On Similarity of Operators to Isometries

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1. Introduction

The problem of the similarity of the operators on Hilbert spaces to isometric operators has been considered by Sz.-Nagy [4]. He proved that an operator A on a Hilbert space $\mathcal{H}$ is similar to an isometric operator if and only if there exist a and $b>0$ such that $b\|h\| \le \|A^n h\| \le a\|h\|$ for each $h \in \mathcal{H}$, $n \in \mathbf{N}$.

In particular, an operator A is similar to a unitary operator if and only if A is invertible and

$$\sup\{\|A^n\|; n \in \mathbf{Z}\} < \infty.$$

Other necessary and sufficient conditions for the similarity to a unitary operator have been obtained [5; 7; 2; 9]. Let us recall from [9] one such condition that will be used in this paper: A power bounded operator A on $\mathcal{H}$ is similar to a unitary operator if and only if A is surjective and if there exists $c>0$ such that $\|(A-\lambda)h\| \ge c(1-|\lambda|)\|h\|$, for each $\lambda \in \mathbf{D} = \{z \in C: |z|<1\}$ and each $h \in \mathcal{H}$.

Concerning the similarity of a contraction T to an isometric operator, necessary and sufficient conditions have been obtained [6; 10] in terms of the characteristic function of T. Recently, Uchiyama [8] has also obtained new criteria for contractions to be similar to isometries.

In [1], Fadeev gives some (necessary *or* sufficient) conditions for the similarity of the contractions to isometric operators in terms of their resolvents. Also, an example illustrating the precision of his conditions is given. The aim of this paper is to give some necessary *and* sufficient conditions under which an operator A on a Hilbert space $\mathcal{H}$ is similar to an isometric operator or to a unilateral shift, in terms of the resolvent of the operator.

In Sections 3, 4, and 5, we shall provide some new necessary and sufficient conditions for a contraction T to be similar to an isometry. It is also shown that if T is similar to an isometry then it is similar to a restriction of the minimal unitary dilation of T to a certain invariant subspace.

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2. Preliminaries

In the following, all the operators are acting on complex Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. We shall need a few definitions. An operator $A \in B(\mathcal{H})$ is said to be *power bounded* if $\sup\{\|A^n\|: N \in \mathbf{N}\} < \infty$. Recall that two operators $A \in B(\mathcal{H})$ and $\tilde{A} \in B(\mathcal{K})$ are said to be *similar* if there exists a bounded invertible operator X from $\mathcal{H}$ onto $\mathcal{K}$ such that $A = X^{-1}\tilde{A}X$.

Let T be a contraction on $\mathcal{H}$; that is, $\|T\| \le 1$. We denote as usual by $D_T = (1 - T^*T)^{1/2}$ and $\mathfrak{D}_T = (D_T\mathcal{H})^-$ the defect operator and the defect space of T, respectively. Let $\mathcal{H}$ be a Hilbert space with the inner product $(\cdot,\cdot)$, and let $(\cdot,\cdot)_o$ be another inner product on $\mathcal{H}$. The norms $\|\ \|$ and $\|\ \|_o$ are equivalent, and we denote this by $\|\ \| \sim \|\ \|_o$, if there exist two constants $a, b > 0$ such that $a\|h\| \le \|h\|_o \le b\|h\|$ for each $h \in \mathcal{H}$.

In this case, the identity mapping ω from $(\mathcal{H}; (\cdot,\cdot))$ to $\mathcal{H}_o := (\mathcal{H}_i; (\cdot,\cdot)_o)$ is an invertible bounded operator. Throughout this paper we shall denote $A_o = \omega A \omega^{-1}$, for every $A \in B(\mathcal{H})$. It is well known that such a norm $\|\ \|_o$ is given by a positive invertible operator $P \in B(\mathcal{H})$ by the relation

$$\|h\|_o^2 = (Ph, h) \quad \text{for each } h \in \mathcal{H}.$$

3. Similarity to Isometries. I

In this section we give necessary and sufficient conditions in terms of the resolvent of the operator for an operator in $B(\mathcal{H})$ to be similar to an isometric operator. For our purpose we need the following lemma.

LEMMA 3.1. *If V is an isometric operator on $\mathcal{H}$, then*

$$\|(V-\lambda)h\|^2 \ge (1-|\lambda|)^2\|h\|^2 + |\lambda|(\|h\|^2 - \|P_{\mathcal{R}(V)}h\|^2)$$

for each $\lambda \in \mathbf{D}$ and $h \in \mathcal{H}$, where $P_{\mathcal{R}(V)}$ stands for the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(V) := \{Vh; h \in \mathcal{H}\}$.

Proof. Writing the Wold decomposition for V, we get the orthogonal sum $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_s$ in which $\mathcal{H}_u$ and $\mathcal{H}_s$ reduce V; the part of V on $\mathcal{H}_u$ is unitary and the part of V on $\mathcal{H}_s$ is a unilateral shift. This decomposition is uniquely determined and we have $\mathcal{H}_u = \bigcap_{n=0} V^n\mathcal{H}$ and $\mathcal{H}_s = \bigoplus_{n=0}^{\infty} V^n\mathcal{L}$, where $\mathcal{L} = \mathcal{H} \ominus \mathcal{R}(V)$.

For $h_u \in \mathcal{H}_u$ and $\lambda \in \mathbf{D}$ we have

$$\begin{aligned}\|(V-\lambda)h_u\|^2 &= \|Vh_u\|^2 + |\lambda|^2\|h_u\|^2 - 2\operatorname{Re}\bar{\lambda}(Vh_u, h_u)\\ &\ge \|h_u\|^2(1+|\lambda|^2) - 2|\lambda|\|h_u\|^2 = (1-|\lambda|)^2\|h_u\|^2.\end{aligned}$$

For $h_s \in \mathcal{H}_s$ and $h_s = \sum_{n=0}^{\infty} V^n l_n$, where $l_n \in \mathcal{L}$ and $\sum_{n=0}^{\infty}\|l_n\|^2 < \infty$, we have

$$\begin{aligned}\|(V-\lambda)h_s\|^2 &= (1+|\lambda|^2)\|h_s\|^2 - 2\operatorname{Re}\bar{\lambda}\sum_{n=0}^{\infty}(l_n, l_{n+1})\\ &\geq (1+|\lambda|^2)\|h_s\|^2 - 2|\lambda|\sum_{n=0}^{\infty}\|l_n\|\|l_{n+1}\|\\ &\geq (1+|\lambda|^2)\|h_s\|^2 - |\lambda|\sum_{n=0}^{\infty}(\|l_n\|^2+\|l_{n+1}\|^2)\\ &= (1-|\lambda|)^2\|h_s\|^2 + |\lambda|\|l_o\|^2 = (1-|\lambda|)^2\|h_s\|^2 + |\lambda|\|P_{\operatorname{Ker} V^*}h_s\|^2.\end{aligned}$$

Now for each $h = h_u + h_s \in \mathcal{H}$, where $h_u \in \mathcal{H}_u$ and $h_s \in \mathcal{H}_s$, we have

$$\begin{aligned}\|(V-\lambda)h\|^2 &= \|(V-\lambda)h_u\|^2 + \|(V-\lambda)h_s\|^2\\ &\geq (1-|\lambda|)^2\|h_u\|^2 + (1-|\lambda|)^2\|h_s\|^2 + |\lambda|\|P_{\operatorname{Ker} V^*}h_s\|^2\\ &= (1-|\lambda|)^2\|h\|^2 + |\lambda|\|P_{\operatorname{Ker} V^*}h\|^2 \quad \text{for any } \lambda \in \mathbf{D}.\end{aligned}$$

This completes the proof. □

For notational simplicity we will denote the restriction of an operator A to a subspace $\mathcal{K}$ by $A|\mathcal{K}$. For $A \in B(\mathcal{H})$ and for any $n \in \mathbf{N}$, let us consider the operator

$$A_n\colon \mathcal{H} \oplus \underbrace{\operatorname{Ker} A^* \oplus \cdots \oplus \operatorname{Ker} A^*}_{n-1 \text{ times}} \to \mathcal{H}$$

defined by the operator matrix

$$A_n = [A^n, A_*^{n-1}, A_*^{n-2}, \ldots, A_*], \quad \text{where } A_*^k = A^k|\operatorname{Ker} A^*.$$

Note that the following two conditions are equivalent:

(i) $\sup\{\|A_n\|: n \in \mathbf{N}\} < \infty$; and
(ii) $\|A^{*n}h\|^2 + \|P_{\operatorname{Ker} A^*}A^{*n-1}h\|^2 + \cdots + \|P_{\operatorname{Ker} A^*}A^*h\|^2 \leq a\|h\|^2$ for any $h \in \mathcal{H}$, $n \in \mathbf{N}$, where $a > 0$.

Indeed, since

$$A_n^* = \begin{bmatrix} A^{*n} \\ P_{\operatorname{Ker} A^*}A^{*n-1} \\ \vdots \\ P_{\operatorname{Ker} A^*}A^* \end{bmatrix}$$

and $\|A_n\| = \|A_n^*\|$, the result follows.

The main result of this section is given in the following theorem.

THEOREM 3.2. *An operator $A \in B(\mathcal{H})$ is similar to an isometry if and only if there exists an inner product $(\cdot,\cdot)_o$ on $\mathcal{H}$ such that the following conditions hold:*

(i) $\|\ \| \sim \|\ \|_o$;
(ii) $\sup\{\|(A_o)_n\|_o: n \in \mathbf{N}\} < \infty$; *and*

(iii) $\|(A_o-\lambda)h\|_o^2 \geq c(1-|\lambda|)^2\|h\|_o^2+|\lambda|(\|h\|_o^2-\|P_{\mathcal{R}(A_o)}h\|_o^2)$ *for each* $\lambda \in \mathbf{D}$ *and* $h \in \mathcal{H}$, *where* $c>0$ *and* $P_{\mathcal{R}(A_o)}$ *is the orthogonal projection of* $\mathcal{H}_o$ *onto* $\mathcal{R}(A_o)$.

Proof. Assume that $A=X^{-1}VX$, where $X\in B(\mathcal{H})$ is a positive invertible operator and $V\in B(\mathcal{H})$ is an isometry. Let $\mathcal{H}_o$ be the Hilbert space obtained from $\mathcal{H}$ by redefining the inner product by setting

$$(h,k)_o=(Xh,Xk) \quad \text{for } h \text{ and } k\in\mathcal{H}.$$

Obviously$\|\ \|\sim\|\ \|_o$ and $A_o=\omega A\omega^{-1}$ is an isometry on $\mathcal{H}_o$. Hence it follows that (ii) holds. Taking into account Lemma 3.1, we have

$$\|(A_o-\lambda)h\|_o^2\geq(1-|\lambda|)^2\|h\|_o^2+|\lambda|(\|h\|_o^2-\|P_{\mathcal{R}(A_o)}h\|_o^2) \quad \text{for each } \lambda\in\mathbf{D},\ h\in\mathcal{H}.$$

Conversely, let us suppose that there exists an inner product $(\cdot,\cdot)_o$ on $\mathcal{H}$ such that conditions (i), (ii), and (iii) are fulfilled.

Let us consider the Hilbert space

$$\mathcal{K}_o=\mathcal{H}_o\oplus\operatorname{Ker}A_o^*\oplus\operatorname{Ker}A_o^*\oplus\cdots$$

and the operator $A_o^{\sim}$ defined on $\mathcal{K}_o$ by

$$A_o^{\sim}(h_1,h_2,h_3,\ldots)=(A_oh_1+h_2,h_3,\ldots)$$

for each $(h_1,h_2,\ldots)\in\mathcal{K}_o$. Note that since (i) holds, the identity mapping ω from $\mathcal{H}$ to $\mathcal{H}_o$ is a bounded invertible operator and $A_o=\omega A\omega^{-1}\in B(\mathcal{H}_o)$.

Now, taking into account (iii), a simple computation shows that for each $\lambda\in\mathbf{D}$ and $k=(h_1,h_2,\ldots)\in\mathcal{K}_o$,

$$\begin{aligned}
&\|(A_o^{\sim}-\lambda)k\|_o^2\\
&\quad=\|(A_o-\lambda)h_1\|_o^2-2\operatorname{Re}[\lambda(h_1,h_2)_o]\\
&\qquad+\sum_{n\geq2}(1+|\lambda|^2)\|h_n\|_o^2-2\operatorname{Re}\left[\lambda\sum_{n\geq2}(h_n,h_{n+1})_o\right]\\
&\quad\geq c(1-|\lambda|)^2\|h_1\|_o^2+|\lambda|(\|h_1\|_o^2-\|P_{\mathcal{R}(A_o)}h_1\|_o^2)-|\lambda|(\|P_{\operatorname{Ker}A_o^*}h_1\|_o^2+\|h_2\|_o^2)\\
&\qquad+\sum_{n\geq2}(1+|\lambda|^2)\|h_n\|_o^2-|\lambda|\sum_{n\geq2}(\|h_n\|_o^2+\|h_{n+1}\|_o^2)\\
&\quad=c(1-|\lambda|)^2\|h_1\|_o^2+(1-|\lambda|)^2\sum_{n\geq2}\|h_n\|_o^2.
\end{aligned}$$

Therefore, there exists $b>0$ such that for each $\lambda\in\mathbf{D}$ and $k\in\mathcal{K}_o$ we have

$$\|(A_o^{\sim}-\lambda)k\|_o^2\geq b(1-|\lambda|)^2\|k\|_o^2.$$

Hence, we obtain for $\lambda=0$ that $\|A_o^{\sim}k\|_o\geq b\|k\|_o$ for each $k\in\mathcal{K}_o$. Thus $A_o^{\sim}$ has closed range. This fact, together with the definition of $A_o^{\sim}$, implies that $A_o^{\sim}$ is surjective. On the other hand, condition (ii) shows that $A_o^{\sim}$ is a power bounded operator. Indeed, with respect to the orthogonal decomposition of $\mathcal{K}_o$, the operator matrix of $A_o^{\sim n}$ for $n\in\mathbf{N}$ is

$$A_o^{\sim n} = \begin{bmatrix} A_o^n & A_{o*}^{n-1} & \dots & A_{o*} & I\,|\,\mathrm{Ker}\,A_o^* & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & I\,|\,\mathrm{Ker}\,A_o^* & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \ddots & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \end{bmatrix},$$

where A_{o*}^k is the restriction of A_o^k to $\mathrm{Ker}\,A_o^*$. Hence we deduce that $\|A_o^{\sim n}\|_o \le \|(A_o)_n\|_o$ for each $n \in \mathbf{N}$. Now condition (ii) shows that $A_o^{\sim}$ is a power bounded operator. According to the theorem stated in the introduction, it follows that $A_o^{\sim}$ is similar to a unitary operator. Since $A_o^{\sim}(\mathcal{H}_o) \subset \mathcal{H}_o$ and $A_o^{\sim} \,|\, \mathcal{H}_o = A_o$, we deduce that A_o is similar to an isometric operator. Hence it follows that A is similar to an isometric operator. This completes the proof. □

REMARK 3.3. If $A \in B(\mathcal{H})$ is similar to an isometry, one take the inner product $(\cdot,\cdot)_o$ to be defined by

$$(h,k)_o = \mathrm{LIM}(A^n h, A^n k) \quad \text{for } h \text{ and } k \in \mathcal{H},$$

where LIM means Banach limit (see [4]).

With respect to this inner product, we have

$$(*) \qquad \|(A_o-\lambda)h\|_o^2 \ge (1-|\lambda|)^2\|h\|_o^2 + |\lambda|(\|h\|_o^2 - \|P_{\mathcal{R}(A_o)}h\|_o^2)$$

for each $\lambda \in \mathbf{D}$ and $h \in \mathcal{H}$. Indeed, with respect to this inner product the operator A is an isometry (see [4]). According to Lemma 3.1, the result follows.

In the particular case when A is a contraction, which is similar to an isometry, one can take the inner product $(h,k)_o = \lim_{n\to\infty}(T^n h, T^n h)$ for $h \in \mathcal{H}$ so that the relation $(*)$ holds. In the case when A is a surjective operator, we get one of the theorems stated in the beginning of the paper. It consists of the following remark.

REMARK 3.4. If A is a surjective operator, then conditions (ii) and (iii) of Theorem 3.2 are equivalent, respectively, to

(ii′) $\sup\{\|A^n\|; n \in \mathbf{N}\} < \infty$; and
(iii′) $\|(A-\lambda)h\| \ge \beta(1-|\lambda|)\|h\|$ for each $\lambda \in \mathbf{D}$ and $h \in \mathcal{H}$, where $\beta > 0$.

It is easy to see that if $\|A_o\| \le 1$ then condition (ii) in Theorem 3.2 is automatically satisfied as is (ii′).

4. Similarity to Isometries. II

In this section we give another characterization of those operators that are similar to isometries. Also, we point out a consequence connected with the previous section.

THEOREM 4.1. *A one-to-one operator $A \in B(\mathcal{H})$ with closed range is similar to an isometry if and only if there exists an inner product $(\cdot,\cdot)_o$ on $\mathcal{H}$ such that the following conditions hold:*

(i) $\| \ \| \sim \| \ \|_o$;

(ii) *there exist k in $\mathbf{N}\cup\{0\}$ such that the operator $A_o^{*k+1}A_o^{k+1}$ belongs to the convex hull of the operators $I, A_o^* A_0, \ldots, A_o^{*k}A_o^k$.*

Proof. If A is similar to an isometry, then there exists inner product $(\cdot,\cdot)_o$ on $\mathcal{H}$ such that $\| \ \| \sim \| \ \|_o$ and $A_o^* A_o = 1$ (see the proof of Theorem 3.2). Conversely, suppose that conditions (i) and (ii) are fulfilled. Then there exist $\lambda_i \geq 0$ $(i=0,1,\ldots,k)$ with $\sum_{i=0}^k \lambda_i = 1$ such that

$$\sum_{i=0}^{k} \lambda_i A_o^{*i} A_o^i = A_o^{*k+1} A_o^{k+1}. \tag{4.1}$$

Let us define a new inner product $(\cdot,\cdot)_1$ on $\mathcal{H}$ by

$$\begin{aligned}(h,h')_1 = \lambda_o(h,h')_o + (\lambda_o+\lambda_1)(Ah,Ah')_o + \cdots \\ + (\lambda_o+\lambda_1+\cdots+\lambda_k)(A^k h, A^k h')_o\end{aligned}$$

and $h, h' \in \mathcal{H}$. Taking into account that the operator A is one-to-one and has a closed range, it follows that there exists $a>0$ such that

$$\|Ah\| \geq a\|h\| \quad \text{for each } h \in \mathcal{H}.$$

Since $\| \ \| \sim \| \ \|_o$, it is easy to see that $\| \ \| \sim \| \ \|_1$. A simple computation shows that $(Ah, Ah')_1 = (h,h')_1$ for h and $h' \in \mathcal{H}$. Hence we infer that A is similar to an isometry on $\mathcal{H}$. The proof is completed. □

As a consequence of this theorem, we obtain a result which is well known.

COROLLARY 4.2. *If A^k for $k \in \mathbf{N}$ is similar to an isometry, then A is also similar to an isometry.*

Indeed, if A^k is similar to an isometry then there exists an inner product $(\cdot,\cdot)_o$ on $\mathcal{H}$ such that $A_o^{*k}A_o^k = I$. Now the result follows from Theorem 4.1.

REMARK 4.3. Taking into account Corollary 4.2, one can state the theorems from Section 3 by replacing (in the necessary and sufficient conditions that occur there) the operator A by A^k for $k \in \mathbf{N}$.

5. A Model for Contractions Similar to Isometries

We recall from [7] a few facts from the theory of unitary dilations. Let $T \in B(\mathcal{H})$ be a contraction, and let $U \in B(\mathcal{K})$ be its minimal unitary dilation. This means that $\mathcal{H} \subset \mathcal{K}$, $T^n = P_{\mathcal{H}} U^n \mid \mathcal{H}$ for $n \geq 0$, and

$$\mathcal{K} = \bigvee_{n=-\infty}^{\infty} U^n \mathcal{H}.$$

The $*$-residual subspace $\mathcal{R}_*$ of $\mathcal{K}$ is defined as

$$\mathcal{R}_* = \bigcap_{k=0}^{\infty} \left(\bigvee_{n=k}^{\infty} U^{*n} \mathcal{H} \right),$$

and the operator $X = P_{\mathcal{R}_*} | \mathcal{H}$ is given by

(5.1) $$Xh = \lim_{n \to \infty} U^{*n} T^n h \quad \text{for } h \in \mathcal{H}.$$

Then, in particular,

(5.2) $$(U \,|\, \mathcal{R}_*) X = XT.$$

The restriction of U to R_* is a unitary operator on $\mathcal{R}_*$.

In the sequel we need the following lemma.

LEMMA 5.1. *Let $A \in B(\mathcal{H})$ be such that there exists*

(5.3) $$P = \mathrm{so} - \lim_{n \to \infty} A^{*n} A^n$$

("so" *means strong operator limit). Then A is similar to an isometry if and only if P is an invertible operator.*

Proof. Suppose $A = X^{-1} V X$, where $V \in B(\mathcal{H})$ is an isometry and $X \in B(\mathcal{H})$ is a positive invertible operator. Using (5.3), we have

$$(\|X^{-1}\| \|X\|)^{-2} \|h\|^2 \leq (Ph, h) = \lim_{n \to \infty} \|A^n h\|^2 \leq (\|X^{-1}\| \|X\|)^2 \|h\|^2$$

for each $h \in \mathcal{H}$; whence it follows that P is a positive invertible operator on $\mathcal{H}$. Conversely, suppose P is an invertible operator. It is obvious that $A^*PA = P$. Setting $Q = P^{1/2}$ we have $\|QAQ^{-1}h\|^2 = (A^*PAQ^{-1}h, Q^{-1}h) = (PQ^{-1}h, Q^{-1}h) = \|h\|^2$ for $h \in \mathcal{H}$; that is, QAQ^{-1} is an isometry and $A = Q^{-1}(QAQ^{-1})Q$, so that T is similar to an isometry. The proof is completed. $\square$

Let us point out the well-known fact that if A is a contraction then $\mathrm{so} - \lim_{n \to \infty} A^{*n} A^n$ exists.

The main result of this section is the following theorem, which gives a model for the contractions similar to isometric operators.

THEOREM 5.2. *Let $T \in B(\mathcal{H})$ be a contraction with minimal unitary dilation $U \in B(\mathcal{K})$ and $*$-residual space $\mathcal{R}_*$. Then the following statements are equivalent:*

(i) *T is similar to an isometric operator; and*
(ii) *$P_{\mathcal{R}_*} | \mathcal{H}$ is one-to-one and has closed range.*

Moreover, if this is the case then T is similar to $U \,|\, P_{\mathcal{R}_*}\mathcal{H}$.

Proof. Suppose (i) is fulfilled; then, by Lemma 5.1, the operator

$$P = \mathrm{so} - \lim_{n \to \infty} T^{*n} T^n$$

is a positive invertible operator. Therefore there exists $a > 0$ such that

$$\lim_{n \to \infty} \|T^n h\| \geq a \|h\| \quad \text{for each } h \in \mathcal{H}.$$

Since, by (5.1), $\|P_{\mathcal{R}_*}h\|=\lim_{n\to\infty}\|T^n h\|$ for each $h\in\mathcal{H}$, we deduce that the operator $P_{\mathcal{R}_*}|\mathcal{H}$ is one-to-one and has closed range.

Conversely, if (ii) holds then consider the operator $Z:\mathcal{H}\to P_{\mathcal{R}_*}\mathcal{H}$ defined by $Zh=P_{\mathcal{R}_*}h$ ($h\in\mathcal{H}$). Obviously, Z is an invertible operator. The subspace $P_{\mathcal{R}_*}\mathcal{H}$ is invariant for the minimal unitary dilation U of T, and $U|P_{\mathcal{R}_*}\mathcal{H}$ is an isometry. As

$$(U|P_{\mathcal{R}_*}\mathcal{H})Z=ZT,$$

it follows that T is similar to an isometry. This completes the proof. □

REMARK 5.3. The restriction of U to $\mathcal{R}_*$ is the minimal unitary dilation of $U|P_{\mathcal{R}_*}\mathcal{H}$.

Indeed, let $\mathcal{R}_o$ denote the reducing subspace generated by $P_{\mathcal{R}_*}\mathcal{H}$, and set $\mathcal{R}'=\mathcal{R}_*\ominus\mathcal{R}_o$. It is easy to check that $U|\mathcal{K}\ominus\mathcal{R}'$ is a unitary dilation of T and, by minimality, we must have $\mathcal{K}\ominus\mathcal{R}'=\mathcal{K}$, that is, $\mathcal{R}'=\{0\}$.

COROLLARY 5.4. *A contraction $T\in B(\mathcal{H})$ is similar to a unitary operator if and only if T is surjective and $P_{\mathcal{R}_*}|\mathcal{H}$ is one-to-one and has closed range. In this case T is similar to the restriction of U to $\mathcal{R}_*$.*

Thus we find again the result obtained by Sz.-Nagy and Foiaş [7, §IX.1].

COROLLARY 5.5. *A contraction $T\in B(\mathcal{H})$ is similar to a unilateral shift if and only if $T^{*n}\to 0$ as $n\to\infty$ and $P_{\mathcal{R}_*}\mathcal{H}$ is one-to-one and has closed range.*

In connection with Theorem 5.2, we mention that this statement follows also from the more general results included in the first section of the paper [3].

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