n-Harmonic Morphisms in Space Are Möbius Transformations

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1. Introduction

Let Ω and Ω' be open subsets of $\mathbf{R}^n$. A mapping $f: \Omega \to \Omega'$ is a harmonic *motion* if whenever $u: \Omega' \to \mathbf{R}^n$ is a harmonic function so too is $v = u \circ f$. We say that f is a harmonic *morphism* if f is continuous and $u \circ f$ is harmonic in $f^{-1}(\mathbf{D})$ whenever $u: \mathbf{D} \to \mathbf{R}$ is harmonic, where $\mathbf{D}$ is an arbitrary open subset of Ω'. Clearly every harmonic morphism is a harmonic motion.

It is easy to see that in the one dimension $n = 1$, harmonic motions are simply affine functions. When $n \geq 2$, by considering the harmonic functions x_i, $i = 1, \dots, n$, and $x_i x_j$, $x_i^2 - x_j^2$ for $i \neq j$, we obtain that each component f_i of f is harmonic, $\langle \nabla f_i, \nabla f_j \rangle = 0$, and $|\nabla f_i| = |\nabla f_j|$ for $i \neq j$. Therefore the differential of f, $Df(x)$, must be a conformal matrix for all $x \in \Omega$. In two dimensions $n = 2$ we conclude that f is analytic or anti-analytic on each component of Ω.

In higher dimensions $n \geq 3$, a classical theorem of Liouville implies that f must be a Möbius map on each component of Ω. Therefore, f can be expressed as a finite composition of similarities (rotations, translations, and reflections on planes) and inversions on spheres. Since these last inversions are not harmonic we conclude that, on each component of Ω,

$$f(x) = \lambda \Theta x + b \tag{1}$$

for certain $\lambda \in \mathbf{R}$, $b \in \mathbf{R}^n$, and Θ an orthogonal matrix. In particular, note that in this Euclidean case harmonic motions are also harmonic morphisms.

More details can be found in [GH], where homeomorphic motions are treated and nonpositive metrics are included, and in [Fu] and [Is], where harmonic morphisms between Riemannian manifolds are studied. Motions of linear partial differential equations with constant coefficients are studied in [Ru]. Finally, a treatment of harmonic morphism from the point of view of abstract potential theory is in [CC].

In this paper we study p-harmonic morphisms. These are defined as above by requiring that they preserve p-harmonic functions, which are solutions of the p-Laplace equation

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$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0, \tag{2}$$

where $1<p<\infty$. A corresponding notion of p-harmonic motion is defined similarly. The functions x_i, $i=1,\dots,n$, are still p-harmonic and so must be the components of f. It is known that p-harmonic motions are $C^{1,\alpha}_{\mathrm{loc}}$ (see [Di; Le]). Thus p-harmonic motions must have Hölder continuous derivatives. We remark that it is not known to us whether every p-harmonic motion is a p-harmonic morphism, although one would certainly conjecture so.

Recently, Heinonen, Kilpeläinen, and Martio [HKM] have studied harmonic morphisms in the general framework of nonlinear potential theory, where it was well known that quasiregular mappings provide examples of A_n-harmonic morphisms. We refer to [HKM] for the general definition of A_p-harmonic morphisms. For our purposes it is enough to remark that p-harmonic morphisms in our sense are A_p-harmonic morphisms for $A(x,h)=|h|^{p-2}h$. The striking result proved in [HKM] is that every sense-preserving A_n-harmonic morphism is a K-quasiregular mapping. Moreover, many properties of A_p-harmonic morphisms are discussed. Our main result, Theorem 1 in Section 1, complements the theory developed in [HKM] by proving that sense-preserving n-harmonic morphisms are indeed 1-quasiregular mappings, and thus Möbius transformations if $n\geq 3$.

The key fact used both in [HKM] and in this paper is that $\log(1/|f(x)|)$ must be n-harmonic in the complement of $f^{-1}(0)$, since $\log(1/|x|)$ is n-harmonic away from 0. One then uses estimates for the growth of singular n-harmonic functions near their singularity to derive properties of f. Since the behavior of solutions of (2) is much better known than that of solutions of the A_n-harmonic equation, we are able to obtain more information in that case, allowing us to conclude that K is indeed 1.

In Section 2 we discuss the case $1<p<n$. It is proved in [HKM] that general A_p-morphisms are open mappings if they are not constants for $1<p\leq n$. In Theorem 2 we establish that discrete p-harmonic morphisms are similarities. Therefore, on each component of Ω they are expressed as in (1). Finally, in Section 3 we extend Theorem 2 to discrete p-harmonic morphisms for $p>n$.

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2. The Case $p=n$

THEOREM 1. *Let $f:\Omega\to\Omega'$ be a nonconstant sense-preserving n-harmonic morphism. Then f is a* 1*-quasiregular mapping. In particular, f is a Möbius transformation if $n\geq 3$.*

We have already established that $f\in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, and by [HKM] we know that f is K-quasiregular for some $K\geq 1$. Choose $x\in\Omega$. We will prove that

$$Df(x)^t Df(x)=\lambda(x)I, \tag{3}$$

where $\lambda(x) \in \mathbf{R}$. Without loss of generality we may assume that $x=0$, $f(0)=0$, $0 \in \Omega'$, and $Df(0)$ is not the zero matrix.

Set $u(x)=\log(1/|x|)$ and $v(x)=\log(1/|f(x)|)$. Since u is n-harmonic in $\Omega' \setminus \{0\}$, we conclude that v is n-harmonic in $\Omega \setminus f^{-1}(0)$. The set $f^{-1}(0)$ is discrete since f is quasiregular. Therefore we can find a small ball B centered at 0 such that v is a nonnegative n-harmonic function in the punctured ball $B \setminus \{0\}$. Such a v is termed a *singular* n-harmonic function; its asymptotic behavior at the singularity has been obtained by Kichenassamy and Veron [KV] who extended previous work by Serrin [Se], where it is proven that u/v is bounded at 0. Then, Theorem 1.1 of [KV] implies that there exists $\gamma \in \mathbf{R}$ such that

$$v(x)-\gamma u(x) \text{ is bounded in } B \setminus \{0\} \tag{4}$$

and

$$\lim_{x \to 0} |x| \nabla (v(x)-\gamma u(x)) = 0. \tag{5}$$

Note that necessarily $\gamma \neq 0$ since otherwise 0 would be a removable singularity for v. We now use the particular form of u and v to extract information about f from (4) and (5).

LEMMA 1. *There exists a constant $C>0$ such that*

$$\frac{1}{C}|x| \leq |f(x)| \leq C|x| \tag{6}$$

for $x \in B \setminus \{0\}$. Moreover, the constant γ in (4) *is* 1.

Proof. By (4) we have

$$\left|\log \frac{1}{|f(x)|} - \gamma \log \frac{1}{|x|}\right| = \left|\log \frac{|x|^\gamma}{|f(x)|}\right| \leq C_2$$

for some $C_2>0$. If $|x|^\gamma \geq |f(x)|$ we deduce that

$$1 \leq \frac{|x|^\gamma}{|f(x)|} \leq e^{C_2}.$$

Otherwise $|x|^\gamma < |f(x)|$ and we have

$$1 \geq \frac{|x|^\gamma}{|f(x)|} \geq e^{-C_2}.$$

Therefore, in either case, we obtain

$$e^{-C_2}|x|^\gamma \leq |f(x)| \leq e^{C_2}|x|^\gamma. \tag{7}$$

Since f is $C^{1,\alpha}$ at 0, for each β with $0<\beta<\alpha$ we have

$$f(x)=Df(0)\cdot x + o(|x|^{1+\beta}). \tag{8}$$

Because of the hypothesis $Df(0) \neq 0$, the lemma follows from (7) and (8).

□

LEMMA 2. *For each $x \in \mathbf{R}^n$ we have*

$$|D^t f(0) Df(0)x||x| = |Df(0)x|^2. \tag{9}$$

Proof. In our situation the limit (5) becomes

$$\lim_{x\to 0} |x| \left| \frac{Df(x)^t \cdot f(x)}{|f(x)|^2} - \frac{x}{|x|^2} \right| = 0. \tag{10}$$

Next we calculate and make use of (8). By squaring (10) and using Lemma 1, we obtain

$$\lim_{x\to 0} \frac{|x|^2 |Df(x)^t f(x)|^2 - 2|f(x)|^2 \langle Df(x)^t f(x), x\rangle + |f(x)|^4}{|x|^5} = 0. \tag{11}$$

Choosing β so that $0 < \beta < \alpha$, we have

$$Df(x)^t = Df(0)^t + o(|x|^\beta). \tag{12}$$

Using (8) and (12), we obtain the following expansions for the numerator in (11):

$$|x|^2 |Df(x)^t f(x)|^2 = |x|^2 |D^t f(0) Df(0)x|^2 + o(|x|^{4+\beta}), \tag{13}$$

$$\begin{aligned} -2|f(x)|^2 \langle Df(x)^t f(x), x\rangle = {} & -2\langle D^t f(0) Df(0)x, x\rangle |Df(0)x|^2 \\ & + o(|x|^{4+\beta}), \end{aligned} \tag{14}$$

and

$$|f(x)|^4 = |Df(0)x|^4 + o(|x|^{4+\beta}). \tag{15}$$

From (11) we deduce that given $\epsilon > 0$ we can find a small $r > 0$ such that for $0 < |x| < r$,

$$\left| |x|^2 |D^t f(x) Df(x)|^2 - 2|f(x)|^2 \langle D^t f(x) f(x), x\rangle + |f(x)|^4 \right| \le \epsilon |x|^5.$$

Using now (13), (14), and (15), we get

$$\begin{aligned} \left| |x|^2 |D^t f(0) Df(0)x|^2 - 2\langle D^t f(0) Df(0)x, x\rangle |Df(0)x|^2 - |Df(0)x|^4 \right| & \\ \le \epsilon |x|^5 + C_3 |x|^{4+\beta} & \end{aligned}$$

for some constant $C_3 > 0$. Therefore we obtain

$$\begin{aligned} \lim_{x\to 0} \frac{|D^t f(0) Df(0)x|^2}{|x|^2} - 2\frac{\langle D^t f(0) Df(0)x, x\rangle}{|x|^4} |Df(0)x|^2 & \\ + \frac{|Df(0)x|^4}{|x|^4} = 0. & \end{aligned} \tag{16}$$

Note that $\langle D^t f(0) Df(0)x, x\rangle |Df(0)x|^2 = |Df(0)x|^4$ and that the expression inside the limit in (16) is homogeneous of degree 0 in x to conclude (9). □

LEMMA 3. *Let A be an $n \times n$ matrix such that*

$$|A^t Ax||x| = |Ax|^2$$

for all $x \in \mathbf{R}^n$. Then $A^t A = \lambda I$ for some $\lambda \in \mathbf{R}$.

Proof. If $A^tAx \neq 0$ then we must have

$$\frac{\langle A^tAx, x\rangle}{|A^tAx||x|} = 1.$$

Thus either A^tAx is parallel to x or $A^tAx = 0$, which implies $A^tAx = \lambda_x x$ for some scalar $\lambda_x \in \mathbf{R}$ depending on x. Write $A^tA = O^{-1}DO$, where O is an orthogonal matrix and D a diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$. We then have $DOx = \lambda_x Ox$. Therefore, for each $i = 1, \ldots, n$, $\lambda_i (Ox)_i = \lambda_x (Ox)_i$. Given $x \in \mathbf{R}^n$ not zero, at least one of $(Ox)_i$ must be nonzero. Thus $\lambda_x = \lambda_i$ for some i. Choose $x \in \mathbf{R}^n$ such that $(Ox)_i \neq 0$ for all $i = 1, \ldots, n$. We conclude that $\lambda_i = \lambda_j = \lambda$ for all i and j and that $\lambda_x = \lambda$ for all $x \in \mathbf{R}^n$. □

Theorem 1 now follows from Lemmas 1, 2, and 3.

REMARK 1. Our proof uses essentially the fact that f is a discrete mapping. As a matter of fact our proof shows that discrete n-harmonic morphisms are 1-quasiregular mappings. It is not known whether n-harmonic morphisms must be sense-preserving or sense-reversing or whether they are discrete. However, as shown in [HKM] in the range $n-1 < p \leq n$, n-harmonic morphisms are discrete if and only if they are sense-preserving or sense-reversing.

3. The Case $1 < p < n$

THEOREM 2. *Let $f: \Omega \to \Omega'$ be a discrete p-harmonic morphism, where $1 < p < n$. Then on each component of Ω we have*

$$f(x) = \lambda \Theta x + b$$

for certain $\lambda \in \mathbf{R}$, $b \in \mathbf{R}^n$, and Θ an orthogonal matrix.

Proof. The proof of this theorem is similar to the proof of Theorem 1. Without loss of generality we assume that Ω is connected, $0 \in \Omega$, $f(0) = 0 \in \Omega'$. Set $u(x) = |x|^{(p-n)/(p-1)}$ and $v(x) = |f(x)|^{(p-n)/(p-1)}$. Since u is p-harmonic in $\Omega' \setminus \{0\}$, we conclude that v is p-harmonic in $\Omega \setminus f(0)$. By the discreteness hypothesis we can find a small ball B centered at zero such that v is a non-negative p-harmonic function in the punctured ball $B \setminus \{0\}$. Once again we appeal to Theorem 1.1 in [KV] to infer the existence of $\gamma \in \mathbf{R} \setminus \{0\}$ such that

$$v(x) - \gamma u(x) \text{ is bounded on } B \setminus \{0\} \tag{17}$$

and

$$\lim_{x \to 0} |x|^{(n-1)/(p-1)} |\nabla(v(x) - \gamma u(x))| = 0. \tag{18}$$

Note that in this case (17) immediately implies that γ is positive and

$$\lim_{x \to 0} \frac{|f(x)|}{|x|} = \gamma^{(p-1)/(p-n)}. \tag{19}$$

The estimate for the gradients (18) now becomes

$$\lim_{x\to 0}|x|^{(n-1)/(p-1)}\left|\left(\frac{p-n}{p-1}\right)|f(x)|^{(p-n)/(p-1)-2}D^tf(x)f(x)\right.$$
$$\left. -\gamma\left(\frac{p-n}{p-1}\right)|x|^{(p-n)/(p-1)-2}x\right|=0. \tag{20}$$

Using (19), we rewrite (20) as follows:

$$\lim_{x\to 0}\frac{|x|^{2+(n-p)/(p-1)}Df^t(x)\cdot f(x)-\gamma|f(x)|^{2+(n-p)/(p-1)}x}{|x|^{3+(n-p)/(p-1)}}=0.$$

Therefore, given any $\epsilon>0$, there exists $r>0$ such that if $0<|x|<r$ we have

$$\left||x|^{2+(n-p)/(p-1)}Df^t(x)f(x)-\gamma|f(x)|^{2+(n-p)/(p-1)}x\right|\leq\epsilon|x|^{3+(n-p)/(p-1)},$$

which combined with (8) and (12) becomes

$$\left||x|^{2+(n-p)/(p-1)}Df^t(0)Df(0)x-\gamma|f(x)|^{2+(n-p)/(p-1)}x\right|$$
$$\leq\epsilon|x|^{3+(n-p)/(p-1)}. \tag{21}$$

Dividing both terms by $|x|^{2+(n-p)/(p-1)}$, we get

$$\left|Df^t(0)Df(0)x-\gamma\left(\frac{|f(x)|}{|x|}\right)^{2+(n-p)/(p-1)}x\right|\leq\epsilon|x|. \tag{22}$$

With the help of (19) we conclude that

$$Df^t(0)Df(0)x=\gamma^{2((p-1)/(p-n))}x$$

for all $x\in\mathbf{R}^n$. Setting $\mu=\gamma^{(p-1)/(n-p)}$, we deduce that $\mu Df(0)$ is an orthogonal matrix.

We have established that for each $x\in\Omega$ there is a positive number μ_x such that $\mu_x Df(x)$ is an orthogonal matrix. At this point we need the fact that open discrete maps in any dimension are either sense-preserving or sense-reversing [Vä], and we will assume that f is sense-preserving, the other case being similar. Thus, we conclude that f is a 1-quasiregular mapping.

Suppose now that $n\geq 3$. Then f must be a Möbius transformation and thus can be expressed as a finite composition of rotations, translations, and inversions on spheres. Since the composition of two p-harmonic morphisms is again a p-harmonic morphism and the components of an inversion on a sphere are not p-harmonic for $p\neq n$, we conclude that these inversions cannot occur. Therefore f must be a similarity, proving Theorem 2 in this case.

It remains to settle the case $n=2$. In this case f must be a holomorphic function. Thus the components of f must be both harmonic and p-harmonic, where $1<p<2$. Write $f=u+iv$ and rewrite the p-Laplacian as

$$\operatorname{div}|\nabla u|^{p-2}\nabla u=|\nabla u|^{p-2}\left\{\Delta u+(p-2)\frac{\Delta_\infty u}{|\nabla u|^2}\right\}, \tag{23}$$

where

$$\Delta_\infty u = \sum_{i,j=1}^{2} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}$$

is the so-called infinite Laplacian. Note that $\nabla u \neq 0$ so that (23) makes sense pointwise. If u is both harmonic and p-harmonic, equality (23) implies that $\Delta_\infty u = 0$. Aronsson [Ar, Thm. 2] proves that any function which is both harmonic and infinite-harmonic must be either linear or

$$u(x,y) = a \arctan\left(\frac{y-y_o}{x-x_o}\right) + b \quad \text{for some } a, b \in \mathbf{R} \text{ and } (x_o, y_o) \in \mathbf{R}^2.$$

In the latter case we may assume without loss of generality that $u(x,y) = \arctan(y/x)$. Therefore $v(x,y) = \log(x^2+y^2)^{1/2} + c$ for some $c \in \mathbf{R}$, since v is the conjugate function of u. An elementary calculation shows that

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = (2-p)(x^2+y^2)^{-p/2}. \tag{24}$$

Therefore v cannot be p-harmonic, since $p \neq 2$. We conclude that u and v must both be linear functions, thereby proving Theorem 2. □

4. The Case $p > n$

THEOREM 3. *Let $f: \Omega \to \Omega'$ be a discrete p-harmonic morphism where $p > n$. Then on each component of Ω we have*

$$f(x) = \lambda \Theta x + b$$

for certain $\lambda \in \mathbf{R}$, $b \in \mathbf{R}^n$, and Θ an orthogonal matrix.

Proof. It follows from Theorem 2.1 in [HKM] that f must be open. As in the proof of Theorem 2, we may assume that Ω is connected, f is orientation-preserving, $0 \in \Omega$, and $f(0) = 0 \in \Omega'$. We define u and v as in the proof of Theorem 2. Observe now that $(p-n)/(p-1) > 0$, so that u and v are bounded in a neighborhood of zero. It follows from (8) that for some $C_4 > 0$,

$$|f(x)| \leq C_4 |x|;$$

hence

$$|v(x)| \leq C_4^{(p-n)/(p-1)} |u(x)|.$$

We can now apply the corresponding isotropy result in Remark 1.6 of [KV] to conclude that (19) and (20) also hold in this case. At this point the rest of the proof continues as in Theorem 2. □

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