting. Let us recall from [4] that a contraction $[T_1, \ldots, T_n]$ is called completely noncoisometric if there is no $h \in \mathcal{H}$, $h \neq 0$, such that

$$\sum_{f \in F(k, \Lambda)} \|T_f^* h\|^2 = \|h\|^2 \quad \text{for any } k \in \{1, 2, \ldots\}.$$

We remark that $[T_1, \ldots, T_n]$ is c.n.c. if there is no subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that the operator

$$\begin{bmatrix} T_1^* \\ \vdots \\ T_n^* \end{bmatrix} : \mathcal{H}_0 \to \underbrace{\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_0}_{n \text{ times}}$$

is an isometry. For this kind of contraction we have a functional model [5] which will be used in what follows.

THEOREM 4.3. *If $(T_1, \ldots, T_n) \in (B(\mathcal{H})^n)_1$ is c.n.c., then for any $f \in F^\infty$ there exists*

$$\text{so-}\lim_{r \to 1} f_r(T_1, \ldots, T_n) \stackrel{\text{def}}{=} f(T_1, \ldots, T_n).$$

*Moreover, the mapping*

$$f \mapsto f(T_1, \ldots, T_n)$$

*is a contractive homomorphism from the Banach algebra $F^\infty$ to $B(\mathcal{H})$.*

*Proof.* According to [5, Thm. 4.1], we can assume that

$$\mathcal{H} \subset \mathcal{K} = l^2(\mathfrak{F}, \mathfrak{D}_*) \oplus \overline{\Delta_3 l^2(\mathfrak{F}, \mathfrak{D})}$$

and

$$T_\lambda = P_{\mathcal{H}} V_\lambda |_{\mathcal{H}}, \lambda \in \Lambda, \tag{4.3}$$

where $V_\lambda = S_\lambda \oplus C_\lambda$, $\lambda \in \Lambda$, and $\{S_\lambda\}_{\lambda \in \Lambda}$ is the $\Lambda$-orthogonal shift on $l^2(\mathfrak{F}, \mathfrak{D}_*)$. $\{C_\lambda\}_{\lambda \in \Lambda}$ is a sequence of isometries on $\overline{\Delta_3 l^2(\mathfrak{F}, \mathfrak{D})}$ defined by

$$C_\lambda(\Delta_3 v) = \Delta_3(S_\lambda v), \quad v \in l^2(\mathfrak{F}, \mathfrak{L}), \tag{4.4}$$

where $\{S_\lambda\}$ is the $\Lambda$-orthogonal shift on $l^2(\mathfrak{F}, \mathfrak{L})$. Let us recall from [5, Thm. 4.1] that $\Delta_3$ is a contraction and

$$\sum_{\lambda \in \Lambda} C_\lambda C_\lambda^* = I.$$

Let $f \in F^\infty$ and $0 < r < 1$. By (4.3) we infer that

$$f_r(T_1, \ldots, T_n) = P_{\mathcal{H}} f_r(V_1, \ldots, V_n) |_{\mathcal{H}}. \tag{4.5}$$

We claim that so-$\lim_{r \to 1} f_r(V_1, \ldots, V_n)$ exists. Because

$$f_r(V_1, \ldots, V_n) = f_r(S_1, \ldots, S_n) \oplus f_r(C_1, \ldots, C_n)$$

and so-$\lim_{r \to 1} f_r(S_1, \ldots, S_n)$ exists by Proposition 4.2, it is enough to prove that so-$\lim_{r \to 1} f_r(C_1, \ldots, C_n)$ exists on $\overline{\Delta_3 l^2(\mathfrak{F}, \mathfrak{D})}$.

By (4.4), for any $v \in l^2(\mathfrak{F}, \mathfrak{D})$ we have

$$f_r(C_1, \ldots, C_n)(\Delta_3 v) = \Delta_3 f_r(S_1, \ldots, S_n) v,$$

which tends to $\Delta_3 f(S_1, \ldots, S_n) v$ as $r \to 1$ (according to Proposition 4.2). Notice that for any $0 < r < 1$ we have

$$\|f_r(C_1, \ldots, C_n)\| = \|f(rC_1, \ldots, rC_n)\| \le \|f(S_1, \ldots, S_n)\| = \|f\|_\infty. \tag{4.6}$$

The above inequality comes from applying the von Neumann inequality (1.5).

Let $x \in \overline{\Delta_3 l^2(\mathfrak{F}, \mathfrak{D})}$ and $v_k \in l^2(\mathfrak{F}, \mathfrak{D})$ be such that $\|\Delta_3 v_k - x\| \to 0$ as $k \to \infty$. There exists $k_0$ such that if $k \geq k_0$ then $\|\Delta_3 v_k - x\| < \epsilon$. We have:

$$\begin{aligned}
&\|f_r(C_1, \dots, C_n)x - f_{r'}(C_1, \dots, C_n)x\| \\
&\quad \leq \|f_r(C_1, \dots, C_n)(x - \Delta_3 v_{k_0})\| \\
&\qquad + \|((f_r(C_1, \dots, C_n) - f_{r'}(C_1, \dots, C_n))\Delta_3 v_{k_0}\| + \|f_{r'}(C_1, \dots, C_n)(x - \Delta_3 v_{k_0})\| \\
&\quad \leq \|f\|_\infty \|x - \Delta_3 v_{k_0}\| + \|f\|_\infty \|x - \Delta_3 v_{k_0}\| \\
&\qquad + \|\Delta_3 (f_r(S_1, \dots, S_n) - f_{r'}(S_1, \dots, S_n)) v_{k_0}\| \\
&\quad \leq 2\epsilon \|f\|_\infty + \|(f_r(S_1, \dots, S_n) - f_{r'}(S_1, \dots, S_n)) v_{k_0}\|.
\end{aligned}$$

Using Proposition 4.2 and the above inequalities, it follows that

$$\lim_{r \to 1} f_r(C_1, \dots, C_n)x$$

exists for any $x \in \overline{\Delta_3 l^2(\mathfrak{F}, \mathfrak{D})}$.

On the other hand, according to (4.6) we have

$$\|\lim_{r \to 1} f_r(C_1, \dots, C_n)x\| \leq \|f\|_\infty \|x\|.$$

Therefore, the operator

$$f(C_1, \dots, C_n) \stackrel{\text{def}}{=} \text{so-}\lim_{r \to 1} f_r(C_1, \dots, C_n)$$

is well-defined and bounded. Thus so-$\lim_{r \to 1} f_r(V_1, \dots, V_n)$ exists and the relation (4.5) implies that so-$\lim_{r \to 1} f_r(T_1, \dots, T_n)$ exists, and the operator $f(T_1, \dots, T_n)$ defined by

$$f(T_1, \dots, T_n) = \text{so-}\lim_{r \to 1} f_r(T_1, \dots, T_n)$$

is bounded and

$$\|f(T_1, \dots, T_n)\| \leq \|f\|_\infty.$$

The fact that the mapping $f \mapsto f(T_1, \dots, T_n)$ is a homomorphism from $F^\infty$ to $B(\mathcal{H})$ is easy to deduce. □

COROLLARY 4.4. *If $n = 1$ we find again the Sz.-Nagy–Foiaş $H^\infty$-functional calculus for completely noncoisometric contractions.*

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