The Existence of 7-fields and 8-fields on $(8k+5)$-dimensional Manifolds

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1. Introduction

Let M be a closed, connected and smooth manifold whose dimension n is congruent to 5 mod 8 with $n \geq 21$. Let η be a spin n-plane bundle over M. We shall investigate the span of η. Recall that the Kervaire mod 2 semi-characteristic of M, $\chi_2(M)$, is defined by

$$\chi_2(M) = \sum_{2i<n} \dim_{\mathbf{Z}_2} H^i(M;\mathbf{Z}_2) \bmod 2.$$

When η is the tangent bundle of M and M is 3-connected mod 2, we have from [12] that $\operatorname{span}(\eta) \geq 6$ if and only if $w_{n-5}(M)=0$ and $\chi_2(M)=0$, where $w_i(M)$ is the ith mod 2 Stiefel–Whitney class of M.

We shall prove the following theorems.

THEOREM 1.1. *If M is 5-connected* mod 2, *then* $\operatorname{span}(M) \geq 7$ *if and only if $\delta w_{n-7}(M)=0$ and $\chi_2(M)=0$, where δ is the Bockstein operator associated with the exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0$.*

THEOREM 1.2. *Suppose M is 5-connected* mod 2 *and $Sq^1 H^{n-7}(M;\mathbf{Z}_2) = 0$. Then* $\operatorname{span}(M) \geq 8$ *if and only if $w_{n-7}(M)=0$, $0 \in \psi_3(w_{n-9}(M))$, and $\chi_2(M)=0$, where ψ_3 is a stable secondary cohomology operation associated with the relation*

$$\psi_3 : Sq^2Sq^2 + Sq^1(Sq^2Sq^1) = 0.$$

Some applications to immersions of manifolds into Euclidean spaces are given in the last section. Throughout the paper we assume that $\dim M = n$ is congruent to 5 mod 8 with $n \geq 21$. All cohomology will be ordinary cohomology with mod 2 coefficients unless otherwise specified.

2. The Modified Postnikov Tower

We shall consider the problem of finding an s-field as a lifting problem. Let $B\hat{SO}_j\langle 8\rangle$ be the classifying space of orientable j-plane bundles ξ satisfying $w_2(\xi) = w_4(\xi) = 0$, where $w_i(\xi)$ is the ith mod 2 Stiefel–Whitney class of

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Table 1. The n-MPT for $\pi: B\hat{S}O_{n-7}\langle 8\rangle \to B\hat{S}O_n\langle 8\rangle$

	k-invariant	Dimension	Defining Relation
Stage 1	k_1^1	$n-6$	$k_1^1=\delta w_{n-7}$
	k_2^1	$n-5$	$k_2^1=w_{n-5}$
Stage 2	k_1^2	$n-5$	$Sq^2\delta w_{n-7}+Sq^1 w_{n-5}=0$
	k_2^2	$n-3$	$Sq^4\delta w_{n-7}+Sq^3 w_{n-5}=0$
	k_3^2	$n-1$	$Sq^4Sq^2\delta w_{n-7}=0$
	k_4^2	n	$\chi Sq^4Sq^2 w_{n-5}=0$
	k_5^2	n	$Sq^6 w_{n-5}=0$
Stage 3	k_1^3	$n-3$	$Sq^2Sq^1k_1^2+Sq^1k_2^2=0$
	k_2^3	$n-1$	$Sq^4Sq^2k_1^2+Sq^1k_3^2=0$
	k_3^3	n	$\chi Sq^4k_2^2+Sq^2k_3^2+Sq^1k_4^2=0$
Stage 4	k^4	$n-1$	$Sq^2Sq^1k_1^3+Sq^1k_2^3=0$

the bundle ξ. Let η be an n-plane bundle over M with $w_4(\eta)=0$. Since M is 3-connected mod 2, the n-plane bundle η is classified by a map $g: M\to B\hat{S}O_n\langle 8\rangle$. Then the problem of finding s linearly independent sections of η is equivalent to lifting η to $B\hat{S}O_{n-s}\langle 8\rangle$. Thus we shall consider the n-MPT [3] for the fibration $\pi: B\hat{S}O_{n-s}\langle 8\rangle\to B\hat{S}O_n\langle 8\rangle$ for $n\geq 21$ and for $s=7$ or 8. The k-invariants for the n-MPT for π for $s=7$ or 8 are listed in Tables 1 and 2.

The computation of the k-invariants for π is now a routine exercise (see, e.g., [15] and [18]). We leave the details for the reader.

For notational purposes we shall use the following diagram as a reference to the n-MPT's defined by Tables 1 or 2, where $\eta=\pi$.

$$\begin{tikzcd}
 & & B' \arrow[dll, "q_2"'] \arrow[dl, "q_1"] \arrow[d, "\eta"] \\
E_2 \arrow[r, "p_2"] & E_1 \arrow[r, "p_1"] & B
\end{tikzcd}$$

Let γ be the appropriate universal n-plane bundle over B. We shall denote by $T(E_i)$ the Thom space of the n-plane bundle over E_i induced from γ by p_1 when $i=1$ or by $p_1\circ p_2$ when $i=2$. The Thom class of $T(E_i)$ is denoted by

Table 2. The n-MPT for $\pi: B\hat{SO}_{n-8}\langle 8\rangle \to B\hat{SO}_n\langle 8\rangle$

	k-invariant	Dimension	Defining Relation
Stage 1	k_1^1	$n-7$	$k_1^1 = w_{n-7}$
	k_2^1	$n-5$	$k_2^1 = w_{n-5}$
Stage 2	k_1^2	$n-6$	$Sq^2 w_{n-7} = 0$
	k_2^2	$n-5$	$Sq^2 Sq^1 w_{n-7} + Sq^1 w_{n-5} = 0$
	k_3^2	$n-4$	$Sq^4 w_{n-7} + Sq^2 w_{n-5} = 0$
	k_4^2	$n-1$	$(Sq^4 Sq^2 Sq^1 + Sq^6 Sq^1 + Sq^7) w_{n-7} + Sq^2 Sq^3 w_{n-5} = 0$
	k_5^2	n	$(Sq^8 + w_8 \cdot) w_{n-7} + \binom{n-8}{8} Sq^6 w_{n-5} = 0$
	k_6^2	n	$Sq^6 w_{n-5} = 0$
Stage 3	k_1^3	$n-5$	$Sq^2 k_1^2 + Sq^1 k_2^2 = 0$
	k_2^3	$n-1$	$Sq^4 Sq^1 k_2^2 + Sq^1 k_4^2 + Sq^3 Sq^1 k_3^2 = 0$
	k_3^3	n	$(Sq^4 Sq^2 Sq^1 + Sq^7) k_1^2 + Sq^2 k_4^2 = 0$
Stage 4	k^4	$n-1$	$Sq^2 Sq^3 k_1^3 + Sq^1 k_2^3 = 0$

$U(E_i)$. The Thom space of the n-plane bundle γ is denoted by $T(B)$ and its Thom class $U(B)$.

We shall denote the tangent bundle of M by τ, the stable normal bundle of M by ν, the Thom space of any vector bundle η and its Thom class by $T(\eta)$ and $U(\eta)$, respectively.

We shall denote the Eilenberg-MacLane space of type $(\mathbf{Z}_2, j)$ by K_j and its fundamental class by ι_j. All manifolds are assumed to be closed, connected, and smooth.

3. 7-frame Fields

Throughout this section M is assumed to be 3-connected mod 2. We shall refer to Table 1 for the k-invariants of π.

Consider cohomology operations ξ_1^* and ξ_2 associated with the relations:

$$(3.1)\quad \begin{cases} \xi_1^*: Sq^2(\delta Sq^{8k-2}) + Sq^1 Sq^{8k} = 0 \text{ on integral classes,} \\ \xi_2: Sq^6 Sq^{8k} + Sq^1(Sq^{8k+4}Sq^1 + Sq^{8k+2}Sq^3) = 0 \\ \qquad\qquad \text{on mod 2 classes of dimension} \le 8k+5. \end{cases}$$

We can choose ξ_1^* and ξ_2 to be of Hughes–Thomas type [4]. Hence we have the following.

LEMMA 3.2.

$$U(E_1)\cdot k_1^2 \in \xi_1^*(U(E_1)) \quad \textit{and} \quad U(E_2)\cdot k_5^2 \in \xi_2(U(E_1)),$$

where E_1 is the first stage of the n-MPT for $\pi: B\hat{SO}_{n-7}\langle 8\rangle \to B\hat{SO}_n\langle 8\rangle$.

Because $Sq^1(U(E_1)) = Sq^2(U(E_2)) = 0$, Lemma 3.2 is an immediate consequence of the choice of ξ_1^* and ξ_2 and the admissible class theorem [8] since $(\delta w_{8k-2}, w_{8k})$ is admissible for k_1^2 and w_{8k} is admissible for k_5^2. We leave the details to the reader.

Next we shall consider the identification of k_3^3.

Consider the stable secondary cohomology operations ζ_1, ζ_2, and ζ_3 of Hughes–Thomas type associated with the relations

$$\text{(3.3)}\quad \begin{cases} \zeta_1: Sq^4(\delta Sq^{8k-2}) + Sq^3Sq^{8k} + Sq^1(Sq^{8k}Sq^2) = 0; \\ \zeta_2: (Sq^4Sq^2)\delta Sq^{8k-2} + Sq^3(Sq^{8k}Sq^2) \\ \qquad\qquad + Sq^2(Sq^{8k}Sq^2Sq^1 + Sq^{8k-1}Sq^3Sq^1) = 0; \\ \zeta_3: (\chi Sq^4Sq^2)Sq^{8k} + \chi Sq^4(Sq^{8k}Sq^2) + (Sq^2Sq^1)(Sq^{8k}Sq^2Sq^1 \\ \qquad\qquad + Sq^{8k-1}Sq^3Sq^1) + Sq^5(Sq^{8k}Sq^1) = 0. \end{cases}$$

Then we have the following theorem.

THEOREM 3.4. *The operations ζ_1, ζ_2, and ζ_3 can be chosen in such a way that on b_{8k-2}, the fundamental class of Y_{8k-2}, the principal fibration over K_{8k-2}, with classifying map $(Sq^1, Sq^2)\iota_{8k-2}$, we have*

$$(Sq^4b_{8k-2} \cup b_{8k-2}, Sq^{8k}Sq^4b_{8k-2}, 0) \in (\zeta_1, \zeta_2, \zeta_3)(b_{8k-2})$$

and

$$(0, Sq^{8k-1}Sq^4Sq^1\iota_{8k-6} + Sq^5Sq^1\iota_{8k-6} \cup Sq^3Sq^1\iota_{8k-6}, 0) \\ \in (\zeta_1, \zeta_2, \zeta_3)(\iota_{8k-6}).$$

The proof of Theorem 3.4 is omitted since it is analogous to that of the proof of Lemma 4.7 in [8].

Furthermore, ζ_3 can be chosen in such a way that $\zeta_3 \subset \tilde{\zeta}_3$, where $\tilde{\zeta}_3$ is a stable operation associated with the relation

$$\text{(3.5)}\quad \tilde{\zeta}_3: \chi Sq^4(Sq^{8k+2} + Sq^{8k}Sq^2) + Sq^2(Sq^{8k+3}Sq^1 + Sq^{8k+1}Sq^2Sq^1) = 0.$$

Indeed, we may take $\tilde{\zeta}_3 = \phi_7 \circ Sq^{8k-2}$, where ϕ_7 is a stable operation associated with the relation

$$\phi_7: \chi Sq^4Sq^4 + Sq^2(Sq^4Sq^2) = 0.$$

Using Theorem 3.4 and the methods of [8], we can prove the next theorem.

THEOREM 3.6. *The operation $(\zeta_1, \zeta_2, \zeta_3)$ can be chosen in such a way that the relation*

$$\text{(3.7)}\quad \Omega: \chi Sq^4\zeta_1 + Sq^2\zeta_2 + Sq^1\zeta_3 = 0$$

holds. Furthermore, associated with the relation Ω we can choose a stable tertiary cohomology operation, also denoted by the same symbol, such that on d_{8k-2}, the fundamental class of D_{8k-2}, the principal fibration over K_{8k-2} with classifying map $(Sq^1, Sq^2, Sq^4)\iota_{8k-2}$, we have

$$d_{8k-2} \cup \phi_7(d_{8k-2}) \in \Omega(d_{8k-2}).$$

LEMMA 3.8. *Let* $E_2 \xrightarrow{p_2} E_1 \xrightarrow{p_1} B\hat{SO}_n\langle 8\rangle$ *be the* n-MPT *for* $\pi : B\hat{SO}_{n-7}\langle 8\rangle \to B\hat{SO}_n\langle 8\rangle$. *Then*

$$U(E_2)\cdot(p_2^* p_1^*(w_{n-7}\cdot Sq^3\theta_4)+k_3^3) \in \Omega(U(E_2)),$$

where $U(E_2)$ *is the Thom class of the n-plane bundle induced on* E_2 *via* $p_2 \circ p_1$ *from the universal n-plane bundle over* $B\hat{SO}_n\langle 8\rangle$, *and*

$$\theta_4 \in H^4(B\hat{SO}_n\langle 8\rangle) \approx \mathbf{Z}_2$$

is a generator.

Proof. For $n>37$, π^* is an epimorphism, and since $\operatorname{Ker}\pi^* \supset \operatorname{Ker} p_1^*$ in dimension $\leq n$, by Theorem 3.6 and the admissible class theorem [8], we have

$$U(E_2)\cdot(p_2^* p_1^*(w_{n-7}\cdot Sq^3\theta_4)+k_3^3) \in \Omega(U(E_2))$$

for $\phi_7(U(B\hat{SO}_{n-7}\langle 8\rangle)) = U(B\hat{SO}_{n-7}\langle 8\rangle)\cdot Sq^3\theta_4$, where $U(B\hat{SO}_{n-7}\langle 8\rangle)$ is the Thom class of the universal $(n-7)$-plane bundle over $B\hat{SO}_{n-7}\langle 8\rangle$. For $n \leq 37$, notice that $\operatorname{Indet}^{2n}(T(B\hat{SO}_{n-7}\langle 8\rangle),\Omega) = (Tq_2)^* \operatorname{Indet}^{2n}(T(E_2),\Omega)$, where $T(B\hat{SO}_{n-7}\langle 8\rangle)$ is the Thom space of the n-plane bundle induced by π and q_2 is a lifting of π to E_2. As $w_{8k-2}\cdot Sq^3\theta_4$ is in the image of π^*, by a slight modification of the admissible class theorem we have the required result. □

For any n-plane bundle η over M with $w_4(\eta)=0$ that is classified by a map $g: M \to B\hat{SO}_n\langle 8\rangle$, define $\theta_4(\eta)$ to be $g^*(\theta_4)$. A similar definition applies when η is a stable bundle over M.

THEOREM 3.9. *Suppose* M *is* 3-*connected* mod 2 *and* $w_4(M)=0$. *Let* η *be an n-plane bundle over M satisfying* $w_4(\eta)=0$. *If* $Sq^2\theta_4(\nu)=0$, *then assume further that* $\theta_4(-\eta-\tau)=0$. *Then* $\operatorname{span}(\eta)\geq 7$ *if and only if* $\delta w_{n-7}(\eta) = 0$, $w_{n-5}(\eta)=0$, $0 \in \xi_1^*(U(\eta))$, $\xi_2(U(\eta))=0$, *and* $\Omega(U(\eta))=0$ *whenever* $Sq^2\theta_4(\nu)=0$.

Proof. Since $w_4(\eta)=w_4(M)=0$, $\operatorname{Indet}^{n,n}(M,(k_4^2,k_5^2))$ is trivial. By the admissible class theorem and Lemma 3.2, $0 \in k_1^2(\eta)$ if and only if $0 \in \xi_1^*(U(\eta))$. Similarly, $k_5^2(\eta)=0$ if and only if $\xi_2(U(\eta))=0$. Now $\zeta_3(U(\eta)) = \tilde{\zeta}_3(U(\eta)) = \phi_7 \circ Sq^{8k-2}(U(\eta))$. The S-duality pairing

$$\begin{aligned}
&\langle \phi_7 \circ Sq^{8k-2}(U(\eta)), U(-\eta-\tau)\rangle \\
&\qquad = \langle Sq^{8k-2}(U(\eta)), \chi\phi_7 U(-\eta-\tau)\rangle \\
&\qquad = \langle Sq^{8k-2}(U(\eta)), U(-\eta-\tau)\cdot Sq^3\theta_4(-\eta-\tau)\rangle \\
&\qquad = \langle Sq^{8k+1}(U(\eta)), U(-\eta-\tau)\cdot\theta_4(-\eta-\tau)\rangle = 0,
\end{aligned}$$

since $Sq^1\theta_4(-\eta-\tau)=0$. Thus $\zeta_3(U(\eta))=0$. Therefore, by the admissible class theorem, $k_4^2(\eta)=0$ whenever it is defined. If $\operatorname{Indet}^n(M,k_3^3)$ is not trivial, then $0 \in k_3^3(\eta)$ and we are done. If $\operatorname{Indet}^n(M,k_3^3)=0$, then by the connectivity condition on M and S-duality it can be shown that $Sq^2\theta_4(\nu)=0$.

Now by the connectivity condition on M, S-duality and Atiyah–James duality (applied to $T(\eta)$), and by the assumption $\theta_4(-\eta-\tau)=0$, we can verify that $\mathrm{Indet}^{2n}(T(\eta),\Omega)=0$ modulo a zero primary piece. Therefore, by Lemma 3.8, $k_3^3(\eta)=0$ if and only if $\Omega(U(\eta))=0$, since $w_{n-7}(\eta)\cdot Sq^3\theta_4(\eta)=Sq^1(w_{n-7}(\eta)\cdot Sq^2\theta_4(\eta))=0$ when $\delta w_{n-7}(\eta)=0$. This completes the proof. □

Now we consider the span of M. Let $g: M\times M\to T(\tau)$ be the map that collapses the complement of a tubular neighbourhood of the diagonal in M to a point. Let $U=g^*(U(\tau)) \bmod 2$. Then $U=A+tA$, where $t: H^*(M\times M)\to H^*(M\times M)$ is the homomorphism induced by the map that interchanges the factors of M and

$$A=\sum_{i\le 4k+2}\sum_{l=1}^{n(i)}\alpha_i^l\otimes\beta_{n-i}^l,\quad \alpha_i^l\in H^i(M),\ n(i)=\dim H^i(M),$$

and $\alpha_i^j\cup\beta_{n-i}^k=\delta_{jk}\mu$, $\mu\in H^n(M)$ is a generator and δ_{jk} is the Kronecker function. As in [19], $A\cup tA=\chi_2(M)\mu\otimes\mu$.

Using Wu-duality and a Cartan formula for Steenrod squares we can easily derive the following lemma.

LEMMA 3.10.

(i) *If M is* 2*-connected* mod 2, *then* $Sq^{8k}Sq^1A=0$, $Sq^{8k}Sq^2Sq^1A=Sq^{8k-1}Sq^3Sq^1A=0$, *and* $Sq^{8k}Sq^2A=0$; *if also* $w_{8k}(M)=0$, *then* $Sq^{8k}A=0$.

(ii) *If M is* 3*-connected* mod 2 *and* $\delta w_{8k-2}(M)=0$, *then* $\delta Sq^{8k-2}A=0$.

THEOREM 3.11. *Suppose M is* 3*-connected* mod 2 *and* $w_4(M)=0$. *Then* $\mathrm{span}(M)\ge 7$ *if and only if* $\delta w_{n-7}(M)=0$, $w_{n-5}(M)=0$, $0\in\xi_1^*(U(\tau))$, *and* $\chi_2(M)=0$.

Proof. This follows essentially from Theorem 3.9. Thus $\mathrm{span}(M)\ge 7$ if and only if $\delta w_{n-7}(M)=0$, $w_{n-5}(M)=0$, $0\in\xi_1^*(U(\tau))$, $\xi_2(U(\tau))=0$, and $0\in k_3^3(\tau)$. By Lemma 3.10(i), ξ_2 is defined on A hence on tA. Thus

$$g^*\xi_2(U(\tau))=\xi_2(U)=\xi_2(A+tA)=\xi_2(A)+\xi_2(tA)+A\cup tA=\chi_2(M)\mu\otimes\mu.$$

Since g^* is injective, $\xi_2(U(\tau))=\chi_2(M)U(\tau)\mu$. We shall next show that Ω is defined on A. By Lemma 3.10(i), ζ_3 is defined on A. By our choice

$$\zeta_3(A)=\tilde{\zeta}_3A=\phi_7\circ Sq^{8k-2}A.$$

Now by using S-duality as in the proof of Theorem 3.9 together with Lemma 3.10(i), we can show that $\phi_7\circ Sq^{8k-2}A=0$. Thus $\zeta_3(A)=0$ modulo zero indeterminancy. So Ω is defined on A. If $\mathrm{Indet}^n(M,k_3^3)\ne 0$, then we have nothing to prove for $0\in k_3^3(\tau)$. Assume now $\mathrm{Indet}^n(M,k_3^3)=0$. Then $Sq^2\theta_4(\nu)=0$ and by Theorem 3.9 we must show that $\Omega(T(\tau))=0$.

Let $P_2\to P_1\to K_n$ be the universal example space for the operation Ω. Let U be represented by a map also denoted by $U: T(\tau)\to K_n$. Let $\bar{U}$ be a lifting of U to P_1 such that $\bar{U}$ also has a lifting $\bar{\bar{U}}$ to P_2. Let $m_1: P_1 \times P_1 \to P_1$ and $m_2: P_2 \times P_2 \to P_2$ be the multiplication maps. Let A be represented by a map $M \times M \to K_n$, also denoted by A. If $w_{n-5}(M)=0$ and $\delta w_{n-7}(M)=0$ then Ω is defined on A. Let $\bar{A}$ be a lifting of A to P_1 and $\bar{\bar{A}}$ a lifting of $\bar{A}$ to P_2. The rest of the proof is similar to Theorem 9.10(ii) of [11]. Then $h=m_1\circ(\bar{A}, \bar{A}\circ t)$ is a lifting of $U\circ g$ to P_1 and $\bar{h}=m_2\circ(\bar{\bar{A}}, \bar{\bar{A}}\circ t)$ is a lifting of h to P_2. Let $f=\bar{U}\circ g$. Then f is also a lifting of $U\circ g$ to P_1. Thus there is a map $l: M \times M \to \Omega C_1$, where $C_1 = K(\mathbf{Z}, 2n-6) \times K_{2n-5} \times K_{2n-4} \times K_{2n-3} \times K_{2n-2}$ is represented as a vector (y, z, c, d, o), $y \in H^{2n-7}(M \times M; \mathbf{Z})$, $z \in H^{2n-6}(M \times M)$, $c \in H^{2n-5}(M \times M)$, and $d \in H^{2n-4}(M \times M)$ such that f and $h_1 = m_1\circ(i_1\circ l, h)$ are homotopic, where $i_1: \Omega C_1 \to P_1$ is the inclusion of the fibre. The class $i_1 \circ l$ is invariant under t since both f and h are obviously invariant under t. Thus the homotopy class $[l]+[l\circ t]$ lies in the image of the homomorphism $[M \times M, K_{n-1}] \to [M \times M, \Omega C_1]$. Thus there is a class θ in $H^{n-1}(M \times M)$ such that

$$[l]+[l\circ t] = (\delta Sq^{8k-2}\theta, Sq^{8k}\theta, Sq^{8k}Sq^1\theta, Sq^{8k}Sq^2\theta, 0).$$

It can be shown that if M is 3-connected mod 2, then $Sq^{8k}H^{n-1}(M \times M) = 0$, $Sq^{8k-1}\theta$ is of the form $\alpha_{4k-1}^2 \otimes \mu + \mu \otimes \alpha_{4k-1}^2$, $Sq^{8k}Sq^1\theta$ is of the form $(Sq^1\alpha_{4k-1})^2 \otimes \mu + \mu \otimes (Sq^1\alpha_{4k-1})^2$, and $Sq^{8k}Sq^2\theta$ is of the form

$$Sq^{4k}Sq^2\alpha_{4k-1} \otimes \mu + \mu \otimes Sq^{4k}Sq^2\alpha_{4k-1}, \quad \text{where } \alpha_{4k-1} \in H^{4k-1}(M).$$

We can write $\rho_2 y = y' \otimes \mu + \mu \otimes y''$, $y', y'' \in H^{n-7}(M)$, and $\rho_2 y + \rho_2 t y = (y'+y'') \otimes \mu + \mu \otimes (y'+y'')$. Since ϕ_7 is defined on y' and y'', $\phi_7(y'+y'') = \phi_7(y') + \phi_7(y'')$. Now by the S-duality pairing,

$$\begin{aligned} \langle \phi_7(\alpha_{4k-1})^2, U(\nu) \rangle &= \langle (\alpha_{4k-1})^2, \chi\phi_7(U(\nu)) \rangle \\ &= \langle (\alpha_{4k-1})^2, U(\nu) \cdot Sq^3\theta_4(\nu) \rangle \\ &= \langle (Sq^1(\alpha_{4k-1})^2, U(\nu) \cdot Sq^2\theta_4(\nu) \rangle = 0. \end{aligned}$$

Thus $\phi_7(\alpha_{4k-1})^2 = 0$ and so $\phi_7(y') = \phi_7(y'')$. Hence $\phi_7(y) = \phi_7(\rho_2(y)) = 0$. Note that by the definition of Ω, $\mathrm{Indet}^{2n}(M \times M, \Omega)$ is given by $\{\phi_7(a_1) + \phi_6(a_2) + \phi_5(a_3) + \chi\phi_4(a_4) \mid a_1 \in H^{n-7}(M \times M; \mathbf{Z}), a_2 \in H^{n-6}(M \times M), a_3 \in H^{n-5}(M \times M)$, and $a_4 \in H^{n-4}(M \times M)\}$, where $\chi\phi_4$, ϕ_5, and ϕ_6 are stable cohomology operations defined by the following relations:

$$\begin{aligned} &\chi\phi_4: \chi Sq^4Sq^1 + Sq^2Sq^3 + Sq^1\chi Sq^4 = 0, \\ &\phi_5: Sq^1Sq^5 = 0, \\ &\phi: \chi Sq^4Sq^3 + Sq^1(\chi Sq^4Sq^2) = 0. \end{aligned}$$

Trivially, since $Sq^{8k}\theta = 0$, z is symmetrical and so $\phi_6(z) = 0$. As before, we can write $c = c' \otimes \mu + \mu \otimes c''$, $c', c'' \in H^{n-5}(M)$. Then

$$c + tc = (c'+c'') \otimes \mu + \mu \otimes (c'+c'') = (Sq^1\alpha_{4k-1})^2 \otimes \mu + \mu \otimes (Sq^1\alpha_{4k-1})^2$$

for some $\alpha_{4k-1} \in H^{4k-1}(M)$. Now the S-duality pairing

$$\langle \phi_5(Sq^1\alpha_{4k-1})^2, U(\nu) \rangle = \langle (Sq^1\alpha_{4k-1})^2, \chi\phi_5(U(\nu)) \rangle = 0,$$

since $\chi\phi_5(U(\nu))=0$ modulo zero indeterminacy. Therefore, by the additivity of the stable operation ϕ_5, $\phi_5(c)=0$. Similarly we can write $d=d'\otimes\mu+\mu\otimes d''$, $d', d''\in H^{n-4}(M)$. Then $d+td=(d'+d'')\otimes\mu+\mu\otimes(d'+d'')=Sq^{4k}Sq^2\alpha_{4k-1}\otimes\mu+\mu\otimes Sq^{4k}Sq^2\alpha_{4k-1}$. Again by the S-duality pairing, we have

$$\begin{aligned}
&\langle\chi\phi_4(Sq^{4k}Sq^2\alpha_{4k-1}), U(\nu)\rangle\\
&\qquad=\langle Sq^{4k}Sq^2\alpha_{4k-1}, \phi_4 U(\nu)\rangle\\
&\qquad=\langle Sq^4Sq^2(Sq^{4k-4}+Sq^{4k-5}Sq^1)\alpha_{4k-1}, U(\nu)\cdot\theta_4(\nu)\rangle\\
&\qquad=\langle(Sq^{4k-4}+Sq^{4k-5}Sq^1)\alpha_{4k-1}, U(\nu)\cdot Sq^6\theta_4(\nu)\rangle=0.
\end{aligned}$$

Thus $\chi\phi_4(Sq^{4k}Sq^2\alpha_{4k-1})=0$ and so by stability $\chi\phi_4(d)=0$. Hence $\phi_7(y)+\phi_6(z)+\phi_5(c)+\chi\phi_4(d)=0$. Now consider the fibre square

$$\begin{array}{ccc}
G_1 & \xrightarrow{\bar{i}_1} & P_2\\
{\scriptstyle \bar{p}_2}\downarrow & & \downarrow{\scriptstyle p_2}\\
\Omega C_1 & \xrightarrow{i_1} & P_1
\end{array}$$

Since both f and h lift to P_2, l must lift to G_1 with a lifting $\bar{l}: M\times M\to G_1$. Now $\bar{\bar{h}}=m_2\circ(\bar{i}_1\circ\bar{l}, \bar{h})$ is a lifting of $m_1\circ(i_1\circ l, h)\sim f$. Let w be a representative for the operation Ω. Then $\bar{\bar{h}}^*w=\bar{h}^*w+\bar{l}^*\bar{i}_1^*w$. Now

$$\bar{l}^*i_1^*w\in\phi_7(y)+\phi_6(z)+\phi_5(c)+\chi\phi_4(d)=0.$$

Thus $\bar{\bar{h}}^*w=\bar{h}^*w=\bar{\bar{A}}^*w+t\bar{\bar{A}}^*w=0$. Now $\bar{\bar{f}}=\bar{\bar{U}}\circ g$ is a lifting of $f\sim m_1\circ(i_1\circ l, h)$. Since the primary piece of the indeterminacy of Ω is trivial, $\bar{\bar{f}}^*w=\bar{\bar{h}}^*w=0$; that is, $g^*\bar{\bar{U}}^*w=0$. Since g^* is injective, $\bar{\bar{U}}^*w=0$ and thus $\Omega(U(\tau))=0$ modulo zero indeterminacy. This completes the proof. □

3.12. PROOF OF THEOREM 1.1. This is now an immediate corollary to Theorem 3.11.

4. 8-frame Fields

In this section we shall assume that M is 4-connected mod 2. We shall refer to Table 2 for the n-MPT for $\pi: B\hat{SO}_{n-8}\langle 8\rangle\to B\hat{SO}_n\langle 8\rangle$. Because of the connectivity condition on M we shall be interested in realizing $(k_1^2, k_2^2, k_5^2, k_6^2)(\tau)$ and $(k_1^3, k_3^3)(\tau)$ whenever they are defined. For simplicity we shall make the assumption that $Sq^1H^{n-6}(M)=H^{n-5}(M)$ so that we can ignore k_2^2 and k_1^3. Recall that $\dim M=n=8k+5,\ k\geq 2$.

Consider the stable cohomology operations θ_1, θ_2, and θ_3 associated with the following relations

$$(4.1)\quad \begin{cases} \theta_1: Sq^2Sq^{8k-2}+Sq^1(Sq^{8k-2}Sq^1)=0; \\ \theta_2: (Sq^4Sq^2Sq^1+Sq^7+Sq^6Sq^1)Sq^{8k-2}+(Sq^2Sq^3)Sq^{8k} \\ \qquad\qquad +(Sq^2Sq^1)(Sq^{8k}Sq^2)+Sq^6(Sq^{8k-2}Sq^1) \\ \qquad\qquad +Sq^2(Sq^{8k-1}Sq^3Sq^1)=0; \\ \theta_3: Sq^2Sq^1(Sq^{8k-1}Sq^3Sq^1)=0. \end{cases}$$

We can choose $(\theta_1,\theta_2,\theta_3)$ such that

$$(4.2)\quad \begin{aligned} &(0,\ (Sq^{8k-1}Sq^4Sq^1)\iota_{8k-6},\ Sq^5Sq^1\iota_{8k-6}\cup Sq^3Sq^1\iota_{8k-6},\ 0) \\ &\qquad\qquad\qquad\qquad \in(\theta_1,\theta_2,\theta_3)(\iota_{8k-6})\cdots \end{aligned}$$

and such that, on the fundamental class b_{8k-3} of the principal fibration Y_{8k-3} over K_{8k-3} with classifying map $(Sq^1,Sq^2)\iota_{8k-3}$,

$$(4.3)\quad (0,Sq^7b_{8k-3}\cup b_{8k-3},0)\in(\theta_1,\theta_2,\theta_3)(b_{8k-3}).$$

Using the above characterization and the methods of [8] and [18], we can prove the next theorem.

THEOREM 4.4. *The operation $(\theta_1,\theta_2,\theta_3)$ can be chosen in such a way that the following relation holds stably:*

$$(4.5)\quad \Theta: (Sq^4Sq^2Sq^1+Sq^7)\theta_1+Sq^2\theta_2+Sq^1\theta_3=0.$$

Hence there is defined a family of tertiary operations associated with the above relation. Furthermore, we can choose a tertiary operation Θ such that on d_{8k-3}, the fundamental class of dimension $8k-3$ of D_{8k-3}, the principal fibration over K_{8k-3} with classifying map $(Sq^1,Sq^2,Sq^4)(\iota_{8k-3})$, we have

$$(4.6)\quad \phi_8(d_{8k-3})\cup d_{8k-3}\in\Theta(d_{8k-3}),$$

where ϕ_8 is a stable operation associated with the relation

$$(4.7)\quad \phi_8: (Sq^4Sq^2Sq^1+Sq^7)Sq^2+Sq^2(Sq^4Sq^2Sq^1+Sq^7+Sq^6Sq^1)=0.$$

THEOREM 4.8. *For the k-invariants defined by Table 2, we have:*

(a) $(w_{8k-2},w_{8k})\in H^{8k-2}(B\hat{SO}_n\langle 8\rangle)\oplus H^{8k}(B\hat{SO}_n\langle 8\rangle)$ *is admissible for* $(k_1^2,k_4^2,0)\in H^{8k-1}(E_1)\oplus H^{8k+4}(E_1)\oplus H^{8k+5}(E_1)$ *via* $(\theta_1,\theta_2,\theta_3)$.

(b) $(k_1^2,k_4^2)\in H^{8k-1}(E_1)\oplus H^{8k+4}(E_1)$ *is admissible for* $k_3^3\in H^{8k+5}(E_2)$ *via Θ. In particular,*

$$(4.9)\quad U(E_1)\cdot(k_1^2,k_4^2)\in(\theta_1,\theta_2)(U(E_1))$$

and, for some $\lambda\in\mathbf{Z}_2$,

$$U(E_2)\cdot(p_2^*p_1^*(w_{8k-3}\cdot(\theta_4^2+\lambda w_8))+k_3^3)\in\Theta(U(E_2)).$$

Proof. The proofs of (a) and (b) are similar to that of Lemma 3.8, and we shall not present them here. The evaluations on $U(E_1)$ and $U(E_2)$ are (respectively) given by the choice of θ_1, θ_2, and θ_3 and the fact that

$$\phi_8(U(B\hat{S}O_{n-8}\langle 8\rangle)) = (U(B\hat{S}O_{n-8}\langle 8\rangle)\cdot(w_{8k+3}\cdot(\theta_4^2+\lambda w_8))$$

for some $\lambda\in\mathbf{Z}_2$, and by the admissible class theorem. □

We shall now consider the identification of k_5^2 and k_6^2. Consider the operations associated with the relations:

$$(4.11)\quad \begin{cases} \theta_5: Sq^8Sq^{8k-2}+Sq^6Sq^{8k} \\ \qquad +Sq^2(Sq^{8k}Sq^4+Sq^{8k-1}Sq^5+Sq^{8k+3}Sq^1)=0, \ k \text{ even}; \\ \theta_5: Sq^8Sq^{8k-2} \\ \qquad +Sq^2(Sq^{8k}Sq^4+Sq^{8k-1}Sq^5+Sq^{8k+2}Sq^2)=0, \ k \text{ odd}; \\ \theta_6: Sq^6Sq^{8k}+Sq^2(Sq^{8k+2}Sq^2+Sq^{8k+3}Sq^1)=0, \\ \qquad\qquad \text{valid on classes of dimension} \le 8k+5. \end{cases}$$

We can choose (θ_5,θ_6) so that on d_{8k-3}, the fundamental class of D_{8k-3},

$$(4.12)\qquad (Sq^8d_{8k-3}\cup d_{8k-3},0)\in(\theta_5,\theta_6)(d_{8k-3}).$$

Now $\pi^*: H^*(B\hat{S}O_{8k+5}\langle 8\rangle)\to H^*(B\hat{S}O_{8k-3}\langle 8\rangle)$ is an epimorphism in dimension $\le 8k+5$ for $k\ge 5$. For $2\le k\le 4$, π^* is an epimorphism in dimension $8k-1$ and $8k+3$. Thus $\mathrm{Indet}^{2n}(T(\pi^*\gamma),\theta_i)=(T\pi)^*\,\mathrm{Indet}^{2n}(T(\gamma),\theta_i)$ for $i=5,6$, where γ is the universal n-plane bundle over $B\hat{S}O_n\langle 8\rangle$. Therefore, by (4.12) and the admissible class theorem, we have the following.

THEOREM 4.13.
(a) $U(E_1)\cdot(p_1^*(w_8\cdot w_{8k-3})+k_5^2)\in\theta_5(U(E_1))$.
(b) $U(E_1)\cdot(k_6^2)\in\theta_6(U(E_1))$.

We shall also need a characterization of θ_1. Consider the following relations:

$$(4.14)\quad \begin{cases} \Gamma_1: Sq^2(Sq^2Sq^{8k-4})+Sq^1(Sq^2Sq^1Sq^{8k-4})=0; \\ \Gamma_2: Sq^2(Sq^{8k-2})+Sq^2(Sq^{8k-3}Sq^1)+Sq^1(Sq^{8k-2}Sq^1)=0; \\ \Gamma_3: Sq^2(Sq^{8k-3}Sq^1)=0. \end{cases}$$

Then we can choose operations associated with the above relations (also denoted by the same symbols) such that

$$(4.15)\qquad \Gamma_3=\psi_4\circ Sq^{8k-6}Sq^1,\quad \Gamma_2+\Gamma_3\subset\theta_1,\quad \text{and}\quad \Gamma_2\subset\Gamma_1\subset\psi_3\circ Sq^{8k-4},$$

where ψ_3 is defined in Theorem 1.1 and ψ_4 is a stable operation associated with the relations

$$(4.16)\qquad \psi_4: Sq^2(Sq^2Sq^1)=0.$$

THEOREM 4.17. *Suppose M is 4-connected* mod 2. *Suppose M satisfies*

(i) $Sq^1H^{n-6}(M)=H^{n-5}(M)$ *and*
(ii) $Sq^1H^{n-7}(M)=0$.

Let η be an n-plane bundle over M with $w_8(\eta)=w_8(M)$. Then $\operatorname{span}(\eta)\geq 8$ *if and only if $w_{n-7}(\eta)=0$, $w_{n-5}(\eta)=0$, $0\in\theta_1(U(\eta))$, $\theta_5(U(\eta))=0$, $\theta_6(U(\eta))=0$, and $\Theta(U(\eta))=0$.*

Proof. Condition (i) implies that $0\in k_2^2(\eta)$ and $0\in k_1^3(\eta)$ whenever $k_1^2(\eta)$ and $k_1^3(\eta)$ are defined. By Theorem 4.13, $k_5^2(\eta)=0$ if and only if $\theta_5(U(\eta))=0$ modulo zero indeterminacy, since

$$w_8(\eta)\cdot w_{8k-3}(\eta)=Sq^1(w_8(\eta)\cdot w_{8k-4}(\eta))=0.$$

Again by Theorem 4.13, $k_6^2(\eta)=0$ if and only if $\theta_6(U(\eta))=0$. Now, by the connectivity condition on M, condition (ii), and S-duality, we can easily deduce that $\operatorname{Indet}^{2n}(T(\tau),\Theta)=0$. Since $\psi\operatorname{Indet}^n(M,k_3^3)\subset\operatorname{Indet}^{2n}(T(\tau),\Theta)$, where ψ is the Thom isomorphism, we see that $\operatorname{Indet}^n(M,k_3^3)=0$. It follows from condition (ii) and Theorem 4.8 that $0\in k_1^2(\eta)$ if and only if $0\in\theta_1(U(\eta))$. Then Theorem 4.8(b) says that $k_3^3(\eta)=0$ if and only if

$$U(\eta)\cdot(w_{8k-3}(\eta)(\theta_4(\eta)^2+\lambda w_8(\eta))=\Theta(U(\eta)).$$

Since M is 4-connected mod 2 and $w_8(\eta)\cdot w_{8k-3}(\eta)=0$, we conclude that $k_3^3(\eta)=0$ if and only if $\Theta(U(\eta))=0$. This completes the proof. □

We shall now apply the above theorem to the span of M.

THEOREM 4.18. *Suppose that M is* 4*-connected* mod 2. *Suppose also that $Sq^1H^{n-7}(M)=0$ and $Sq^1H^{n-6}(M)=H^{n-5}(M)$. Then* $\operatorname{span} M\geq 8$ *if and only if $w_{n-7}(M)=0$, $w_{n-5}(M)=0$, $0\in\theta_1(U(\tau))$, and $\chi_2(M)=0$.*

Before proving this we need a preliminary lemma. Recall the decomposition $U(M)=A+tA$.

LEMMA 4.19. *Suppose that M is* 4*-connected* mod 2, *$w_{n-7}(M)=0$, and $w_{n-5}(M)=0$. Then:*

(a) *θ_1, θ_2, and θ_3 are defined on A. In particular, if $Sq^1H^{n-7}(M)=0$ and $0\in\theta_1(U(\tau))$, then Θ is defined on A and hence on tA;* $\Theta(U(\tau))=0$ *modulo zero indeterminacy.*
(b) *θ_5 and θ_6 are defined on A. $\theta_5(U(\tau))=0$ modulo zero indeterminacy and $\theta_6(U(\tau))=\chi_2(M)U(\tau)\cdot\mu$.*

Proof. Part (a). If $w_{n-5}(M)=0$, then by the definition of $\theta_1,\theta_2,\theta_3$ and Lemma 3.10(i), it only remains to show that $Sq^{8k-2}A=0$ and $Sq^{8k-2}Sq^1A=0$. By a Cartan formula and Wu duality we can show that

$$Sq^{8k-2}A=w_{8k-2}(M)\otimes\mu,$$

since M is 3-connected mod 2 and $Sq^{8k-2}Sq^1A=0$. Thus θ_1, θ_2, and θ_3 are defined on A. Now we can choose θ_3 such that $\theta_3(A)=Sq^2\theta_3'(A)=0$, where

θ_3' is a stable cohomology operation associated with the relation

$$Sq^1(Sq^{8k-1}Sq^3Sq^1)=0.$$

By the connectivity condition on M, $\theta_2(A)=0$. By the characterization (4.15),

$$\theta_1(A)=\psi_4(Sq^{8k-6}Sq^1A)+\psi_3(Sq^{8k-4}A)$$

modulo $Sq^{8k-3}Sq^2A\oplus Sq^{8k-4}Sq^3A\oplus Sq^{8k-4}Sq^2Sq^1A$. The above group can be shown to be trivial since M is 3-connected mod 2. By the connectivity condition on M we have

$$\begin{aligned}\psi_4(Sq^{8k-6}Sq^1A)&=\psi_4\left(\sum_{l=1}^{n(4k+2)}Sq^{4k-3}Sq^1\alpha_{4k+2}^l\otimes Sq^{4k-3}\beta_{4k+3}^l\right)\\&=\sum_{l=1}^{n(4k+2)}\{\psi_4(Sq^{4k-3}Sq^1)\alpha_{4k+2}^l\otimes Sq^{4k-3}\beta_{4k+3}^l\\&\qquad+Sq^{4k-3}Sq^1\alpha_{4k+2}^l\otimes\psi_4Sq^{4k-3}\beta_{4k+3}^l)\}\\&=0.\end{aligned}$$

Now suppose that $0\in\theta_1(U(\tau))$. Then $0\in\theta_1(U(M))$. Now $\psi_3(Sq^{8k-4}A)=\{\psi_3(w_{8k-4}(M))\otimes\mu\}$ and

$$H^{n+8k-1}(M\times M)\approx H^n(M)\otimes H^{8k-1}(M)\oplus H^{8k-1}(M)\otimes H^n(M).$$

From this we see that $0\in\theta_1(U(\tau))$ implies $g^*(\theta_1(U(\tau)))$, and that $\theta_1(A)+\theta_1(tA)$ differ by symmetric elements. Thus $0\in\theta_1(A)$. Thus Θ is defined on A hence on tA. The rest of the proof is exactly the same as that used in Theorem 3.11 to show that $\Theta(U(\tau))=0$, and we leave the details to the reader.

Part (b). By the definitions of θ_5 and θ_6 and the connectivity condition on M, θ_5 and θ_6 are defined on A and hence on tA. Since $\mathrm{Indet}^{2n}(M\times M,\theta_6)$ is trivial, $g^*\theta_6(U(\tau))=\theta_6(A+tA)=\theta_6(A)+\theta_6(tA)+A\cup tA=\chi_2(M)\mu\otimes\mu$ for θ_6 is nonstable. By the methods of [11, Thm. 9.5], using the fact that θ_5 is defined on A and on tA, we can show that $\theta_5(U(\tau))=0$ modulo zero indeterminacy. This completes the proof of part (b). □

4.20. PROOF OF THEOREM 4.18. This is now an immediate consequence of Theorem 4.17 and Lemma 4.19.

4.21. PROOF OF THEOREM 1.2. This is now an immediate consequence of Theorem 4.18 since $0\in\theta_1(U(\tau))$ is equivalent to $0\in\psi_3(w_{n-9}(M))$. (For $w_{n-9}\in H^{n-9}(B\hat{SO}_n\langle 8\rangle)$ is a generating class for k_1^2 via ψ_3, which is spin-trivial. Thus $0\in k_1^2(\tau)$ is equivalent to $0\in\psi_3(w_{n-9}(M)$.)

5. Applications to Immersions

Suppose that M is a spin manifold of dimension n with $n\equiv 5 \bmod 8$ and $n\geq 21$. Then by Massey and Peterson [5], $\bar{w}_{n-i}(M)=0$ for $i=0,1,2,\dots,5$; $\delta\bar{w}_{n-7}(M)=0$ unless $\alpha(n)=3$, where $\alpha(n)$ is the number of terms in the dyadic development of n; $\bar{w}_{n-7}(M)=0$ unless $\alpha(n)\leq 4$; and $\bar{w}_{n-8}(M)=0$ unless $\alpha(n)=4$ or 5. If M is a spin manifold satisfying $w_4(M)=0$, then $\bar{w}_{n-7}(M)=0$ unless $\alpha(n)\leq 4$ and $n\equiv 5 \bmod 16$; $\bar{w}_{n-9}(M)=0$ unless $\alpha(n)\leq$ 6 and $n\equiv 5 \bmod 16$. Since we are now seeking the stable span of ν we shall ignore the nonstable k-invariants (k_5^2 in Table 1 and k_6^2 in Table 2).

THEOREM 5.1. *Suppose M is 4-connected* mod 2. *Suppose M satisfies*

(i) $Sq^1H^{n-6}(M)=H^{n-5}(M)$ *and*

(ii) $Sq^1H^{n-7}(M)=0$.

If $\alpha(n)\leq 4$ and $n\equiv 5 \bmod 16$ assume $\bar{w}_{n-7}(M)=0$. Then M immerses in $\mathbf{R}^{2n-8}$ if and only if $0\in\psi_3(\bar{w}_{n-9}(M))$.

Proof. We can use Theorem 4.17 ignoring the nonstable operation θ_6. We remark that $0\in\psi_3(\bar{w}_{n-9}(M))$ is equivalent to $0\in\theta_1(U(\nu))$. Therefore, if $0\in\theta_1(U(\nu))$ then $\Theta(U(\nu))$ is defined, and $\Theta(U(\nu))=0$ since Θ is a stable cohomology operation mapping into the top class of $H^*(T(\nu))$. Similarly, $\theta_5(U(\nu))=0$. Thus M immerses in $\mathbf{R}^{2n-8}$. □

We now have the following.

COROLLARY 5.2. *Suppose M is a 5-connected* mod 2 *manifold of dimension $n\equiv 13$* mod $16\geq 29$. *Suppose $H_6(M;\mathbf{Z})$ has no 2-torsion. Then M immerses in $\mathbf{R}^{2n-8}$.*

Proof. Plainly, conditions (i) and (ii) of Theorem 5.1 are satisfied. By the remark preceding Theorem 5.1, $\bar{w}_{n-9}(M)=0$. Trivially, $0\in\psi_3(\bar{w}_{n-9}(M))$. Therefore, by Theorem 5.1, M immerses in $\mathbf{R}^{2n-8}$. □

REMARK 5.3. Theorem 5.1 and Corollary 5.2 give a new result only if $\alpha(n)<8$, in view of Cohen's immersion theorem.

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