The Bers–Nielsen Kernels and Souls of Open Surfaces with Negative Curvature

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Dedicated to the memory of Professor Lipman Bers

0. Introduction

The purpose of this paper is to study the infinite Nielsen kernel for certain noncompact and complete hyperbolic surfaces. The precise description of the infinite Nielsen kernel for an open surface was first introduced by Bers [Be] (see Definition 0.1 below). For any given open Riemann surface X of constant negative curvature with a finitely generated fundamental group, we can assign the infinite Nielsen kernel $N^{\infty}(X)$ to X such that $N^{\infty}(X)$ is the limiting set of a nested sequence of certain Riemann surfaces $\{N^k(X)\}_{k=1}^{+\infty}$ associated with X. We shall show in this paper that the infinite Nielsen kernel $N^{\infty}(X)$ of an open surface X has no interior points. Furthermore, we will prove that the infinite Nielsen kernel is equal to the limit of a sequence of homeomorphic 1-dimensional graphs in a given surface X, with respect to the Gromov–Hausdorff distance function on the space of all subsets in X.

In order to develop a theory to study the Teichmüller space $\mathfrak{T}(S)$ for a noncompact surface S, Bers [Be] in 1976 called for an investigation of the Nielsen kernel. Let X be a Riemann surface of finite type (n, k, m), with $m \geq 1$. This means that X can be conformally embedded in a closed Riemann surface Y of genus n so that $Y \backslash X$ consists of $m \geq 1$ disjoint closed disks and $k \geq 0$ additional points, called the punctures (or cusps) of X. It is well known that there is a unique, conformally equivalent, complete hyperbolic metric on X (cf. [Ne; G1]) which has m expanding tubes provided that $2n-2+k+m>0$. In fact, the condition $2n-2+k+m>0$ ensures that the Euler number $e(X)$ of the surface X is negative, and hence the universal cover of X with the lifted metric is conformal to the unit disk with the Poincaré metric, a metric of constant curvature. Therefore, in what follows we always assume that the Riemann surface X is of type (n, k, m) with $2n-2+k+m>0$, and that X has a complete hyperbolic metric of constant negative curvature -1.

For each expanding tube in X, there is a boundary loop C_i that is freely homotopic in $\bar{X} = X \cup X(\infty)$ to a unique simple closed geodesic $\hat{C}_i$ in X, and $\hat{C}_i$ and $\hat{C}_j$ are disjoint if $1 \leq i \neq j \leq m$ (cf. [Th, §5.3.3]). The Nielsen kernel

Received June 7, 1991. Revision received September 24, 1993.
The author was supported in part by NSF grants DMS-9102212 and DMS-9303711.
Michigan Math. J. 41 (1994).

$N(X)$ of X is the interior of the Riemann surface with boundary obtained from X by removing the m annuli bounded by pairs C_i and $\hat{C}_i$ for $1 \le i \le m$. It is known that the Nielsen kernel $N(X)$ is the smallest open, non-empty convex subset of X with respect to the hyperbolic metric.

Moreover, $N(X)$ has the same finite type (n, k, m) as X. One can therefore iterate the construction above, forming the nested sequence of Riemann surfaces $\{N^k(X)\}_{k=1}^{+\infty}$ where $N^1(X) = N(X)$, $N^{k+1}(X) = N(N^k(X))$, and

$$N^{k+1}(X) \subset N^k(X) \subset \cdots \subset N^1(X) \subset X$$

for $k = 1, 2, \ldots$.

DEFINITION 0.1 [Be]. Let X be a Riemann surface and let $N(X), N^2(X), \ldots, N^k(X)$ be as above. Then the set $N^\infty(X) = \bigcap_{k=1}^\infty N^k(X)$ is called the *infinite Nielsen kernel* of X.

The infinite Nielsen kernel has been studied extensively by Bers, Wason, Halpern, Earle and others (cf. [Wa; H1; H2; E1; E2]). The study of the infinite Nielsen kernel can provide some useful information about the Teichmüller space $\mathfrak{T}_{n,k,m}$ of type (n, k, m), where $\mathfrak{T}_{n,k,m}$ is the set of all open hyperbolic surfaces of type (n, k, m).

Earle [E2] recently computed the infinite Nielsen kernel $N^\infty(X)$ explicitly for a number of special cases in which $N^\infty(X)$ turns out to be a 1-dimensional spline consisting of geodesic arcs. However, for an arbitrary Riemann surface X we know very little about the structure of $N^\infty(X)$. For instance, it is still not known whether the Hausdorff dimension of $N^\infty(X)$ is equal to 1 for an arbitrary open surface X. Our main results in this paper will indicate that the infinite Nielsen kernel $N^\infty(X)$ is a very thin subset in X. In the first part of this paper we will prove the following theorem.

THEOREM A. *Let X be a Riemann surface of finite type (n, k, m) with $m \ge 1$ and $2n-2+k+m > 0$. Then the infinite Nielsen kernel $N^\infty(X)$ of X has no interior points.*

In order to describe further geometric and topological properties of $N^\infty(X)$, we would like to recall the definition of the Gromov–Hausdorff distance function introduced by Gromov [G2, p. 144]. For a subset A of a metric space X, we denote the ϵ-neighborhood of A by $U_\epsilon(A)$. The Gromov–Hausdorff distance between subsets A and B is given by

$$d_X(A, B) = \inf\{\epsilon \mid U_\epsilon(A) \supset B,\ U_\epsilon(B) \supset A\}. \tag{0.1}$$

In the second part of this paper, we will show that the infinite Nielsen kernel is the limit of a sequence of 1-dimensional graphs in the Gromov–Hausdorff topology described here.

Let X be a complete Riemann surface of type (n, k, m) with $2n-2+k+m > 0$. Suppose that X has constant negative curvature -1. We are particularly interested in 1-dimensional graphs that are "cut loci" of certain geodesic cycles in X. If A is a subset of a hyperbolic surface X, then the cut locus $\mathrm{cut}(A)$ of A in X is simply the set of all x in X for which there are at least two geometrically distinct length-minimizing geodesics joining x and A. This definition is given in [CE], where some geometric properties of the cut locus can also be found.

In the early 1970s, Cheeger and Gromoll [CG] carried out an extensive study of the structure of a complete manifold X of nonnegative curvature. They assigned a lower-dimensional, compact, totally geodesic submanifold S to each noncompact, complete manifold X of nonnegative curvature. The lower-dimensional submanifold S in X above is called a *soul* of the given manifold X, according to Cheeger and Gromoll. One of the theorems in [CG] asserts that the soul S is a strong deformation retraction of the manifold X. Inspired by Bers [Be] and Cheeger–Gromoll [CG], we would like to find appropriate 1-dimensional souls for 2-dimensional open surfaces of negative curvature. Hence, we would like to introduce the following definition of the soul of a surface X.

DEFINITION 0.2. Let X be a complete hyperbolic surface of type (n, k, m) with $2n-2+k+m>0$, and let $\hat{C}_1, \ldots, \hat{C}_m$ be m disjoint, simple closed geodesics corresponding to m expanding tubes. The cut locus of the subset $\sigma = \bigcup_{j=1}^{m} \hat{C}_j$ is said to be the *soul* of X, denoted by $S(X)$, and

$$S(X) = \mathrm{cut}(\sigma). \tag{0.2}$$

There are several reasons for us to introduce the notion of “soul”. First, for a given surface X, we are trying to find a smaller-dimensional set of X in a relatively canonical way which carries a lot of the topology and is distinguished geometrically. For instance, our soul $S(X)$ of X has the same negative Euler number as X does, and $S(X)$ is a spline consisting of geodesic arcs in X. Second, our construction of souls of open hyperbolic surfaces is similar to the construction of souls of [CG], because it uses iterated intersections of compact sets to reduce the manifold X to the smaller representative subspaces. Third, for a given hyperbolic surface X, we note that its soul $S(X)$ has interesting properties by comparing with the results of [CG]. For example, there is *one and only one* soul $S(X)$ for a given hyperbolic surface X. Furthermore, the soul $S(X)$ is a strong deformation retraction of surface X. The soul $S(X)$ is a compact graph if X has no cusp points. A careful inspection shows that, for the examples of Riemann surface X described by Earle in [E2, §4], the infinite Nielsen kernel $N^\infty(X)$ coincides with the soul of X.

For an arbitrary Riemann surface X, we note that one can construct a useful sequence of souls as follows. Let the metric g_j be the complete and conformal hyperbolic metric defined on the jth Nielsen kernel $N^j(X)$ and let $S_j(X)$, the jth soul of X, be the soul of the Riemann surface $(N^j(X), g_j)$. That is,

$$S_j(X) = S(N^j(X)). \tag{0.3}$$

It will be shown in Section 3 that all of the souls $S_j(X)$ have the same topological structure and are 1-dimensional splines consisting of geodesic arcs.

There are interesting relations between souls and the infinite Nielsen kernel, which can be described by the following theorem.

THEOREM B. *Let X be an open surface of finite type $(n, 0, m)$ with $m \geq 1$ and without any cusp points. Then the infinite Nielsen kernel of X is equal to the limit of souls*

$$\lim_{j\to+\infty} d_X(N^\infty(X), S_j(X)) = 0, \tag{0.4}$$

where $S_j(X)$ is the jth *soul of X and d_X is the distance function of X.*

We would like to make some comments on Theorem B. Suppose that g_0 is a metric on X and that $L_j = L_{g_0}(S_j(X))$ is the 1-dimensional Hausdorff measure of $S_j(X)$ with respect to the metric g_0. It is still unknown whether or not the terms of $\{L_j\}_{j=1}^{+\infty}$ are uniformly bounded. If one could find a uniform upper bound for the L_j, then using Theorem B one could show that (i) the infinite Nielsen kernel $N^\infty(X)$ is a 1-dimensional rectifiable set and (ii) $N^\infty(X)$ is Lipschitz homeomorphic to $S_1(X)$.

If an open hyperbolic surface X has finitely many cusps, then it will also be shown in Section 4 that

$$\lim_{j\to+\infty} d_X(N^\infty(X)\cap F, S_j(X)\cap F) = 0 \tag{0.5}$$

for each bounded compact set $F \subset X$.

The paper follows this plan: Section 1 will investigate a singular elliptic problem for a Nielsen kernel; Section 2 considers the so-called Fermi radius and conformal changes on a Nielsen kernel. We also derive some estimates for the conformal factor in terms of the Fermi radius. Sections 3 and 4 contain the proofs of Theorems B and A, respectively.

Throughout the paper, we always assume that the surface X has the negative Euler number of type (n, k, m) with m expanding tubes, where $m \geq 1$.

1. A Singular Elliptic Problem for the Nielsen Kernel

In this section, we will set the stage for the study of Nielsen kernels by introducing a singular elliptic problem.

Let us begin with the canonical, complete hyperbolic metric g_0 on a given surface X of finite type (n, k, m). Suppose that g_1 is the canonical, complete hyperbolic metric on the Nielsen kernel $N(X)$, which is conformal to g_0. Then there exists a function u defined on $N(X)$ so that $g_1 = e^{2u} g_0$. Since both metrics g_0 and g_1 have the constant curvature -1, it is known that the conformal factor e^{2u} satisfies the differential equation (cf. [KW])

$$\Delta_{g_0} u = e^{2u} - 1, \tag{1.1}$$

where Δ_{g_0} is the Laplacian operator on the metric g_0. Our Laplacian has the sign so that $\Delta u = +u''$ for all functions u on $\mathbf{R}^1$.

Note that the subset $\partial N(X)$ is a compact subset in (X, g_0). As $(N(X), g_1)$ is a complete and noncompact metric space, our solution u of (1.1) is unbounded on $N(X)$. More precisely, for any $q \in \partial N(X) \subset X$, $\lim_{p\to q} u(p) = +\infty$. All known a priori estimates of (1.1) are only applied to bounded, smooth solutions on closed manifolds. Therefore, we need to make additional efforts to derive some totally new estimates for solutions u of (1.1), which depend only on the topological structure of X and its injectivity radius.

First, let us recall an elementary but useful estimate for singular solutions of (1.1).

1.1. The Schwarz Lemma

LEMMA 1.1. *Let (X, g_0) and $(N(X), g_1)$ be complete hyperbolic surfaces with $g_1 = e^{2u} g_0$ and $N(X) \subset X$. Then*

$$e^{2u(q)} \geq 1 \tag{1.2}$$

for all q in $N(X)$.

Proof. In order to illustrate why Lemma 1.1 is true, we will present a short proof for a special case.

Special case: The surface X does not have any cusp points.

Let $\partial N(X)$ denote the boundary of $N(X)$ in X with respect to the metric g_0. Since $(N(X), g_1)$ is a complete Riemannian metric space, the conformal factor e^{2u} must be infinity along $\partial N(X)$. Recall that, for any $q \in \partial N(X) \subset X$, $\lim_{p\to q} u(p) = +\infty$. Therefore, the minimum points of u lie in the interior of $N(X)$. Suppose that q_0 is a minimum point of u such that $u(q_0) = \min\{u(q) | q \in N(X)\}$. Then we notice that

$$0 \leq \Delta u|_{q_0} = e^{2u(q_0)} - 1.$$

It follows that Lemma 1.1 is true if X does not have any cusp points. □

The proof of Lemma 1.1 for the general case uses a version of the Schwarz lemma, which was conveyed to me by Professor Clifford Earle.

General case: The Riemann surface X may have some cusp points.

We represent X as D/Γ, where D is the open unit disk with the Poincaré metric and $\Gamma = \pi_1(M)$. On the unit circle S^1, there is an open set $G = \bigcup_{\beta\in\Lambda} I_\beta$ of S^1 such that each open arc I_β corresponds to some expanding tube of X. In fact, for each I_β, the stabilizer G_β of I_β in $\pi_1(X)$ is generated by a hyperbolic element whose axis $\hat{C}_\beta = A(\beta)$ is a simple closed geodesic on an expanding tube. We denote by B_β^- the hyperbolic half-plane bounded by I_β and $\tilde{A}(\beta)$. See Figure 1.

The convex set $\bigcap_{\beta\in\Lambda}(D - B_\beta^-)$ is called the *Nielsen region* of X. It is not hard to see that $\tilde{N} = \bigcap_{\beta\in\Lambda}(D - B_\beta^-)$ is the universal cover of $N(X)$ with the

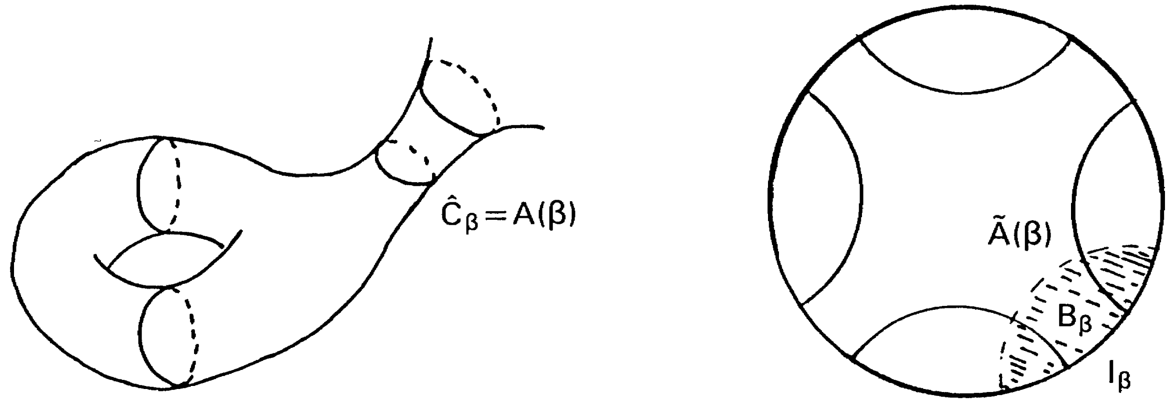

Figure 1

lifted metric $\tilde{g}_0$. The Riemann mapping theorem tells us that there is a unique biholomorphic map $f\colon \tilde{N} \to D$.

A simple computation shows that the lifted conformal factor $e^{\tilde{u}}$ is exactly equal to the function

$$h(z) = \frac{|f'(z)|[1-|z|^2]}{1-|f(z)|^2}.$$

It is sufficient to show that

$$h(z) \geq 1 \tag{1.3}$$

for all $z \in \tilde{N}$.

Since the Möbius transforms are isometries of the Poincaré disk of D, and act on D transitively, we need only verify (1.3) for the special case $f(0)=0$. In this case, we observe that

$$f^{-1}\colon D \to \tilde{N} \subset D$$

is also a holomorphic function. It follows from the Schwarz lemma that

$$\frac{1}{|f'(0)|} = \left|\frac{df^{-1}}{dz}(0)\right| \leq 1.$$

Hence, we obtain

$$h(0) = |f'(0)| \geq 1. \tag{1.3'}$$

Equality holds if and only if $\tilde{N} = D$. This completes the proof of (1.3) and the inequality of the lemma

$$e^{u(q)} \geq 1 \tag{1.2'}$$

for all $q \in N(X)$. □

The following corollary strengthens the result of Lemma 1.1.

COROLLARY 1.1. *Let $W_1 \subset W_2$ be two simply connected open domains in the Poincaré disk with its usual hyperbolic metric g_0. Suppose that f is a biholomorphic map from W_1 to W_2. Then f is a distance-increasing map, which means that*

$$d_{g_0}(f(p), f(q)) \geq d_{g_0}(p, q) \tag{1.4}$$

for all p and q in W_1. *In particular, the pull-back metric* $g_2 = f^* g_0 = e^{2w} g_0$ *satisfies the inequality* $e^{2w(q)} \geq 1$ *for all* $q \in W_1$. *Equality holds if and only if* $W_1 = W_2$.

Proof. Let $F: W_2 \to D$ be a biholomorphic map given by the Riemann mapping theorem, and let $F^* g_0$ be the pull-back metric defined on W_2 via the map F. Hence, $F^* g_0$ can be written as

$$F^* g_0 = e^{2u_2} g_0,$$

since F is conformal. For the same reason, we have

$$(F \circ f)^* g_0 |_q = e^{2u_1} g_0 |_q$$

for all $q \in W_1$. Notice that the Riemann surface $(W_2, F^* g_0)$ is isometric to the Poincaré disk. Applying Lemma 1.1 to the map $f: (W_1, F^* g_0) \to (W_2, F^* g_0)$, one sees that $e^{2w} = e^{2(u_1 - u_2)} \geq 0$ for all $q \in W_1$. □

To illustrate the usefulness of Lemma 1.1, we present the following application.

Corollary 1.2. *Let* (X, g_0) *and* $(N(X), g_1)$ *be as in Lemma 1.1. Then the inclusion map* $i: N(X) \hookrightarrow X$ *is a distance-decreasing map, that is,*

$$d_{g_1}(p, q) \geq d_{g_0}(p, q) \tag{1.5}$$

for all $p, q \in N(X)$, *where* d_g *is the distance function of the metric* g.

We will derive more refined estimates for the conformal scalar function e^{2u} in Sections 2 and 3. At the moment, we would like to make some observations on relations between the infinite Nielsen kernel and a priori estimates of e^{2u}, in order to demonstrate the main strategy for our approach.

1.2. Estimates of Conformal Factors and the Infinite Nielsen Kernel

In this subsection, we shall discuss the geometrical interpretations of a priori estimates for the conformal scalar functions. Specifically, we will use estimates to study the structure of the infinite Nielsen kernel.

For simplicity, we denote the canonical, complete hyperbolic metric on the jth Nielsen kernel $N^j(X)$ by g_j and its conformal scalar factor by e^{2u_j}, so that

$$g_j = e^{2u_j} g_{j-1} \tag{1.6}$$

for $j = 1, 2, \ldots$.

Our approach is motivated by the following observation.

Theorem 1.3. *Let* X, $N(X), \ldots, N^j(X)$, g_j, *and* u_j *be as above. If there is a constant number* $C > 1$ *such that*

$$e^{2u_j(q)} \geq C > 1 \tag{1.7}$$

for all $q \in N^j(X)$ *and* $j = 1, 2, \ldots$, *then the infinite Nielsen kernel* $N^\infty(X)$ *has no interior points.*

Proof. Suppose, to the contrary, there were an interior point $q_\infty \in N^\infty(X)$. Then there would be a non-empty open subset $U \subset N^\infty(X)$.

Let $A_j = \text{area}(U, g_j)$ stand for the area of U with respect to the metric g_j. By assumption

$$A_j = \int_U e^{2u_j}\, dA_{g_{j-1}}, \tag{1.8}$$

where $dA_{g_{j-1}}$ is the area element of the metric g_{j-1}. It follows immediately from (1.8) that

$$A_j \geq C^j A_0. \tag{1.9}$$

On the other hand, we notice that

$$\begin{aligned} A_j &= \text{area}(U, g_j) \\ &\leq \text{area}(N^{j+1}(X), g_j) \\ &= 2\pi |e(X)| \end{aligned} \tag{1.10}$$

by the Gauss–Bonnet formula, where $e(X)$ is the Euler number of X.

Combining (1.9) and (1.10) we would have

$$C^j \leq \frac{2\pi |e(X)|}{A_0}$$

for all $j \geq 1$, which contradicts our assumption that $C > 1$. □

2. The Fermi Radius and Conformal Factors of the Nielsen Kernel

We will introduce the notion of the Fermi radius in order to measure the thickness of the Nielsen kernel. Some relevant a priori estimates for the conformal factors on Nielsen kernels will be derived in terms of the Fermi radius.

DEFINITION 2.1. Let W be a domain of a Riemann surface (Y, g). Then the Fermi radius $\gamma_g(W)$ of W is given by

$$\gamma_g(W) = \sup\{d_g(q, \partial W) \mid q \in W\}. \tag{2.1}$$

The name "Fermi" came into Definition 2.1 because the geometric quantity $\gamma_g(W)$ is clearly related to the Fermi coordinate chart along the boundary ∂W.

For applications of the Fermi radius in this paper, the domain W will be the Nielsen kernel of X. If a complete hyperbolic surface X has k cusps, it is easy to see that $\gamma_g(N(X)) = +\infty$. Therefore, we need to treat two separate cases of complete hyperbolic surfaces.

Case 1: A hyperbolic surface is of type $(n, 0, m)$, which means that the surface does not have any cusp points. Equivalently, a hyperbolic surface X satisfies the condition $\gamma_g(N(X)) < +\infty$, where $N(X)$ is the Nielsen kernel. When X does not have any cusp points, its fundamental group $\pi_1(X)$ is an axial group.

Case 2: A hyperbolic surface X is of type (n, k, m) with $k \geq 1$, which means that the surface has at least one cusp point. In this case, the hyperbolic surface X also satisfies the condition $\gamma_g(N(X)) = +\infty$. When X has a cusp point, its fundamental group $\pi_1(X)$ has at least one parabolic element.

For relevant background, see [G1]. In order to use the Fermi radius, we must develop some preliminaries.

2.1. The Fermi Radius and Estimates for Conformal Scalar Functions

Let us recall that the structure of the infinite Nielsen kernel $N^\infty(X)$ is related to the uniform estimates for the conformal functions (cf. Theorem 1.3). The purpose of this subsection is to derive some a priori estimates for conformal functions in terms of our new geometric quantity—the Fermi radius.

One of our basic estimates is given in the following theorem.

THEOREM 2.1. *Let (X, g_0) be a complete, hyperbolic surface of type $(p, 0, m)$ and let $N(X)$ be its Nielsen kernel. Suppose that the metric g_1 is the complete, conformal, hyperbolic metric on $N(X)$ with $g_1 = e^{2u} g_0$. Then*

$$e^{u(q)} \geq \sqrt{1 + 4e^{-2\gamma_0}} \tag{2.2}$$

for all q in $N(X)$, where $\gamma_0 \geq \gamma_{g_0}(N(X))$, the Fermi radius of $N(X)$.

Proof. Given any point q in $N(X)$, there is a simple closed g_0 geodesic $\sigma \subset \partial N(X)$ such that

$$d_{g_0}(\sigma, q) \leq \gamma_{g_0}(N(X)) \leq \gamma_0. \tag{2.3}$$

In order to prove that $e^{2u(q)} \geq 1 + 4e^{-2\gamma_0}$, we use the universal cover $\tilde{X}$ of X with lifted metric $\tilde{g}_0$. For simplicity, we may choose $\tilde{X}$ to be the Poincaré upper half-plane so that the positive y-axis covers the geodesic σ and the second quadrant covers the expanding tube corresponding to σ. Therefore, the first quadrant contains the Nielsen region $\tilde{N}(X)$, which is the universal cover of $N(X)$. By assumption, there is a lift $\tilde{q}$ of q with

$$d_{\tilde{g}_0}(\tilde{q}, \tilde{\sigma}) = d_{g_0}(q, \sigma). \tag{2.4}$$

If $\tilde{g}_1$ is the lifted metric of g_1 then it is clear that $\tilde{g}_1(\tilde{q}) = e^{2u(\pi(\tilde{g}))}\tilde{g}_0$, where $\tilde{u}(\tilde{q}) = u(\pi(\tilde{q}))$ and $\pi: \tilde{X} \to X$ is the covering map. It is sufficient to verify that

$$e^{\tilde{u}(\tilde{q})} \geq \sqrt{1 + 4e^{-2\gamma_0}}. \tag{2.5}$$

The verification of assertion (2.5) takes two steps. First, we let $F: \tilde{N}(X) \to \tilde{X}$ be the biholomorphic map given by the Riemann mapping theorem. It is

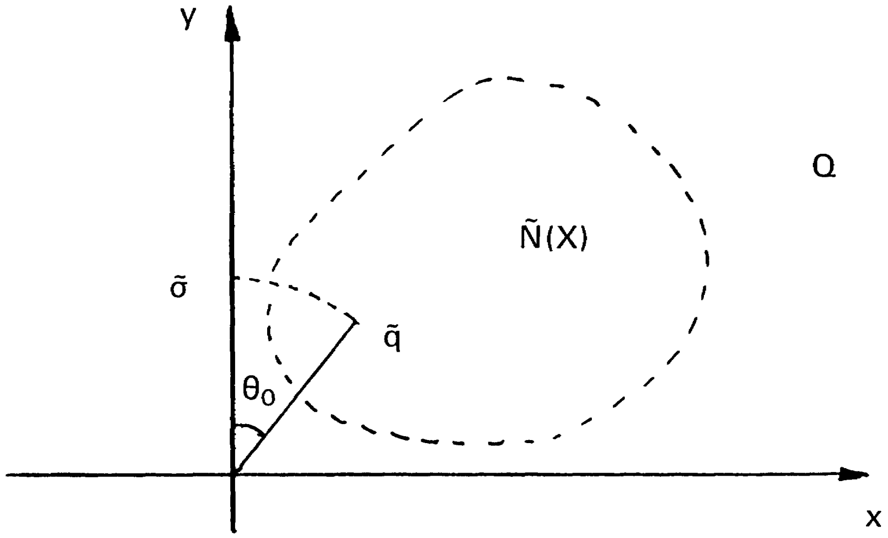

Figure 2

easy to see that

$$\tilde{g}_1 = F^*\tilde{g}_0 = e^{2\tilde{u}}\tilde{g}_0, \tag{2.6}$$

where $F^*\tilde{g}_0$ is the pull-back Poincaré metric via the map F. See Figure 2.

Following Bers and Halpern (cf. [Be; H1]), we can decompose the Riemann map F as $F = f_1 \circ f_2$:

$$\tilde{N}(X) \xrightarrow{f_1} Q \xrightarrow{f_2} \tilde{X},$$

where Q is the first quadrant, $\tilde{X}$ is the Poincaré upper half-plane, and f_2: $Q \to \tilde{X}$ is given by $z \to z^2$. The map $f_1 : \tilde{N}(X) \to Q$ is biholomorphic. The existence of maps f_1 and f_2 is guaranteed by the Riemann mapping theorem (cf. [Ne]).

It follows from Corollary 1.1 that $f_1 : (\tilde{N}(X), \tilde{g}_0) \to (Q, \tilde{g}_0)$ is a distance-increasing map. In particular, the pull-back metric $f_1^* g_0$ has the property

$$f_1^*\tilde{g}_0 = e^{2w}\tilde{g}_0 \tag{2.7}$$

with

$$e^{2w(\tilde{q})} \geq 1 \tag{2.8}$$

for all $\tilde{q} \in \tilde{N}(X)$.

Notice that $F^*\tilde{g}_0 = (f_1 \circ f_2)^*\tilde{g}_0 = f_2^* f_1^* \tilde{g}_0 \geq f_2^*\tilde{g}_0$. Hence, it is sufficient to show that $f_2^*\tilde{g}_0 = e^{2v}\tilde{g}_0$ satisfies the condition

$$e^{v(\tilde{q})} \geq \sqrt{1 + \frac{4e^{2\gamma_0}}{(e^{2\gamma_0} - 1)^2}} \tag{2.9}$$

for the lifted point $\tilde{q}$. It is easy to see that $\tilde{u} = v + w$, $w \geq 0$.

Now, we can evaluate the conformal factor $e^{2v(\tilde{q})}$ in terms of $\tilde{q}$ explicitly, since $f_2(z) = z^2$ for $z \in Q$. A simple calculation shows that

$$e^{v(\tilde{q})} = \frac{|f_2'(z)|}{\operatorname{Im} f_2(z)} \operatorname{Im} z$$

$$= \frac{2|z|y}{2xy}$$
$$= \sqrt{1+(y/x)^2}$$
$$= \sqrt{1+[\cot(\theta_0)]^2}, \tag{2.10}$$

where $\tilde{q} = z = x + iy$, $x = \eta \sin\theta_0$, and $y = \eta \cos\theta_0$ for some $\eta > 0$.

The angle $\theta_0 = \pi/2 - \arg\tilde{q}$ is clearly related to the hyperbolic distance between $\tilde{q}$ and the positive y-axis $\tilde{\sigma}$. Recall that $d_{\tilde{g}_0}(\tilde{q}, \tilde{\sigma}) \le \gamma_0$, and observe that one can choose the non-arclength parameterized geodesic α joining $\tilde{q}$ and $\tilde{\sigma}$ given by

$$\alpha(\theta) = (\eta \sin\theta, \eta\cos\theta)$$

for $0 \le \theta \le \theta_0$. Furthermore, a calculation shows that

$$\gamma_0 \ge d_{g_0}(\tilde{\sigma}, \tilde{q})$$
$$= \int_0^{\theta_0} \|\alpha'(\theta)\|_{g_0}\, d\theta$$
$$= \int_0^{\theta_0} \frac{\|\alpha'(\theta)\|_{\mathbf{R}^2}}{\operatorname{Im}\alpha(\theta)}\, d\theta$$
$$= \int_0^{\theta_0} \frac{d\theta}{\cos\theta}$$
$$= \log|\cot[\tfrac{1}{2}(\pi/2-\theta_0)]|. \tag{2.11}$$

It follows from (2.11) that the following inequalities are true:

$$\cot[\tfrac{1}{2}(\pi/2-\theta_0)] \le e^{\gamma_0}, \tag{2.12}$$
$$\tfrac{1}{2}(\pi/2-\theta_0) \ge \cot^{-1}(e^{\gamma_0}), \tag{2.13}$$

and

$$\theta_0 \le 2\cot^{-1}(e^{\gamma_0}) - \pi/2, \tag{2.14}$$

where $\cot^{-1} = \operatorname{arc\,cot}$ is the inverse of the cotangent function satisfying $-\pi/2 < \cot^{-1}(x) < \pi/2$ for all $x \in \mathbf{R}$.

Using (2.14), one can see that

$$\cot\theta_0 \ge \cot[2\cot^{-1}(e^{\gamma_0}) - \pi/2]$$
$$= \tan[2\cot^{-1}(e^{\gamma_0})]$$
$$= \frac{2\tan[\cot^{-1} e^{\gamma_0}]}{1-\{\tan[\cot^{-1} e^{\gamma_0}]\}^2}$$
$$= \frac{2e^{-\gamma_0}}{e^{2\gamma_0}-1}. \tag{2.15}$$

Combining (2.10) and (2.15), we finally get

$$\begin{aligned} e^{\bar{u}(\bar{q})} &\geq e^{v(\bar{q})} \\ &\geq \sqrt{1+\frac{4e^{2\gamma_0}}{(e^{2\gamma_0}-1)^2}} \\ &\geq \sqrt{1+4e^{-2\gamma_0}}. \end{aligned}$$

This completes the proof of Theorem 2.1. □

2.2. *Open Hyperbolic Surfaces without Cusp Points*

We shall estimate the Fermi radius of the Nielsen kernel by a method which we learned from C. Croke (cf. [Cr, p. 428]).

When an open and complete hyperbolic surface X is of type $(n,0,m)$, the surface X does not have any cusp points. Furthermore, the injectivity radius of X is given by

$$\rho(X)=\mathrm{inj}(X)=\tfrac{1}{2}\min\{L(\sigma)\mid \sigma \text{ is a closed geodesic of } X\}$$

(see [CE, p. 95]), where by $L(\sigma)$ we mean the length of σ.

LEMMA 2.3. *Let (X,g_0) be an open complete hyperbolic surface of type $(n,0,m)$ with $N(X)$ its Nielsen kernel. Then the Fermi radius $\gamma_{g_0}(N(X))$ of $N(X)$ has an upper bound*

$$\gamma_{g_0}(N(X))\leq 16|e(X)|[1+1/\rho], \tag{2.16}$$

where $e(x)$ is the Euler number of X.

Proof. Let $d(x,y)$ denote the distance between x and y in X. Since $\overline{N(X)}$ is a compact subset of X, we can choose a point $q\in N(X)$ such that

$$\begin{aligned} d(q_0,\partial N) &= \gamma_{g_0}(N(X)) \\ &= \max\{d(q,\partial N(X))\mid q\in N(X)\}. \end{aligned}$$

As we pointed out earlier, $\overline{N(X)}$ is a convex subset of X. Hence there is a length-minimizing geodesic $\sigma\colon [0,L]\to\overline{N(X)}$ joining $\partial N(X)$ and q_0, where $L=L(\sigma)=d(q_0,\partial N)$ and σ is arclength parameterized.

We divide our proof into two separate cases.

Case 1: The injectivity radius $\rho(X)$ of X is at least 1.

If $\gamma_{g_0}(N(X))\leq 1$, the estimate (2.16) clearly holds. Let k be the integer such that $k+1\geq\gamma_{g_0}(N(X))=L\geq k\geq 1$. Choose $\sigma(0)=q_0,q_1,\dots,q_k=\sigma(L)$ along σ such that $d_{g_0}(q_i,q_{i+1})\geq 1$. Then the geodesic balls $B_{1/2}(q_i)$ will be disjoint and have area at least $\frac{1}{2}\pi(\frac{1}{2})^2=\pi/8$. Thus, we see that the following is true:

$$\begin{aligned} 2\pi|e(X)| &\geq \mathrm{area}(N(X),g_0) \\ &\geq (k+1)(\pi/8)\geq(\pi/8)\gamma_0(N(X)). \end{aligned}$$

This shows that

$$\gamma_{g_0}(N(X)) \le 16|e(X)|. \tag{2.17}$$

Case 2: The injectivity radius is at most 1.

We may assume that $L = \gamma_{g_0}(N(X)) \ge \rho$; otherwise, $\gamma_{g_0}(N(X)) \le \rho \le 1 < 16$ and (2.16) holds. Therefore, we may choose k to be the integer with $k+1 \ge L/\rho \ge k \ge 1$. Pick $\sigma(0) = q_0, q_1, \ldots, q_k = \sigma(L)$ along σ so that $d(q_i, q_{i+1}) \ge \rho(X)$. Then the geodesic balls $B_{\rho/2}(q_i)$ will be disjoint and have area at least $\frac{1}{2}\pi(\rho/2)^2$. Thus we also have

$$\begin{aligned} 2\pi|e(X)| &\ge (k+1)(\pi/8)\rho^2 \ge (L/\rho)(\pi/8)\rho^2 \\ &= \gamma_{g_0}(N(X))(\pi\rho/8), \end{aligned}$$

and hence

$$\gamma_{g_0}(N(X)) \le (16/\rho)|e(X)|.$$

This completes the proof of Lemma 2.2. □

The following theorem follows from Lemmas 1.1 and 2.3.

THEOREM 2.4. *Let X be an open complete hyperbolic surface of type $(n, 0, m)$ with $N^j(X)$ its jth Nielsen kernel. Then there is a uniform estimate*

$$\gamma_{g_j}(N^{j+1}(X)) \le 16|e(X)|\{1 + 1/\rho\}$$

for all $j \ge 0$, where $e(X) = 2 - 2n - m < 0$, $m \ge 1$, and ρ is the injectivity radius of X.

Proof. It follows from Lemma 1.1 that

$$\rho_{j+1} = \rho(N^{j+1}(X), g_j) \ge \rho(N^j(X), g_{j-1}) \ge \rho(X, g_0) = \rho.$$

Using Lemma 2.3, one has

$$\begin{aligned} \gamma_{g_i}(N^{j+1}(X)) &\le 16|e(N^j(X))|[1 + 1/\rho_{j+1}] \\ &\le 16|e(X)|[1 + 1/\rho]. \end{aligned}$$ □

Now we are ready to give a uniform estimate for all the conformal scalar functions e^{2u_j}, where $g_j = e^{2u_j} g_{j-1}$.

COROLLARY 2.5. *Let (X, g_0) be a complete hyperbolic surface of type $(n, 0, m)$ with g_j the conformal, complete, hyperbolic metric on its jth Nielsen kernel $N^j(X)$. Suppose that $g_j = e^{2u_j} g_{j-1}$. Then*

$$e^{2u_j(q)} \ge C > 1 \tag{2.18}$$

for all $q \in N^j(X)$, $j = 1, 2, \ldots$, where $C > 1$ and C is independent of j and q.

Proof. It follows from Theorem 2.1 that

$$e^{2u_j(q)} \ge 1 + 4e^{-2\gamma_j},$$

where $\gamma_j = \gamma_{g_{j-1}}(N^j(X))$ is the Fermi radius of $N^j(X)$ in the surface $(N^{j-1}(X), g_{j-1})$.

Now Theorem 2.4 says that there is a uniform upper bound for γ_j, namely

$$\gamma_j \le 16|e(X)|(1+1/\rho).$$

Hence we choose $C=1+4\exp[-32|e(X)|(1+1/\rho)]>1$ as the desired constant. □

3. Souls of Open Surfaces

In this section we shall discuss the relations between the infinite Nielsen kernel and the limit of souls. We will make some additional comments after the completion of the proof of Theorem B.

3.1. Proof of Theorem B

Let (X, g_0) be a complete hyperbolic surface of type $(n, 0, m)$ with g_j the conformal, complete, hyperbolic metric on its jth Nielsen kernel $N^j(X)$. Hence, $g_j = e^{2w_j} g_0$ for some smooth function w_j defined on $N^j(X)$. It follows from Corollary 2.5 that

$$g_j \ge C g_{j-1} \ge \cdots \ge C^j g_0. \tag{3.1}$$

This is to say

$$e^{2w_j(q)} = e^{2\sum_{n=1}^{j} u_i} \ge C^j \tag{3.2}$$

for all $q \in N^j(X)$, where $C>1$ and C is a constant number independent of $j=1,2,\ldots$.

Recall that the Hausdorff distance d_X with respect to the metric g_0 is given by

$$d_X(A,B) = \inf\{\epsilon \mid d_X(x,B) \le \epsilon,\ d_X(A,y) \le \epsilon \text{ for all } x \in A,\ y \in B\}.$$

For any given integer j, it is easy to see that both $N^\infty(X)$ and the $(j+1)$th soul $S_{j+1}(X)$ are contained in $N^{j+1}(X)$. Thus, it is clear that

$$d_X(N^\infty(X), S_{j+1}(X)) \le 2\gamma_{g_0}(N^{j+1}(X)) \le (2/C^j)\gamma_{g_j}(N^{j+1}(X)). \tag{3.3}$$

It follows immediately from Theorem 2.4 and (3.3) that

$$d_X(N^\infty(X), S_{j+1}(X)) \le (32/C^j)|e(X)|\{1+1/\rho\}, \tag{3.4}$$

where ρ is the injectivity radius of (X, g_0) and $e(X)$ is the Euler number of X.

By letting $j \to +\infty$ in (3.4), we have

$$\lim_{j\to+\infty} d_X(N^\infty(X), S_{j+1}(X)) = 0$$

since $C>1$. This finishes the proof of Theorem B. □

3.2. Lipschitz Convergence and the Hausdorff Dimension

We would like to address a problem that is related to the Hausdorff dimension of the infinite Nielsen kernel. It will be shown that all souls $S_j(X)$ are

1-dimensional rectifiable subsets of X, and that all have the same topological structure.

PROPOSITION 3.1. (i) *Let σ and η be two geodesics in the Poincaré disk. The set $E=\{q \mid d(q,\sigma)=d(q,\eta)\}$ forms either a geodesic or the union of two geodesics.*

(ii) *If X is an open hyperbolic surface of type (n,k,m), then $S(X)$ is the union of finitely many geodesic segments. Hence, $S(X)$ is a rectifiable subset. Furthermore, the number of geodesic segments in $S(X)$ is at most $4(8k+2m+1)(4k+m)$.*

Proof. (i) We may assume that both σ and η have unit speed. It follows from the convexity of the distance function that σ and η intersect in at most one interior point (or a point at infinity) unless $\sigma=\eta$.

Case 1: σ and η meet at an interior point p.

In this case, E is the union of two geodesics which pass through p and have equal angles with σ and η.

Case 2: σ and η meet a point at infinity.

Let α be an arc on a horocycle with center $\sigma(\infty)=\eta(\infty)$ so that α is connected with σ and η. Then E is the geodesic line perpendicular to α, which passes through the midpoint of α.

Case 3: σ and η form a hyperbolic strip with a positive width.

Let α be the shortest geodesic segment joining σ and η, $L(\alpha)=d(\sigma,\eta)$. Then E is the unique geodesic which is perpendicular to α at the midpoint of α.

(ii) The second assertion is an easy consequence of (i). □

4. Hyperbolic Surfaces with Cusps

We shall modify our approach in order to carry out the proof of Theorem A for hyperbolic surfaces with cusps. Before getting into the details of the proof, we make the following observation on a characterization of interior points.

A subset U is said to be *precompact* in X if $\bar{U}$ is compact in X. A subset N^∞ of X has no interior points if and only if $N^\infty \cap \bar{U}$ has no interior points for any precompact open set U in X. Therefore, for any compact set F in X, we would like to introduce partial estimates, a partial Fermi radius, and a partial injectivity radius of F. The higher-dimensional cut-off technique is related to the Margulis–Jorgensen decomposition (cf. [G1]).

4.1. Estimates on a Part of a Surface

Let (X,g_0) be a hyperbolic surface of type (n,k,m) with $km \geq 1$. Thus X has k cusps. In what follows, we will play the same game as in Sections 1–3 on a given compact domain F, with some alterations.

DEFINITION 4.1. Let W be an arbitrary domain with boundary ∂W, and let F be a compact subset in the metric space (Y, g). The partial Fermi radius of W restricted to F is denoted by

$$\gamma_g(W|_F) = \max\{d_{g_0}(q, \partial W) \mid q \in W \cap F\}. \tag{4.1}$$

DEFINITION 4.2. (i) Let (Y, g) be a complete Riemann surface with negative curvature. For any point $q \in Y$, the injectivity radius of q is given by (cf. [CE])

$$\rho_g(q) = \inf\{L(\sigma) \mid \sigma \text{ is a geodesic loop based on } q\},$$

where $L(\sigma)$ denotes the length of σ.

(ii) The injectivity radius of F is defined to be

$$\rho_g(F) = \inf\{\rho_g(y) \mid y \in F\}. \tag{4.2}$$

With the terminology of Definitions 4.1 and 4.2, we can strengthen Theorem 1.3 in the following way.

PROPOSITION 4.3. *Let X, $N(X), \ldots, N^j(X)$, g_j, and u_j be as in Theorem 1.3, and let F be a compact domain with interior U. If there is a constant number $C = C_F > 1$, which depends on F, such that*

$$e^{2u_j(q)} \geq C > 1 \tag{4.3}$$

for all $q \in N^j(X) \cap F$ and $j = 1, 2, \ldots$, then the infinite Nielsen kernel $N^\infty(X)$ has no interior points which lie on U.

The proof of Proposition 4.3 is identical to the proof of Theorem 1.3. Hence we omit it here.

PROPOSITION 4.4. *Let (X, g_0) be a complete, hyperbolic surface of type (n, k, m) and let $N(X)$ be its Nielsen kernel. Suppose that g_1 is the complete, conformal and hyperbolic metric on $N(X)$ with $g_1 = e^{2u} g_0$ and that F is a compact domain of X. Then*

$$e^{2u(q)} \geq 1 + 4e^{-2\hat{\gamma}_0} \tag{4.4}$$

for all $q \in N(X) \cap F$, where $\hat{\gamma}_0 \geq \gamma_{g_0}(N(X)|_F)$, the partial Fermi radius of $N(X)$ restricted to F.

For a proof of Proposition 4.4, we refer to the proof of Theorem 2.1.

PROPOSITION 4.5. *Let X be an open complete hyperbolic surface of type (n, k, m) with $N^j(X)$ its* jth *Nielsen kernel. Suppose that F is a compact domain of X. Then there is a uniform estimate of F such that*

$$\gamma_{g_j}(N^{j+1}(X)|_F) \leq 16|e(X)|\{1 + 1/\rho_F\} \tag{4.5}$$

for all $j \geq 0$, where $e(X) = 2 - (2n + k + m) < 0$, $m \geq 1$, and ρ_F is the injectivity radius of X restricted to F.

The proof of Proposition 4.5 follows that of Theorem 2.4.

After this lengthy preparation, we are ready to prove Theorem A.

4.2. *Proof of Theorem A*

For any given compact domain F with interior U, one can show by using Propositions 4.3–4.5 that the infinite Nielsen kernel $N^{\infty}(X)$ does not have any interior point within U.

Let us choose a filtration, or a sequence of compact domains $F_1 \subset F_2 \subset \cdots$, such that

$$X = \bigcup_{i=1}^{\infty} \operatorname{int}(F_i) \tag{4.6}$$

and

$$\bar{F}_j \subset \operatorname{int}(F_{j+1}).$$

It follows from the argument above that $N^{\infty}(X)$ has no interior points in each F_j. Therefore, $N^{\infty}(X)$ does not have any interior points in X. This completes the proof of Theorem A. □

REMARK. It follows from Proposition 4.5 and the proof of Theorem B (cf. Section 3.1) that

$$\lim_{j \to +\infty} d_X(N^{\infty}(X) \cap F, S_j(X) \cap F) = 0$$

for all compact domains $F \subset X$, where $S_j(X)$ is the jth soul of the surface X.

ACKNOWLEDGMENTS. I would like to express my deep gratitude to Professor Clifford Earle, from whom I learned of the Bers–Nielsen kernel and received many valuable suggestions. I also appreciate two referees for their careful review of this manuscript. In particular, I am grateful for their valuable comments concerning souls of open surfaces and other problems in this paper. Professor Doug Dickson, the editor of the Michigan Mathematical Journal, generously spent his time correcting both my English and mathematics in a preliminary version of the manuscript. His sharp criticism and suggestions made this paper much more clear and readable. Finally, I would like to thank Tom Stiadle for his careful proofreading of the manuscript.

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