The Basic Geometry of the Manifold of Riemannian Metrics and of its Quotient by the Diffeomorphism Group

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Introduction

Let M be a compact, oriented, smooth n-manifold and consider the collection $\mathrm{Met}(M)$ of all Riemannian metrics on M. Although $\mathrm{Met}(M)$ is a contractible open cone inside the space $\Gamma(S^2T^*M)$ of symmetric rank-2 tensor fields, its natural metric (described later) is nonconstant and yields interesting geometry. Furthermore, the group $\mathrm{Diff}^+(M)$ of orientation-preserving diffeomorphisms acts isometrically (by pullback) on $\mathrm{Met}(M)$, and is free on the subset $\mathrm{Met}'(M)$ of metrics which admit no nontrivial isometries. Hence there is an induced metric on the quotient $\mathrm{Met}'(M)/\mathrm{Diff}^+(M)$. In this paper we derive formulas for the curvature and geodesics of $\mathrm{Met}(M)$ and of $\mathrm{Met}'(M)/\mathrm{Diff}^+(M)$.

The metric on $\mathrm{Met}(M)$ is an example of an "L^2 metric" on a mapping space. More generally, suppose M is a compact (finite-dimensional) manifold endowed with a measure μ, and let N be a Riemannian manifold with metric g. Then the space of (smooth) maps $\mathrm{Map}(M,N)$ inherits an L^2 metric as follows. A tangent vector at $\phi\in\mathrm{Map}(M,N)$ is a cross-section of the pulled-back tangent bundle $\phi^*TN\to M$, and the inner product of two tangent vectors A and B is

$$\langle A,B\rangle=\int_M g(A(x),B(x))\,\mu(x).$$

For this metric one easily calculates that the curvature $R(X,Y)Z$ is, pointwise, simply the curvature of N; it does not depend on the measure μ. Moreover, a geodesic in the mapping space corresponds to a family of geodesics in N. We discuss these matters in the appendix.

Although $\mathrm{Met}(M)$ is not, strictly speaking, a space of maps of the type above, it is the space of sections of a fiber bundle, and similar principles

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apply. The typical fiber is $GL^+(n)/SO(n)$ (viewed as the space of positive-definite $n \times n$ symmetric matrices), and it suffices to compute the curvature of this finite-dimensional space in an appropriate metric (see the appendix). Our computation in Section 1 proceeds by splitting this space into a product of the symmetric space $SL(n)/SO(n)$ and the flat space $\mathbf{R}^+$. However, the metric is *not* a product, and so second fundamental form terms enter the final formula. Although (for concreteness) we derive our formulas directly for the infinite-dimensional space, they equally well give the curvature and geodesics of the finite-dimensional space.

The curvature of the quotient $\mathrm{Met}'(M)/\mathrm{Diff}^+(M)$ can be computed from the curvature of the principal fibration $\mathrm{Met}'(M) \to \mathrm{Met}'(M)/\mathrm{Diff}^+(M)$, and the geodesics of the quotient can be computed from those of $\mathrm{Met}'(M)$.

When M is a Riemannian surface of genus greater than 1, one may form the quotient $\mathrm{Met}_{-1}(M)/\mathrm{Diff}^0(M)$ of the subspace of metrics of constant curvature -1 by the identity component of the diffeomorphism group. This quotient may be naturally identified with the Teichmüller space of M, and the induced metric is the Weil-Petersson metric. This approach to Teichmüller theory has been studied extensively by Fischer and Tromba [3; 4]. The basic differential topology of $\mathrm{Met}(M)/\mathrm{Diff}^+(M)$ was studied by Ebin in his doctoral thesis (see [1]). His main theorem asserts that the action of the diffeomorphism group on the space of metrics admits a slice. Furthermore, he defines the L^2 metric that we use here, and computes the Levi-Civita connection of $\mathrm{Met}(M)$. DeWitt [2] computed the curvature of $\mathrm{Met}(M)$, implicitly using the principle about L^2 metrics mentioned above. He wanted these formulas for his Hamiltonian approach to general relativity. One of our original motivations for this paper was a desire to simplify his calculation and make the geometric content more self-evident. Finally, we understand that Bourguignon and collaborators have made computations similar to those we undertake in Section 3. We owe I. M. Singer the suggestion that we undertake these curvature computations.

In Sections 1–3 we discuss only formal aspects of the problem, essentially treating $\mathrm{Met}(M)$, $\mathrm{Met}'(M)$, and $\mathrm{Diff}^+(M)$ as finite-dimensional manifolds. These spaces are of course infinite-dimensional, but are Fréchet manifolds (see [10]). The maps and group actions we use are smooth in the Fréchet sense, and our formal treatment can be rigorously justified in this setting.

Where convenient, we will abbreviate $\mathrm{Met}(M)$, $\mathrm{Met}'(M)$, and $\mathrm{Diff}^+(M)$ as $\mathfrak{M}$, $\mathfrak{M}'$, and $\mathfrak{D}$, respectively.

1. The Curvature of Met(M)

Since $\mathfrak{M}$ is an open subset of the vector space $\Gamma(S^2T^*M)$, the tangent space $T_g\mathfrak{M}$ (for any $g \in \mathfrak{M}$) is $\Gamma(S^2T^*M)$ itself. Since g determines a pointwise inner product on tensors, an L^2 inner product on tensor fields is induced. Specifically, for any $A, B \in \Gamma(T^*M \otimes T^*M)$, we set

$$\langle A, B\rangle_g = \int_M \operatorname{tr}_g(AB^t)\mu(g), \tag{1.1}$$

where, in local coordinates $\{x^i\}$, $A = A_{ij}dx^i \otimes dx^j$ (and similarly for B, g), $B^t = B_{ji}dx^i \otimes dx^j$ (the "transpose" of B), $g = g_{ij}dx^i \otimes dx^j$, $\{g^{ij}\}$ is the matrix inverse of $\{g_{ij}\}$, $\operatorname{tr}_g(AC) = A_{ij}g^{jk}C_{kl}g^{li}$, and $\mu(g)$ is the volume form $\sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$. The restriction of this quadratic form to symmetric A, B is positive definite. [More generally, one can define a metric by using $\operatorname{tr}_g(A^0B^0) + W \operatorname{tr}_g(A)\operatorname{tr}_g(B)$, with an arbitrary weight W, in place of $\operatorname{tr}_g(AB)$ in (1.1); here $\operatorname{tr}_g(A)$ means $A_{ij}g^{ij}$. The metric (1.1) is equivalent to choosing $W = 1/n$.]

Let $\operatorname{Vol}(M) \subset \Omega^n(M)$ denote the space of volume forms on M consistent with the orientation. For $\alpha \in \Omega^n(M)$ and $\nu \in \operatorname{Vol}(M)$, let (α/ν) denote the function satisfying $\alpha = (\alpha/\nu)\nu$. Let $p: \mathfrak{M} \to \operatorname{Vol}(M)$ denote the projection carrying g to $\mu(g)$, and let $\mathfrak{M}_\nu = \operatorname{Met}_\nu(M) = p^{-1}(\nu)$ for any $\nu \in \operatorname{Vol}(M)$. The fibration p is trivial, and each $g \in \mathfrak{M}$ determines a section $s_g \in \operatorname{Vol}(M) \to \mathfrak{M}$ given by

$$s_g(\nu) = (\nu/\mu(g))^{2/n} g. \tag{1.2}$$

More generally, each volume form μ determines a splitting

$$i_\mu: \operatorname{Vol}(M) \times \mathfrak{M}_\mu \cong \mathfrak{M}, \qquad (\nu, h) \mapsto (\nu/\mu)^{2/n} h. \tag{1.3}$$

Since $\operatorname{Vol}(M)$ is an open subset of the vector space $\Omega^n(M)$, the tangent bundle of $\operatorname{Vol}(M)$ is canonically isomorphic to $\operatorname{Vol}(M) \times \Omega^n(M)$. A vector field β over any subset $U \subset \operatorname{Vol}(M)$ may therefore be naturally identified with a function $\beta: U \to \Omega^n(M)$, and we implicitly make this identification henceforth. Given such a β defined over an open set U, any $\nu \in U$, and any $\alpha \in \Omega^n(M)$, we define the directional derivative

$$\frac{\delta\beta}{\delta\alpha}(\nu) = \frac{d}{dt}\beta(\nu + t\alpha)\bigg|_{t=0},$$

where the right-hand side is computed pointwise on M. Directional derivatives of functions on $\operatorname{Vol}(M)$ are defined analogously.

Each choice of μ and h in (1.3) gives us an embedding of $\operatorname{Vol}(M)$ in $\mathfrak{M}$ as $i_\mu(\operatorname{Vol}(M) \times \{h\})$, and therefore via (1.1) induces a metric on $\operatorname{Vol}(M)$. For $\alpha, \beta \in T_\nu \operatorname{Vol}(M)$, the induced inner product is

$$\langle \alpha, \beta\rangle = \frac{4}{n}\int \left(\frac{\alpha}{\nu}\right)\left(\frac{\beta}{\nu}\right)\nu, \tag{1.4}$$

independently of μ and h, that is, $4/n$ times the "natural" inner product on $\operatorname{Vol}(M)$.

The tangent space to $\mathfrak{M}_\mu$ at h is the set of h-traceless symmetric tensor fields; that is, $\{A \in \Gamma(S^2T^*M) \mid h^{ij}A_{ij} = 0\}$. From this and (1.1), it follows that the splitting (1.3) is everywhere orthogonal.

Our first observation is that, in the metric (1.4), $\operatorname{Vol}(M)$ is flat.

PROPOSITION 1.5. *At $\mu \in \mathrm{Vol}(M)$, the Levi-Civita connection and the Riemannian curvature of the metric* (1.4) *are given by*

(i) $\nabla^{\mathrm{vol}}_\alpha \beta = \delta\beta/\delta\alpha - \frac{1}{2}(\alpha/\mu)\beta$;
(ii) $R^{\mathrm{vol}}(\alpha, \beta) = 0$.

Proof. The Levi-Civita connection on any Riemannian manifold is determined by the "six-term formula"

$$\begin{aligned} 2(\nabla_X Y, Z) &= X(Y, Z) + Y(X, Z) - Z(X, Y) \\ &\quad - (X, [Y, Z]) - (Y, [X, Z]) + (Z, [X, Y]), \end{aligned} \tag{1.6}$$

where brackets denote a Lie derivative. Let α, β, γ be vector fields on $\mathrm{Vol}(M)$; let $\mu_t = \mu + t\gamma(\mu)$, $\alpha_t = \alpha(\mu_t)$, and $\beta_t = \beta(\mu_t)$. Then, at μ,

$$\begin{aligned} \frac{n}{4}\gamma\langle\alpha, \beta\rangle &= \frac{\delta}{\delta\gamma}\int_M \left(\frac{\alpha}{\mu}\right)\left(\frac{\beta}{\mu}\right)\mu \\ &= \int\left[\left(\frac{d}{dt}\left(\frac{\alpha_t}{\mu_t}\right)\right)\left(\frac{\beta}{\mu}\right)\mu + \left(\frac{\alpha}{\mu}\right)\frac{d}{dt}\left(\frac{\beta_t}{\mu_t}\right)\mu + \left(\frac{\alpha}{\mu}\right)\left(\frac{\beta}{\mu}\right)\frac{d}{dt}\mu_t\bigg|_{t=0}\right]. \end{aligned}$$

However, from $\alpha = (\alpha/\mu)\mu$ we find

$$\begin{aligned} \frac{d}{dt}\left(\frac{\alpha_t}{\mu_t}\right)\bigg|_{t=0} &= \left(\frac{\delta\alpha}{\delta\gamma}\Big/\mu\right) - \left(\frac{\alpha}{\mu}\right)\left(\frac{\gamma}{\mu}\right), \\ \frac{n}{4}\gamma\langle\alpha, \beta\rangle &= \int\left[\left(\frac{\beta}{\mu}\right)\left(\frac{\delta\alpha}{\delta\gamma}\right) + \left(\frac{\alpha}{\mu}\right)\left(\frac{\delta\beta}{\delta\gamma}\right) - \left(\frac{\alpha}{\mu}\right)\left(\frac{\beta}{\mu}\right)\gamma\right]. \end{aligned} \tag{1.7}$$

Using this equation, (1.6), and the fact that $[\alpha, \beta] = \delta\beta/\delta\alpha - \delta\alpha/\delta\beta$, we find

$$\langle\nabla^{\mathrm{vol}}_\alpha \beta, \gamma\rangle\,|_\mu = \frac{4}{n}\int\left[\left(\frac{\gamma}{\mu}\right)\left(\frac{\delta\beta}{\delta\alpha}\right) - \frac{1}{2}\left(\frac{\alpha}{\mu}\right)\left(\frac{\beta}{\mu}\right)\gamma\right],$$

and the formula for ∇^{vol} follows. As for curvature, applying the above to *constant* vector fields α, β, γ gives

$$\begin{aligned} \nabla_\alpha\nabla_\beta\gamma &= \nabla_\alpha\left(-\frac{1}{2}\left(\frac{\beta}{\mu}\right)\gamma\right) \\ &= -\frac{1}{2}\left\{\left[-\left(\frac{\beta}{\mu}\right)\left(\frac{\alpha}{\mu}\right)\right]\gamma + \frac{\beta}{\mu}\left[-\frac{1}{2}\left(\frac{\alpha}{\mu}\right)\gamma\right]\right\} \\ &= \frac{3}{4}\left(\frac{\alpha}{\mu}\right)\left(\frac{\beta}{\mu}\right)\gamma, \end{aligned}$$

and thus $R(\alpha, \beta)\gamma = \nabla_\alpha\nabla_\beta\gamma - \nabla_\beta\nabla_\alpha\gamma = 0$. □

Thus the "pure trace" directions of $\mathrm{Met}(M)$ carry no curvature. However, we will see that $\mathfrak{M}$ is not flat in the "traceless" directions [i.e., those tangent to $\mathfrak{M}_\mu$ under the splitting (1.3)].

Given $g \in \mathfrak{M}_\mu$, for each $h \in \mathfrak{M}_\mu$ and $x \in M$ there is a unique endomorphism $\boldsymbol{h}$ of $T_x M$ satisfying $h(X, Y) = g(\boldsymbol{h}(X), (Y))$; moreover, $\boldsymbol{h}$ is self-adjoint with respect to g and of determinant 1. [In terms of local coordinates, $\boldsymbol{h}$ is obtained from h by "raising an index"; i.e., if $h = h_{ij}\,dx^i \otimes dx^j$ and $h = h^i_j(\partial/\partial x^i)\otimes dx^j$ then $h^i_j = g^{ik}h_{kj}$.] Thus $\mathfrak{M}_\mu$ consists of sections of a bundle whose fiber can be identified with the noncompact *symmetric space* $SL(n)/SO(n)$.

In general, given a semisimple, noncompact group G, a maximal compact subgroup H determines a symmetric space $N = G/H$. The Lie algebra decomposes into $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, with $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$. The Killing form, negative-definite on $\mathfrak{h}$ and positive-definite on $\mathfrak{p}$, induces a G-invariant metric on N. Let $\rho(S)$ denote the isometry of N corresponding to left-translation by $S \in G$, and $\dot\rho(X)$ the Killing vector field on N induced by $X \in \mathfrak{p}$. Then the curvature of N is given by

$$R(\dot\rho(X), \dot\rho(Y))\dot\rho(Z) = -\dot\rho([[X, Y], Z]) \tag{1.8}$$

(see [8]). The formula for the connection is more complicated. The isotropy representation of H on $T_{[e]}N$ is given by the adjoint action of H on $\mathfrak{p}$, and the vector bundle associated to the principal H-bundle $G \to G/H$ via this representation is isomorphic to the tangent bundle of N. Thus a vector field on N can be represented by an H-covariant function X from G to $\mathfrak{p}$—more precisely, by $\dot\rho(X)$. With this in mind,

$$\nabla_{\dot\rho(X)}\dot\rho(Y)\,|_{[S]} = \dot\rho(-\tfrac{1}{2}(Ad(S^{-1})[X, Y])_{\mathfrak{p}}), \tag{1.9}$$

where $Z_{\mathfrak{p}}$ is the $\mathfrak{p}$-component of $Z \in \mathfrak{g}$. [Note in particular that $\nabla_{\dot\rho(X)}\dot\rho(Y)$ vanishes at the base point $[e]$ of N, and that $\nabla_{\dot\rho(X)}\dot\rho(X) \equiv 0$; i.e., $\rho(\exp(tX))$ traces out geodesics.]

For the case of relevance to us, the Lie algebra $\mathfrak{sl}(n) = \mathfrak{so}(n) \otimes Sym^0(n)$ decomposes into the direct sum of the orthogonal and traceless symmetric matrices, and we may identify $\mathfrak{so}(n)$ with $\mathfrak{h}$ and $Sym^0(n)$ with $\mathfrak{p}$. Let $SPosSym(n)$ denote the set of positive-definite symmetric $n \times n$ matrices of determinant 1. The action of $SL(n)$ we are considering is the change-of-basis formula for the matrix of a quadratic form. Thus, for $S \in \mathfrak{sl}(n)$ and $P \in SPosSym(n)$,

$$\rho(S)P = SPS^t, \tag{1.10}$$

and so for $X \in Sym^0(n)$,

$$\dot\rho(X)\,|_P = XP + PX. \tag{1.11}$$

In particular, $\dot\rho(X)\,|_{\mathrm{id}} = 2X$, and (1.8) becomes

$$R^{SPosSym(n)}(A, B)C = -\tfrac{1}{4}[[A, B], C], \quad A, B, C \in T_{\mathrm{id}}\, SPosSym(n). \tag{1.12}$$

A useful result of Ebin is the following.

PROPOSITION 1.13 (cf. [1, Thm. 8.9]). $\mathrm{Met}_\mu(M)$ *is itself a (globally) symmetric space.*

Proof. The volume form μ gives a reduction of the $GL(n)$-bundle of linear frames of TM to $SL(n)$. Any $g \in \mathfrak{M}_\mu$ gives a further reduction to $SO(n)$. Let $SL(TM)$ and $SO(TM, g)$ denote (respectively) the automorphism and gauge transformation groups of these reduced bundles. Let $SPosSym(g)$ denote the subset of $\Gamma(\mathrm{End}(TM))$ whose elements are everywhere self-adjoint and positive-definite with respect to g and of determinant 1. The orbit of the identity section under the action of $SL(M)$ (via $P \mapsto SPS^\dagger$, where "$\dagger$" means adjoint with respect to g) is precisely $SPosSym(g)$, and the stabilizer of this section is $SO(TM, g)$. [Surjectivity follows from the fact that each $\boldsymbol{h}(x)$, a positive-definite element of $SL(T_xM)$ which is self-adjoint with respect to $g(x)$, has a unique positive self-adjoint square root, which is again in $SL(T_xM)$.] Hence $SPosSym(g) \cong SL(TM)/SO(TM, g)$. But $\mathfrak{M}_\mu \cong SPosSym(g)$ by the correspondence $h \mapsto \boldsymbol{h}$ discussed above, so $\mathfrak{M}_\mu \cong SL(TM)/SO(TM, g)$. To see that this quotient is a symmetric space, observe that the metric

$$\{A, B\} = \int_M \operatorname{tr}(AB^\dagger)\mu \tag{1.14}$$

on the Lie algebra $\mathfrak{sl}(TM) = \{$everywhere traceless sections of $\operatorname{End}(TM)\}$ is $Ad(SO(TM, g))$-invariant, and induces the metric (1.1) on $\mathfrak{M}_\mu$ under the isomorphism with $SL(TM)/SO(TM, g)$. Now, for each $x \in M$, the group $SL(T_xM, g_x)$ possesses an involutive automorphism σ_x which is the identity on $SO(T_xM, g_x)$ and whose differential at the identity is $+1$ on $\mathfrak{so}(T_xM, g_x)$ and -1 on $Sym^0(T_xM, g_x)$ (see [9]). For any $S \in SL(TM) \subset \Gamma(\operatorname{End}(TM))$, define $\sigma(S)$ to be that section of $\operatorname{End}(TM)$ whose value at x is $\sigma_x \circ S_x$. Then σ is an involutive automorphism of $SL(TM)$ whose fixed-point set is $SO(TM, g)$. Since (1.14) is $Ad(SO(TM, g))$-invariant, it follows that $SL(TM)/SO(TM, g)$ is a symmetric space, and hence so is $\mathfrak{M}_\mu$. □

Since the group operations in $SL(TM)$ are carried out pointwise, and since the metric (1.14) is an integral of a constant metric on $SL(n)$ times the *fixed* volume form $\mu = \mu(g)$, the curvature formula (1.12) carries over to give the following.

COROLLARY 1.15.

$$R^{SPosSym(g)}(A, B)C = -\tfrac{1}{4}[[A, B], C]$$

for $A, B, C \in T_{\mathrm{id}}(SPosSym(M, g)) = Sym^0(M, g)$.

REMARK. For each g, the index-raising isomorphism between $T^*M \otimes T^*M$ and $\operatorname{End}(TM)$ identifies $T_g\mathfrak{M}_\mu$ with $Sym^0(M, g)$. Therefore $T_g\mathfrak{M}_\mu$ inherits an associative product and a Lie bracket operation. We will use these algebraic structures henceforth without further comment, and will write $Sym_0(M, g)$ for T_gM_μ. Then we also have the following.

COROLLARY 1.15′.

$$R^{\mathrm{Met}_\mu(M)}(A, B)C = -\tfrac{1}{4}[[A, B], C] \quad \textit{for } A, B, C \in Sym_0(M, g).$$

The fibration $\mathfrak{M} \to \operatorname{Vol}(M)$, together with the metrics (1.1) and (1.4), comprise a *Riemannian submersion.* We will use this fact to compute the curvature of $\mathfrak{M}$ from (1.5) and (1.15). The result will be as follows.

THEOREM 1.16. *Under the identification* (1.3) *of* $\operatorname{Met}(M)$ *with* $\operatorname{Vol}(M) \times \operatorname{Met}_\mu(M)$,

$$R^{\mathrm{Met}(M)}(A,B)C=-\frac{1}{4}[[A,B],C]+\frac{n}{16}\{(\mathrm{tr}_g(AC))B-(\mathrm{tr}_g(BC))A\}$$

for $A,B,C\in Sym_0(M,g)=T_g\,\mathrm{Met}_\mu(g)$, *and all components of the* $R^{\mathrm{Met}(M)}$ *involving pure trace directions* (*i.e.,* $\mathrm{Vol}(M)$ *directions*) *vanish.*

In particular, $\langle R^{\mathrm{Met}(M)}(A,B)C,D)\rangle|_g=0$ if any of A,B,C,D is pure trace (i.e., a function times the metric g).

The formula in (1.16) can be derived quickly from the six-term formula (1.6). However, such a derivation leaves mysterious the vanishing of the curvature components in pure trace directions. For this reason we choose a derivation which relies explicitly on the splitting (1.3). This derivation is somewhat longer but clarifies the origins of the various terms in (1.16).

COROLLARY 1.17. *The sectional curvatures of* $\mathrm{Met}(M)$ *at g are given by*

$$K^{\mathrm{Met}(M)}(A,B):=\langle R^{\mathrm{Met}(M)}(A,B)B,A\rangle$$

$$=\int\left\{\frac{1}{4}\mathrm{tr}_g([A,B]^2)+\frac{n}{16}((\mathrm{tr}_g(AB))^2-\mathrm{tr}_g(A^2)\,\mathrm{tr}_g(B^2))\right\}\mu(g)$$

for $A,B\in Sym_0(M,g)$ *orthonormal with respect to* (1.1). *In particular,*

$$K^{\mathrm{Met}(M)}(A,B)\le 0.$$

Also, $K^{\mathrm{Met}(M)}(\alpha,A)=K^{\mathrm{Met}(M)}(\alpha,\beta)=0$ *for* $\alpha,\beta\in T_g(i_{\mu(g)}\mathrm{Vol}(M))$ [with notation as in (1.3)].

Proof of 1.17 (assuming 1.16). For any square matrices E,F,G,H,

$$\mathrm{tr}([[E,F]G]H)=\mathrm{tr}([E,F][G,H]). \tag{1.18}$$

Let $x\in M$, choose local coordinates, and let g^{-1},A_x,B_x denote (respectively) the matrices $\{g^{ij}\},\{A_{ij}\},\{B_{ij}\}$ at x. Then, by definition, $\mathrm{tr}_g(AB)|_x=\mathrm{tr}((g^{-1}A_x)(g^{-1}B_x))$, so the formula in (1.17) follows immediately from (1.16), (1.1), and (1.18), as does the vanishing of K if either direction is pure trace. The Cauchy-Schwarz inequality implies $(\mathrm{tr}(EF^t))^2\le\mathrm{tr}(EE^t)\,\mathrm{tr}(FF^t)$ for any matrices E,F and hence that $(\mathrm{tr}(EF))^2\le\mathrm{tr}(E^2)\,\mathrm{tr}(F^2)$ if E and F are symmetric. Therefore $(\mathrm{tr}_g(AB))^2-\mathrm{tr}_g(A^2)\,\mathrm{tr}_g(B^2)\le 0$ pointwise. Since $g^{-1}A$ and $g^{-1}B$ are symmetric matrices, their commutator is antisymmetric and hence $\mathrm{tr}_g([A,B]^2)\le 0$ pointwise. It follows that $K(A,B)\le 0$. $\square$

For the rest of this section we fix $\mu\in\mathrm{Vol}(M)$ and, via (1.3), regard $\mathfrak{M}$ as $\mathrm{Vol}(M)\times\mathfrak{M}_\mu$. We represent any tangent vector at $(\nu,h)\in\mathrm{Vol}(M)\times\mathfrak{M}_\mu$ as $\alpha+A$, where $\alpha\in\Omega^n(M)\cong T_\nu\,\mathrm{Vol}(M)$ and $A\in T_h\mathfrak{M}_\mu\subset\Gamma(S^2T^*M)$. Writing the metric (1.1) relative to this splitting, we have

$$\langle\alpha,\beta\rangle|_{(\nu,h)}=\frac{4}{n}\int\left(\frac{\alpha}{\nu}\right)\left(\frac{\beta}{\nu}\right)\nu, \tag{1.19a}$$

$$\langle\alpha,A\rangle|_{(\nu,h)}=0, \tag{1.19b}$$

(1.19c) $$\langle A, B\rangle|_{(\nu,h)} = \int \operatorname{tr}_h(AB^t)\nu.$$

As a preliminary to proving (1.16) we review some basic geometry of submanifolds. Let W be a submanifold of a Riemannian manifold V and give W the induced metric. Let X, Y, Z, N be locally defined vector fields along W, with X, Y, Z tangent to W and N normal to W. Then, for $\nabla = \nabla^V$ (the Levi-Civita connection on V), the equations

(1.20a) $$\nabla_X^W Y = (\nabla_X Y)^{\mathrm{TAN}} := \text{tangential component of } \nabla_X Y,$$

(1.20b) $$\mathrm{II}(X, Y) = (\nabla_X Y)^{\mathrm{NOR}} := \text{normal component of } \nabla_X Y,$$

(1.20c) $$\mathfrak{I}_N(X) = -(\nabla_X N)^{\mathrm{TAN}},$$

(1.20d) $$\mathrm{D}_X N = (\nabla_X N)^{\mathrm{NOR}},$$

define (respectively) the Levi-Civita connection ∇^W on W, the second fundamental form II, a tensor field $\mathfrak{I}$, and the induced connection D on the normal bundle of $W \hookrightarrow V$. The tensors II and $\mathfrak{I}$ are related by the *Weingarten equation*

(1.21) $$(\mathfrak{I}_N(X), Y) = (\mathrm{II}(X, Y), N).$$

The *Gauss equation, Codazzi equation,* and *Ricci equation* express the various components of the curvature $R(X, Y) = R^V(X, Y)$ in terms of the curvatures R^W of W, R^{D} of the normal connection D, and the quantities defined above [8]:

(1.22) $$(R(X, Y)Z, T) = (R^W(X, Y)Z, T) + (\mathrm{II}(X, Z), \mathrm{II}(Y, T)) - (\mathrm{II}(X, T), \mathrm{II}(Y, Z)),$$

(1.23) $$(R(X, Y)Z)^{\mathrm{NOR}} = (\mathbf{D}_X \mathrm{II})(Y, Z) - (\mathbf{D}_Y \mathrm{II})(X, Z),$$

(1.24) $$(R(X, Y)N)^{\mathrm{NOR}} = R^{\mathrm{D}}(X, Y)N + \mathrm{II}(\mathfrak{I}_N(X), Y) - \mathrm{II}(\mathfrak{I}_N(Y), X),$$

where, in (1.23), $\mathbf{D}$ represents the tensor product connection induced by ∇^W and D.

We apply these equations to $V = \mathrm{Vol}(M) \times \mathfrak{M}_\mu$ and $W = \mathrm{Vol}(M) \times \{g\}$ at the point (μ, g). For $\alpha, \beta, \ldots \in T_\mu \mathrm{Vol}(M) = \Omega^n(M)$, let $\hat{\alpha}, \hat{\beta}, \ldots$ denote the "constant" vector fields on $\mathrm{Vol}(M) \times \mathfrak{M}_\mu$ corresponding to these elements. Let $A, B, \ldots \in Sym_0(M, g)$ and use the same letters to denote the corresponding elements of $Sym^0(M, g)$ obtained by raising an index. We let $\hat{A}, \hat{B}, \ldots$ denote the Killing vector fields on $\mathfrak{M}_\mu$ obtained from $\frac{1}{2}A$, $\frac{1}{2}B, \ldots$ by using the $SL(TM)$-action; thus $\hat{A}(h) = \frac{1}{2}(Ah + hA)$, where $Ah = A_{ik} h_j^k dx^i \otimes dx^j$ locally, and so forth [cf. (1.11)]. We further extend $\hat{A}, \hat{B}, \ldots$ to vector fields on $\mathrm{Vol}(M) \times \mathfrak{M}_\mu$ by declaring them constant in the $\mathrm{Vol}(M)$ direction. Finally, we let ∇ and R denote (respectively) the Levi-Civita connection and curvature of $\mathfrak{M}$.

PROPOSITION 1.25. *In the notation of* (1.20), *with* $W = \mathrm{Vol}(M) \times \{g\}$ *and at the point* $(\mu, g) \in W$:

(i) $\nabla_\alpha^W \beta = \delta\beta/\delta\alpha - \frac{1}{2}(\alpha/\mu)\beta$;
(ii) $\mathrm{II} \equiv 0$ (*i.e.*, $\mathrm{Vol}(M)\times\{g\}$ *is totally geodesic*);
(iii) $\mathfrak{J}_A \equiv 0$ *for all* A; *and*
(iv) $\mathrm{D}_\alpha A = \delta A/\delta\alpha + \frac{1}{2}(\alpha/\mu)A$.

(*In* (i) *and* (iv) *we allow general vector fields* β, A *as well as the constant vector fields* $\hat\beta, \hat A$.)

Proof. Statement (i) is part of Proposition (1.5). For the remaining statements, we need consider only constant α, β. Applying the six-term formula (1.6), noting that $[\hat\alpha, \hat\beta] = [\hat\alpha, \hat A] = 0$ and that $\hat A\langle\hat\alpha, \hat\beta\rangle = 0$, it follows that $2\langle\nabla_\alpha\hat\beta, \hat A\rangle = -A\langle\hat\alpha, \hat\beta\rangle = 0$. This proves (ii), and (iii) then follows from (1.21). Finally, $2\langle\nabla_\alpha\hat A, \hat B\rangle = \alpha\langle\hat A, \hat B\rangle = \int \mathrm{tr}_g(AB)\alpha$, proving (iv) for the "constant" vector field $\hat A$, and the general case of (iv) follows immediately. □

COROLLARY 1.26. *On* $\mathrm{Met}(M)$,

(i) $\langle R(\alpha,\beta)\gamma, \delta\rangle = 0$,
(ii) $\langle R(\alpha,\beta)\gamma, A\rangle = 0$, *and*
(iii) $\langle R(\alpha,\beta)A, B\rangle = 0$,

where α, β, γ *are any directions tangent to* $\mathrm{Vol}(M)\times\{g\}$ *and* A, B *are any directions tangent to* $\mathrm{Met}_{\mu(g)}(M)$ *under the splitting* (1.3).

Proof. By (1.5), $\mathrm{Vol}(M)\times\{g\}$ is flat, and by (1.25) $\mathrm{II}\equiv 0$. Hence (i) follows from the Gauss equation (1.22). Similarly, the Codazzi equation (1.23) implies (ii). For (iii) we use (1.25) and (1.7) to compute

$$\mathrm{D}_{\hat\alpha}\mathrm{D}_{\hat\beta}\hat A = \mathrm{D}_{\hat\alpha}\left(\frac{1}{2}\left(\frac{\beta}{\mu}\right)\hat A\right) = \frac{1}{2}\left\{\left(\frac{\delta}{\delta\hat\alpha}\left(\frac{\hat\beta}{\mu}\right)\right)\hat A + \left(\frac{\hat\beta}{\mu}\right)\mathrm{D}_{\hat\alpha}\hat A\right\}$$
$$= \frac{1}{2}\left\{-\left(\frac{\hat\alpha}{\mu}\right)\left(\frac{\hat\beta}{\mu}\right)\hat A + \frac{1}{2}\left(\frac{\hat\alpha}{\mu}\right)\left(\frac{\hat\beta}{\mu}\right)\hat A\right\},$$

from which

$$R^{\mathrm{D}}(\hat\alpha, \hat\beta)\hat A = \mathrm{D}_{\hat\alpha}\mathrm{D}_{\hat\beta}\hat A - \mathrm{D}_{\hat\beta}\mathrm{D}_{\hat\alpha}\hat A = 0.$$

Since R^{D} is tensorial, we conclude that $R^{\mathrm{D}}(\alpha,\beta)A = 0$ in general. The Ricci equation (1.24) then implies (iii). □

Next we investigate the traceless directions.

PROPOSITION 1.27. *Let notation be as in* (1.20), *with* $W = \{\mu\}\times\mathfrak{M}_\mu$ *and* $V = \mathfrak{M}$. *Then, at the point* $g\in\mathfrak{M}_\mu$,

(i) $\mathrm{II}(A,B) = -(n/8)\,\mathrm{tr}_g(AB)\mu$,
(ii) $\mathfrak{J}_{\hat\alpha}(A) = -\frac{1}{2}(\alpha/\mu)A$,
(iii) $\mathrm{D}_A\alpha = \delta\alpha/\delta A$, *and*
(iv) $\nabla_A^W\hat B = 0$,

where $\delta\alpha/\delta A$ *denotes derivative of the* $\Omega^n(M)$*-valued function* α (*viewed as a vector field normal to* W) *along a curve tangent to* A.

Proof. Again we use the six-term formula (1.6) together with (1.19), this time obtaining

$$2\langle \nabla_A \hat{B}, \hat{\alpha}\rangle = -\hat{\alpha}\langle \hat{A}, \hat{B}\rangle = -\int \operatorname{tr}_g(AB)\left(\frac{\alpha}{\mu}\right)\mu$$
$$= -\left\langle \frac{n}{4}(\operatorname{tr}_g(AB))\mu, \alpha\right\rangle,$$

from which (i) follows. Statement (ii) then follows from Weingarten, (1.21). For constant vector fields $\hat{\alpha}$, (1.6) implies $\langle \nabla_A \hat{\alpha}, \beta\rangle = 0$, and then (iii) follows easily for general α. Statement (iv) follows from the parenthetical sentence following (1.9). □

COROLLARY 1.28. *At* $(\mu, g) \in \operatorname{Vol}(M) \times \mathfrak{M}_\mu$,

(i) $\langle R(A,B)\alpha, \beta\rangle = 0$,
(ii) $\langle R(A,B)C, \alpha\rangle = 0$, *and*
(iii)

$$\langle R(A,B)C, \mathrm{D}\rangle = \int\left\{-\frac{1}{4}\operatorname{tr}_g([A,B][C,\mathrm{D}]) + \frac{n}{16}[\operatorname{tr}_g(AC)\operatorname{tr}_g(B\mathrm{D}) - \operatorname{tr}_g(A\mathrm{D})\operatorname{tr}_g(BC)]\right\}\mu.$$

Proof. Statement (i) follows from the Ricci equation (1.24) (also from (1.26.iii) and the symmetries of the Riemann tensor). Next, let $W = \{\mu\} \times \mathfrak{M}_\mu$ in (1.20)–(1.24). Since the integrand defining the inner product on $\mathfrak{M}_\mu$ is $SL(TM)$-invariant pointwise, (1.27.i) holds at all points of W even if we replace A, B by $\hat{A}, \hat{B}$ only on the left-hand side; that is, $\operatorname{tr}_g(\hat{A}\hat{B})$ is constant as a function of $g \in \mathfrak{M}_\mu$. Also, $\nabla^W_A \hat{B} = 0$ at g by (1.27.iv). Thus, at g,

$$\begin{aligned}(\mathbf{D}_A \mathrm{II})(B, C) &= \mathrm{D}_A(\mathrm{II}(\hat{B}, \hat{C})) - \mathrm{II}(\nabla^W_A \hat{B}, C) - \mathrm{II}(B, \nabla^W_A \hat{C})\\ &= \mathrm{D}_A(\text{constant } n\text{-form}) - 0 - 0\\ &= 0.\end{aligned}$$

Statement (ii) now follows from Codazzi, (1.23). For (iii) we apply (1.27.i) and the Gauss equation (1.22) to (1.15):

$$\begin{aligned}\langle R(A,B)C, D\rangle = &\int -\operatorname{tr}_g([[A,B],C]D)\mu\\ &+ \frac{4}{n}\int\left(-\frac{n}{8}\right)\operatorname{tr}_g(AC)\left(-\frac{n}{8}\right)\operatorname{tr}_g(BD)\mu\\ &+ \frac{4}{n}\int\left(-\frac{n}{8}\right)\operatorname{tr}_g(BC)\left(-\frac{n}{8}\right)\operatorname{tr}_g(AD)\mu,\end{aligned}$$

from which the desired formula is immediate. □

There is one component of the curvature not covered by (1.26) or (1.28)—namely, the mixed component $\langle R(\alpha, A)\beta, B\rangle$. This can be determined neither by applying symmetries of the Riemann tensor to components already discussed nor from the submanifold equations (1.22)-(1.24). However, the map $p \circ i_\mu : \mathrm{Vol}(M) \times \mathfrak{M}_\mu \to \mathrm{Vol}(M)$ is a Riemannian submersion; that is, $(p \circ i_\mu)_*$ is surjective and preserves the metric on "horizontal" vectors (the orthogonal complement to the fiber tangent spaces ["vertical spaces"]; in our case, those vectors tangent to $\mathrm{Vol}(M) \times \{\text{fixed metric}\}$). This allows us to apply a formula of O'Neill [5, Thm. 3]. The result is the following.

PROPOSITION 1.29. *On* $\mathrm{Met}(M)$, $\langle R(\alpha, A)\beta, B\rangle = 0$.

Proof. O'Neill defines tensor fields τ, σ by

(1.30a) $$\tau_X Y = hor(\nabla_{\mathrm{vert}(X)}\, vert(Y)) + vert(\nabla_{\mathrm{vert}(X)}\, hor(Y)),$$

(1.30b) $$\sigma_X Y = vert(\nabla_{\mathrm{hor}(X)}\, hor(Y)) + hor(\nabla_{\mathrm{hor}(X)}\, vert(Y)),$$

and proves that if X, Y are horizontal and V, W vertical then

(1.31) $$\begin{aligned}\langle R(X,V)Y, W\rangle &= \langle \sigma_X V, \sigma_Y W\rangle - \langle \tau_V X, \tau_W Y\rangle \\ &\quad + \langle (\nabla_X \tau)_V W, Y\rangle + \langle (\nabla_V \sigma)_X Y, W\rangle.\end{aligned}$$

In our situation, if we regard some fixed $\{\mu\} \times \mathfrak{M}_\mu$ as the submanifold defining II and $\mathfrak{J}$, then

$$\tau_X Y = \mathrm{II}(vert(X), vert(Y)) - \mathfrak{J}_{hor(Y)}(vert(X)).$$

If we regard a fixed $\mathrm{Vol}(M) \times \{g\}$, $g \in \mathfrak{M}_\mu$, as the submanifold defining II and $\mathfrak{J}$, then

$$\sigma_X Y = \mathrm{II}(hor(X), hor(Y)) - \mathfrak{J}_{vert(Y)}(hor(X)).$$

Thus, from (1.27) we have

$$\tau_{\alpha+A}(\beta+B)\,|_{(\mu,g)} = -\frac{n}{8}\operatorname{tr}_g(AB) + \frac{1}{2}\left(\frac{\beta}{\mu}\right)A$$

and, from (1.25),

$$\sigma_{\alpha+A}(\beta+B) = 0.$$

Hence

(1.32) $$\langle R(\alpha, A)\beta, B\rangle = -\langle \mathfrak{J}_\alpha(A), \mathfrak{J}_\beta(B)\rangle + \langle (\nabla_\alpha \tau)_A B, \beta\rangle.$$

Defining $\mathrm{II}(X, Y) = \mathrm{II}(vert(X), vert(Y))$ for X, Y not vertical, we have

$$\begin{aligned}(\nabla_\alpha \tau)_A B &= \nabla_\alpha(\tau_A B) - \tau_{\nabla_\alpha A} B - \tau_A(\nabla_\alpha B) \\ &= \nabla_\alpha(\mathrm{II}(A,B)) - \mathrm{II}(\mathrm{D}_\alpha A, B) - \mathrm{II}(A, \mathrm{D}_\alpha B) \\ &= -\frac{n}{8}\left\{\nabla_\alpha(\operatorname{tr}_g(AB)\mu) - \operatorname{tr}_g\left(\frac{1}{2}\left(\frac{\alpha}{\mu}\right)AB\right)\mu\right. \\ &\qquad \left. -\operatorname{tr}_g\left(A\left(\frac{1}{2}\left(\frac{\alpha}{\mu}\right)B\right)\right)\mu\right\}\end{aligned}$$

(by (1.27.i) and (1.25.iv))

$$= -\frac{n}{8}\left\{\mathrm{tr}_g(AB)\left(\alpha - \frac{1}{2}\left(\frac{\alpha}{\mu}\right)\mu\right) - \mathrm{tr}_g(AB)\left(\frac{\alpha}{\mu}\right)\right\}$$

$$= \frac{n}{16}\,\mathrm{tr}(AB)\,\alpha.$$

Since $\mathfrak{J}_\alpha(A) = -\frac{1}{2}(\alpha/\mu)A$, (1.32) therefore implies

$$\begin{aligned}\langle R(\alpha, A)\beta, B\rangle &= -\left\langle \frac{1}{2}\left(\frac{\alpha}{\mu}\right)A, \frac{1}{2}\left(\frac{\beta}{\mu}\right)B\right\rangle + \left\langle \frac{n}{16}\,\mathrm{tr}_g(AB)\,\alpha, \beta\right\rangle \\ &= -\frac{1}{4}\int\left(\frac{\alpha}{\mu}\right)\left(\frac{\beta}{\mu}\right)\mathrm{tr}_g(AB)\,\mu \\ &\quad + \left(\frac{n}{16}\right)\left(\frac{4}{n}\right)\int\left(\frac{\alpha}{\mu}\right)\left(\frac{\beta}{\mu}\right)\mathrm{tr}_g(AB)\,\mu \quad \text{(by (1.19))} \\ &= 0. \end{aligned}$$

□

Proof of Theorem 1.16. The result now follows from (1.26), (1.28), and (1.29). □

2. Geodesics on Met(M)

Throughout this section we fix $\mu \in \mathrm{Vol}(M)$ and implicitly identify $\mathrm{Met}(M)$ with $\mathrm{Vol}(M) \times \mathrm{Met}_\mu(M)$ as in (1.3). Also, for

$$A = A^i_j(\partial/\partial x^i) \otimes dx^j \in \Gamma(\mathrm{End}(TM))$$

and

$$h = h_{ij}\,dx^i \otimes dx^j \in \Gamma(T^*M \otimes T^*M),$$

we define

$$hA \in \Gamma(T^*M \otimes T^*M)$$

by the local formula $(hA)_{ij} = h_{ik}A^k{}_j$. Similarly, for $B \in \Gamma(T^*M \otimes T^*M)$ and $g \in \mathrm{Met}(M)$, we define $g^{-1}B \in \Gamma(\mathrm{End}(TM))$ by $(g^{-1}B)^i{}_j = g^{ik}B_{kj}$. Observe that for $h, g \in \mathrm{Met}_\mu(M)$, $h \mapsto g^{-1}h$ is simply the identification of h with an element of *SPosSym*(g) (i.e., $g^{-1}h$ is what we called h in §1).

In this section we solve the geodesic equation on $\mathrm{Met}(M)$. Since the formulas on each piece ($\mathrm{Vol}(M)$ and $\mathrm{Met}_\mu(M)$) of our splitting may be of some independent interest, we state these special cases first, although the following is a corollary of our general formula.

PROPOSITION 2.1. *The geodesic in* $\mathrm{Vol}(M)$ *with initial position* μ *and initial velocity* $\alpha \in T_\mu(\mathrm{Vol}(M)) = \Omega^n(M)$ *is*

$$\mu(t) = \mu_t = \left(1 + \frac{1}{2}\left(\frac{\alpha}{\mu}\right)t\right)^2 \mu.$$

Notice that the total integral of μ changes in time, which is expected since, by (1.33), $\mathrm{Vol}_\lambda(M)$ is not totally geodesic. Also, if $(\alpha/\mu)<0$ at some point of M then the geodesic $\mu(t)$ can only be continued for finite time. This is also as expected, since $\Gamma(\Lambda^n T^*M\,|_{\text{point}})$ is just $\mathbf{R}^+$ and, if one starts anywhere in $\mathbf{R}^+$ and heads towards zero, one hits the origin in finite time.

PROPOSITION 2.2. *The geodesic in* $\mathrm{Met}_\mu(M)$ *with initial position g and initial velocity* $A\in T_g(\mathrm{Met}_\mu(M))=Sym_0(M,g)$ *is*

$$g(t)=g_t=ge^{t(g^{-1}A)}.$$

Moreover, $\mathrm{Met}_\mu(M)$ *is geodesically complete.*

Proof. Since $\mathrm{Met}_\mu(M)$ is a symmetric space, the geodesic formula in (2.2) is more or less automatic [cf. the parenthetic sentence following (1.9)]. Completeness follows immediately from this formula. □

Since $\mathrm{Met}_\mu(M)$ is not a totally geodesic submanifold of $\mathrm{Met}(M)$, one expects some "mixing" between (2.1) and (2.2) for geodesics in $\mathrm{Met}(M)$. This is indeed the case.

THEOREM 2.3. *The geodesic in* $\mathrm{Met}(M)=\mathrm{Vol}(M)\times\mathrm{Met}_\mu(M)$ *with initial position* (μ,g) *and initial velocity* $(\alpha,A)\in\Omega^n(M)\times Sym_0(M,g)$ *is*

$$\text{(a)}\qquad g(t)=i_\mu(g_t)=(q(t)^2+r^2t^2)^{2/n}g\exp\left(\frac{\tan^{-1}(rt/q)}{r}g^{-1}A\right),$$

where $q(t)=1+\frac{1}{2}(\alpha/\mu)t$, $r=\frac{1}{4}(n\,\mathrm{tr}((g^{-1}A)^2))^{1/2}$, *and* i_μ *is as in* (1.3). (*If* $r=0$, *replace the exponential term by* 1.) *The change in the volume form of $g(t)$ is given by the formula*

$$\text{(b)}\qquad \mu(g(t))=(q(t)^2+r^2t^2)\mu.$$

While the results of Section 1 may be used to prove this theorem, we give the direct proof below to avoid some technical points required by that approach.

Proof of (2.3). The six-term formula (1.6), applied to constant vector fields B,C,E on $\mathrm{Met}(M)$, quickly leads to

$$\nabla_B C\,|_g=-\tfrac{1}{2}(Bg^{-1}C+Cg^{-1}B)+\tfrac{1}{4}\{(\mathrm{tr}_g(C))B+(\mathrm{tr}_g(B))C-\mathrm{tr}_g(BC)g\}.$$

Now let $\{g(t)\}$ be the geodesic and let $B=g'=dg/dt$. Then the above formula, applied to the geodesic equation $\nabla_{g'}g'=0$, yields

$$B'-Bg^{-1}B+(1/2)(\mathrm{tr}_g(B))B-(1/4)\,\mathrm{tr}_g(BB)g=0.$$

Setting $C=g^{-1}B\in\Gamma(\mathrm{End}(TM))$ and left-multiplying the above line by g^{-1}, we get the simpler equation

$$\text{(2.4)}\qquad C'+(1/2)(\mathrm{tr}(C))C+(1/4)(\mathrm{tr}(C^2))I=0.$$

(Here I is the identity endomorphism of TM.) Let $f=\operatorname{tr}(C)$, $E=C-(f/n)I$, and $v=\operatorname{tr}(E^2)$. Separating (2.4) into its trace and traceless components, we obtain the coupled equations

$$f'+(1/4)f^2-(n/4)v=0. \tag{2.5a}$$

$$E'+(1/2)fE=0. \tag{2.5b}$$

It is more enlightening to write these equations in terms of quantities directly connected to the splitting (1.3). Write $g_t=(\mu_t,h_t)\in \operatorname{Vol}(M)\times \operatorname{Met}_\mu(M)$. Let $p(t)=(\mu_t/\mu)$. Then, using (1.3), we find $B_t=(2/n)(p'/p)g_t+g_th_t^{-1}h_t'$. Therefore $C_t=(2/n)(p'/p)I+h^{-1}h'$, $f=2p'/p$, and $E=h^{-1}h'$. We rewrite (2.5) as the system

$$u=p'/p, \tag{2.6a}$$

$$E=h^{-1}h', \tag{2.6b}$$

$$v=\operatorname{tr}(E^2), \tag{2.6c}$$

$$u'+(1/2)u^2=(n/8)v, \tag{2.6d}$$

$$E'=-uE. \tag{2.6e}$$

Differentiating (2.6c) and using (2.6e), we obtain

$$v'=2\operatorname{tr}(EE')=-2uv. \tag{2.7}$$

Differentiating (2.6d) and plugging in (2.7), (2.6d), and (2.6a) then gives

$$\begin{aligned}0&=u''+uu'+u(n/4)v\\&=u''+uu'+2u(u'+u^2/2)\\&=u''+3uu'+u^3\\&=p'''/p.\end{aligned}$$

Therefore $p'''=0$, so $p=a+bt+ct^2$ for some functions a,b,c on M. The initial conditions $p(0)=1$, $\mu'(0)=\alpha$, and $h'(0)=A$ lead, with the aid of (2.6), to the equalities $p'(0)=(\alpha/\mu)$ and $p''(0)=(1/2)(\alpha/\mu)^2+2r^2$, where r is as in the statement of the theorem. It follows that $a=1$, $b=(\alpha/\mu)$, $c=(1/4)(\alpha/\mu)^2+r^2$, and therefore

$$\begin{aligned}p(t)&=1+\left(\frac{\alpha}{\mu}\right)t+\left(\frac{1}{4}\left(\frac{\alpha}{\mu}\right)^2+r^2\right)t^2\\&=q(t)^2+r^2t^2.\end{aligned} \tag{2.8}$$

Next, from (2.6e) and (2.6a), we have $E'+(\log p)'E=0$, implying that $E(t)=E(0)/p(t)=g^{-1}A/p(t)$. It is straightforward (albeit tedious) to integrate the resulting equation $(\log(g^{-1}h))'=h^{-1}h'=g^{-1}A/(q(t)^2+r^2t^2)$ (which one may view as a local equation for matrices), finding

$$g^{-1}h_t=\exp\left(\frac{\tan^{-1}(rt/q)}{r}g^{-1}A\right)\quad \text{if } r\neq 0; \tag{2.9a}$$

(2.9b) $\qquad g^{-1}h_t = I, \quad \text{if } r=0$ (equivalently, if $A=0$).

Part (a) of the theorem now follows from (2.8), (2.9), and (1.3). For part (b), simply observe that $\det(\exp(\lambda g^{-1}A)) = \exp(\operatorname{tr}(\lambda g^{-1}A)) = 1$, because (by hypothesis) $0 = \operatorname{tr}_g(A) = \operatorname{tr}(g^{-1}A)$. □

REMARK. A direct consequence of equation (2.6e) in the proof above is that the change in the traceless part of g along a geodesic is *abelian*—the commutator of E and E' is zero. This is reflected in (2.3a); the traceless part of g varies along the one-parameter subgroup $\{\exp(sg^{-1}A)\}$.

One can deduce some elementary aspects of the qualitative behavior of geodesics from (2.1)–(2.3). First, as noted above, geodesics in $\mathrm{Vol}(M)$ cannot be continued beyond $t_{\text{crit}} = \inf\{-2(\mu(x)/\alpha(x)) \mid \alpha(x) < 0\}$. The same is true of a geodesic in $\mathrm{Met}(M)$ with $A \equiv 0$, since each $\mathrm{Vol}(M) \times \{g\}$ is totally geodesic. However, if A is nowhere zero then a geodesic in $\mathrm{Met}(M)$ runs forever, since the r in (2.3a) is everywhere positive. If, in addition, α is nowhere zero, then as $t \to \infty$ the projection of the geodesic onto $\mathrm{Met}_\mu(M)$ approaches a limiting value, namely

$$g \exp\left\{\frac{1}{r} \tan^{-1}\left(\frac{2r}{\alpha/\mu}\right) g^{-1}A\right\}.$$

We also remark that (2.1) implies that distinct geodesics in $\mathrm{Vol}(M)$ emanating from the same point never intersect (roughly, because the two distinct geodesics from a point in $\mathbf{R}^+$ don't intersect). The same holds for geodesics in $\mathrm{Met}_\mu(M)$, again reflecting the behavior of the associated finite-dimensional (and negatively curved) space $SL(n)/SO(n)$.

3. The Quotient $\mathrm{Met}'(M)/\mathrm{Diff}^+(M)$

$\mathrm{Met}'(M)/\mathrm{Diff}^+(M)$ is an example of what one may call a "principal Riemannian submersion"—a Riemannian submersion $\pi: P \to X$, defined as in Section 1, where in addition P is a principal bundle with (say) group G, and the G-action is isometric. [Fischer and Marsden [3] use the term "(weak-) Riemannian principal fiber bundle." "Weak" applies to infinite-dimensional cases of the sort we are considering, because spaces of C^∞ sections are not complete with respect to L^2 metrics.] A familiar example of this is the quotient of a Lie group with a right-invariant metric by a closed subgroup. The gauge-theorists' fibration {irreducible connections on some principal bundle}/{gauge transformations} is an infinite-dimensional example. It is not hard to find a general formula relating the curvatures of P and M in the general case. Let $\tilde{R}, R$ be the Riemann curvature tensors of P, M, respectively. Let $\tilde{Y}, \tilde{Z}$ be horizontal vector fields on P (not necessarily G-invariant). Let $p \in P$, $x = \pi(p)$, $Y = \pi_*(\tilde{Y}_p)$, and $Z = \pi_*(\tilde{Z}_p)$. The general formula (see O'Neill [5] and Cheeger and Ebin [6]) then states

(3.1) $\qquad (R(Y,Z)Z,Y)\,|_x = [(\tilde{R}(\tilde{Y},\tilde{Z})\tilde{Z},\tilde{Y}) + \tfrac{3}{4}|\mathrm{vert}\{\tilde{Y},\tilde{Z}]|^2]_p.$

The last term in this formula is related to the curvature of the principal connection defined by the horizontal distribution. If $\iota_p: \mathfrak{g} \to V_p$ is the canonical isomorphism from the Lie algebra of G to the vertical space $V_p \subset T_p P$, then this curvature F, viewed as an $\operatorname{Ad} G$-equivariant $\mathfrak{g}$-valued 2-form on P, is given by

$$F(U, V)|_p = -\iota_p^{-1}(\mathit{vert}[\mathit{hor}(U), \mathit{hor}(V)]), \tag{3.2}$$

where U and V are extended arbitrarily to local vector fields on P, and of course "*vert*" and "*hor*" are the orthogonal projections onto the vertical and horizontal spaces, respectively. Thus, in (3.1), $|\mathit{vert}[\tilde{Y}, \tilde{Z}]|_p^2 = |\iota_p F(\tilde{Y}, \tilde{Z})|^2$.

In our situation, $\mathfrak{D}$ acts freely and isometrically on $\mathfrak{M}'$, and, as we will see below, it is not hard to write explicitly the associated splitting of $T\mathfrak{M}'$ into horizontal and vertical. A metric on $\mathfrak{M}'/\mathfrak{D}$ is given by restricting the inner product (1.1) to the horizontal distribution. With these data, $\pi: \mathfrak{M}' \to \mathfrak{M}'/\mathfrak{D}$ becomes an infinite-dimensional principal Riemannian submersion. To compute the curvature of the quotient, we need to exhibit the maps ι, *vert*, and *hor*.

The Lie algebra of $\mathfrak{D}$ is $\mathfrak{X}(M)$, the space of vector fields on M. Since M is compact, each $X \in \mathfrak{X}(M)$ generates a one-parameter subgroup $\{\exp(tX)\}$ of $\mathfrak{D}$, and the tangent vector to this curve in $\mathfrak{D}$ at $t = 0$ is X. Hence, for $g \in \mathfrak{M}'$,

$$\begin{aligned} V_g &= \left\{ \frac{dt}{d} (\exp(tX)^* g)_{t=0} \,|\, X \in \mathfrak{X}(M) \right\} \\ &= \{ L_X g \,|\, X \in \mathfrak{X}(M) \}, \end{aligned} \tag{3.3}$$

where L denotes Lie derivative. The map $\iota_g: \mathfrak{X}(M) \to V_g$ is simply

$$\iota_g(X) = L_X g. \tag{3.4}$$

Since

$$\begin{aligned} (L_X g)(Y, Z) &= X(g(Y, Z)) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= g(\nabla_X Y - [X, Y], Z) + g(Y, \nabla_X Z - [X, Z]) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X), \end{aligned}$$

ι_g can be expressed in terms of the Levi-Civita connection ∇ of g as the following composition:

$$\begin{array}{ccccccc} \iota_g: \mathfrak{X}(M) & \xrightarrow{\nabla} & \Gamma(TM \otimes T^*M) & \xrightarrow{\text{index-raise}} & \Gamma(T^*M \otimes T^*M) & \xrightarrow{\text{symmetrize}} & \Gamma(S^2 T^*M) \\ X & \mapsto & \nabla X & \mapsto & g^{-1}(\nabla X) & \mapsto & (1+\tau) g^{-1} \nabla X, \end{array}$$

where τ denotes transpose and g^{-1} has the same meaning as in Section 2.

It is useful to express this in local coordinates. If $X = X^i (\partial/\partial x^i)$ is a local vector field, we write $\nabla X = (X^i{}_{;j})(\partial/\partial x^i) \otimes dx^j$, and similarly use semicolon notation for covariant derivative of any tensor field. Also, we will lower indices on tensors, just as on vectors, using the matrix $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$; indices will be raised using g^{ij}. With this notation in mind, the analysis above yields

$$\iota_g(X) = (X_{i;j} + X_{j;i})dx^i \otimes dx^j. \tag{3.5}$$

Next we identify the horizontal spaces. For $C \in T_g \mathfrak{M}' = \Gamma(S^2T^*M)$, we have

$$\begin{aligned}\langle L_X g, C\rangle|_g &= \int (X_{i;j} + X_{j;i})C^{ij}\mu(g)\\ &= \int (X_{i;j} + X_{j;i})C^{ij}\mu(g)\\ &= -2\int X_i C^{ij}{}_{;j}\,\mu(g)\end{aligned}$$

by Stokes' theorem. Giving $\mathfrak{X}(M)$ the L^2 metric induced by g, analogously to (1.1), the map $\iota_g \colon \mathfrak{X}(M) \to T_g\mathfrak{M}'$ acquires a formal adjoint $(\iota^*)_g$, and the computation above shows that

$$(\iota^*)_g(C_{ij}\,dx^i \otimes dx^j) = -2C^{ij}{}_{;j}\frac{\partial}{\partial x^i}. \tag{3.6}$$

(It is important to remember that the "$*$" in "ι^*" depends on g, so we have until now avoided the simpler notation ι_g^*. The reader now having been warned, we henceforth adopt the simpler notation.)

For each $g \in \mathfrak{M}'$, $\ker_g \iota = 0$, since otherwise (M, g) would have at least a one-parameter group of isometries. The operator $\iota_g^*\iota_g$ is elliptic, and hence $(\iota_g^*\iota_g)^{-1}$ maps C^∞ vector fields to C^∞ vector fields, continuously with respect to the L^2 norm on $\mathfrak{X}(M)$. Therefore $\mathfrak{X}(M) = \operatorname{Im}(\iota_g) \oplus \ker(\iota_g^*)$, orthogonally and continuously with respect to the L^2 inner product. This fact justifies many of the otherwise formal statements below, in which the vertical and horizontal subspaces of $T\mathfrak{M}'$ are treated essentially as if they were finite-dimensional.

For any rank-2 tensor field T, let us define $(div_2 T)(g) = T^{ij}{}_{;j}(\partial/\partial x^i)$, where the covariant derivative is the one determined by g. The orthogonal complement H_g of V_g is simply $\ker(\iota_g^*)$, so by (3.6)

$$H_g = \{B \in \Gamma(S^2T^*M) \mid (div_2 B)(g) = 0\}. \tag{3.7}$$

Looking at (3.1), we see that we need to exhibit some horizontal vector fields and compute their brackets. Given $B \in H_g$, define a horizontal extension $\tilde{B}$ of B to a neighborhood of $g \in \mathfrak{M}'$ by $\tilde{B}(g+A) = hor_{g+A}(B) \in H_{g+A}$; this is defined for small A since $\mathfrak{M}'$ is an open subset of $\mathfrak{M}$ (see [1]). The Lie bracket of two such extensions $\tilde{B}, \tilde{C}$ is given by

$$[\tilde{B}, \tilde{C}]_g = \left(\frac{d}{dt}\tilde{C}_{g+tB} - \frac{d}{dt}\tilde{B}_{g+tC}\right)\bigg|_{t=0}. \tag{3.8}$$

To compute the first term, remember that $\iota_g^* C_g = 0$; hence

$$\begin{aligned}\frac{d}{dt}\tilde{C}_{g+tB}\bigg|_{t=0} &= -\frac{d}{dt}(vert_{g+tB}(\tilde{C}_g))\bigg|_{t=0}\\ &= -\iota_g(\iota_g^*\iota_g)^{-1}\frac{d}{dt}(\iota_{g+tB}^*\tilde{C}_g)\bigg|_{t=0}.\end{aligned} \tag{3.9}$$

In local coordinates, the Christoffel symbols of $g+A$ are related to those of g by

$$(\Gamma(g+A))^i{}_{jk}=(\Gamma(g))^i{}_{jk}+\tfrac{1}{2}((g+A)^{-1})^{il}(A_{lj;k}+A_{lk;j}+A_{jk;l})$$

where ";" is with respect to g and the matrix $(g+A)^{-1}$ is the inverse of $\{g_{ij}+A_{ij}\}$. Substituting $A=tB$, using $C^{ij}{}_{;k}=(\partial C^{ij}/\partial x^k)+C^{aj}\Gamma^i{}_{ak}+C^{ia}\Gamma^j{}_{ak}$, and remembering to raise the indices of the symmetric tensor $C_{ij}\,dx^i\otimes dx^j$ with the metric $(g+tB)$ *before* taking d/dt, one finds (eventually) that

$$\begin{aligned}\frac{d}{dt}C^{ij}{}_{;k}\Big|_{t=0}&=\frac{1}{2}\{B^i{}_{a;k}+B^i{}_{k;a}-B_{ka;}{}^i\}^{aj}\\&\quad+\frac{1}{2}\{B^j{}_{a;k}+B^j{}_{k;a}-B_{ka;}{}^j\}C^{ai}-(B^{ia}C_a^j+C^{ia}B_a^j)_{;k},\end{aligned}$$

where all indices on the right are raised with g. Inserting this expression into (3.6), using (3.9) yields

$$\begin{aligned}&\frac{d}{dt}\tilde{C}_{g+tB}\Big|_{t=0}\\&=\iota(\iota^*\iota)^{-1}\left\{[(2B^i{}_{a;j}-B_{ja;}{}^i)C^{aj}+B^j{}_{j;a}C^{ai}-2(BC+CB)^{ij}{}_{;j}]\frac{\partial}{\partial x^i}\right\},\end{aligned}$$

where $\iota=\iota_g$ and $(BC)^{ij}=B^{ia}C_a^j$. Using $\iota^*B=\iota^*C=0$, the expression in square brackets simplifies to

$$-B_{ja;}{}^iC^{aj}+\mathrm{tr}_g(B)_{;a}C^{ai}-2(CB)^{ij}{}_{;j}.$$

Let $[B,C]$ denote the algebraic commutator $BC-CB$ (not to be confused with the Lie bracket $[\tilde{B},\tilde{C}]$), let $b=\mathrm{tr}_g(B)$, and let $c=\mathrm{tr}_g(C)$. Then, putting the last few equations together,

$$\frac{d}{dt}\tilde{C}_{g+tB}\Big|_{t=0}=\iota(\iota^*\iota)^{-1}\left([-B_{jk;}{}^iC^{kj}+b_{;k}C^{ki}-2(CB)^{ij}{}_{;j}]\frac{\partial}{\partial x^i}\right)$$

and

$$[\tilde{B},\tilde{C}]\,|_{t=0}=\Lambda(B,C)=\iota(\iota^*\iota)^{-1}\mathrm{P}(B,C), \tag{3.10a}$$

where

$$\begin{aligned}\mathrm{P}(B,C)=&\{(b_{;k}C^{ki}-c_{;k}B^{ki})+(B^{jk}C_{jk;}{}^i-C^{jk}B_{jk;}{}^i)\}\frac{\partial}{\partial x^i}\\&+2\,div_2([B,C],g).\end{aligned} \tag{3.10b}$$

Conveniently, $\Lambda(B,C)$ is already vertical, and we may plug directly into (3.1) to find the sectional curvatures of $\mathfrak{M}'/\mathfrak{D}$ in terms of those of $\mathfrak{M}'$. Thus

$$(R(B,C)C,B)\,|_{\pi(g)}=[(\tilde{R}(\tilde{B},\tilde{C})\tilde{C},\tilde{B})+\tfrac{3}{4}\|\Lambda(B,C)\|_{L^2}^2]\,|_g, \tag{3.11}$$

where we have abused notation by writing B, C on the left for π_*B, π_*C.

Recall that the sectional curvatures determine the full Riemann tensor by polarization. Polarizing (3.11), one obtains

$$(3.12)\qquad \begin{aligned}(R(A,B)C,D) = (\tilde{R}(A,B)C,D) + \tfrac{1}{3}\langle \Lambda(A,D), \Lambda(B,C)\rangle_{L^2} \\ -\tfrac{1}{3}\langle \Lambda(A,C), \Lambda(B,D)\rangle_{L^2} - \tfrac{2}{3}\langle \Lambda(A,B), \Lambda(C,D)\rangle_{L^2}.\end{aligned}$$

Summarizing, we have the following.

PROPOSITION 3.1. *Equations* (3.12) *and* (3.10), *together with Corollary* (1.28.iii), *constitute a formula for the Riemannian curvature of* $\mathfrak{M}'/\mathfrak{D}$.

REMARKS. (i) There is a minor simplification that occurs when one inserts (3.10a) into (3.11) or (3.12); namely,

$$\langle \Lambda(A,B), \Lambda(C,D)\rangle_{L^2} = \langle \mathrm{P}(A,B), (\iota_g^* \iota_g)^{-1}\mathrm{P}(C,D)\rangle_{L^2}.$$

(ii) The expression $-(\iota_g^*\iota_g)^{-1}\mathrm{P}(B,C)$ is precisely the curvature F of (3.2). To see this, observe that $(\iota^*\iota)^{-1}\iota^*|_{\mathrm{Im}(\iota)}$ plays the role played by ι^{-1} in (3.2). Hence

$$\begin{aligned} F(B,C) &= (\iota^*\iota)^{-1}\iota^*\Lambda(B,C) \\ &= -(\iota^*\iota)^{-1}\mathrm{P}(B,C). \end{aligned}$$

(iii) $\mathrm{P}(B,C)$ can be written purely in terms of the traceless parts of B and C. For if $\mathring{B} = B - (b/n)g$ and $\mathring{C} = C - (c/n)g$, then one finds

$$\mathrm{P}(B,C) = (b_{;k}\mathring{C}^k{}_i - c_{;k}\mathring{B}^k{}_i)\frac{\partial}{\partial x^i} + 2\,div_2([\mathring{B},\mathring{C}],g)$$

$$+ ((\mathring{B})^{jk}(\mathring{C})_{jk;}{}^i - (\mathring{C})^{jk}(\mathring{B})_{jk;}{}^i)\frac{\partial}{\partial x^i}.$$

One can then eliminate b and c from this expression, since

$$\begin{aligned} 0 = B^{ij}{}_{;j} &= \left(\mathring{B} + \frac{b}{n}g\right)^{ij}{}_{;j} \\ &= \mathring{B}^{ij}{}_{;j} + \frac{1}{n}b_;{}^i, \end{aligned}$$

implying

$$b_{;k} = -n\mathring{B}_{kj;}{}^j, \qquad c_{;k} = -n\mathring{C}_{kj;}{}^j.$$

It is not clear, however, whether this is a useful simplification.

Finally, we consider geodesics on $\mathfrak{M}'/\mathfrak{D}$. Given any Riemannian submersion $\pi: P \to M$, the horizontal lift of a geodesic is a geodesic (see [6]). Since geodesics are determined uniquely by initial conditions, it follows that any geodesic on P which is anywhere horizontal projects to a geodesic on M and is in fact everywhere horizontal.

In our situation, therefore, the geodesics on $\mathfrak{M}'/\mathfrak{D}$ are simply the π-images of those curves in Theorem 2.3 with horizontal initial vectors. In the notation of that theorem, (α, A) is horizontal at (μ, g) if and only if $A^{ij}{}_{;j} + (2/n)(\alpha/\mu)_;{}^i = 0$. Hence we have the following.

COROLLARY 3.2. *The geodesics on $\mathfrak{M}'/\mathfrak{D}$ are of the form $\pi(g(t))$, where $g(t)$ is as in Theorem* (2.3a) *and, in the notation of* (2.3),

$$grad\left(\frac{\alpha}{\mu}\right) = -\frac{n}{2}(div_2 A)$$

(*covariant derivatives being taken with respect to g*).

REMARK. It is possible for a geodesic in $\mathfrak{M}$ to start in $\mathfrak{M}'$ and run for infinite time in $\mathfrak{M}$, but to leave $\mathfrak{M}'$ (by reaching a metric with nontrivial isometry group) in finite time. Thus in general the geodesics of $\mathfrak{M}'/\mathfrak{D}$ are projections only of geodesic *segments* in $\mathfrak{M}$.

Appendix. L^2 Metrics on Mapping Spaces

In this appendix we demonstrate how the basic Riemannian geometry of L^2 metrics on mapping spaces reduces to the (finite-dimensional) Riemannian geometry of the target. We need to deal with the more general situation of fiber bundles. Thus let $S \to M$ be a smooth fibering of manifolds. We denote the fiber over $x \in M$ by N_x. Let g be a Riemannian metric along the fibers, that is, a metric on the vertical tangent bundle VTS. Further suppose that M comes equipped with a measure μ.

Our basic mapping space is the space $\mathfrak{M}$ of smooth sections of $S \to M$. (For the trivial fibration $M \times N \to M$, the space $\mathfrak{M}$ is simply $\mathrm{Maps}(M, N)$.) Suppose that $\phi \in \mathfrak{M}$ is such a smooth section. Then the tangent space to $\mathfrak{M}$ at ϕ may be naturally identified with the space of vertical vector fields along the image of ϕ (more precisely, with the space of sections of the pulled-back bundle $\phi^* VTS$). There are special vector fields we can consider in a neighborhood of ϕ. Specifically, let A be a vertical vector field defined on a neighborhood in S of the image of ϕ. Then for any ψ close enough to ϕ the restriction of A to the image of ψ is a tangent vector at ψ. Suppose we are given two such vector fields A and B.

LEMMA A.1. *The bracket $[A, B]$ in $\mathfrak{M}$ is given by the pointwise bracket in the fibers: $[A, B](x) = [A|_{N_x}, B|_{N_x}](\psi(x))$.*

The proof follows easily from the observation that the flows of $\{A|_{N_x}\}_{x \in M}$ combine to give the flow of A in $\mathfrak{M}$. We omit the details.

The L^2 metric on $\mathfrak{M}$ is defined by integrating the vertical metric g over M using the measure μ. Specifically, suppose A and B are tangent vectors to $\mathfrak{M}$. Then

$$\langle A, B \rangle = \int_M g(A(x), B(x))\, \mu(x). \tag{A.2}$$

From this and the six-term formula (1.6), the "Christoffel symbols" $\langle \nabla_A B, C \rangle$ of the Levi-Civita connection of g are easily determined. We leave it for the reader to verify the following proposition.

PROPOSITION A.3. *Let $\phi(t)$ be a curve in $\mathfrak{M}$ and $A(t)$ a vectorfield along this curve. For each $x \in M$, let ϕ_x denote the induced curve in the fiber N_x, with A_x the induced vector field along ϕ_x. Then, for all x and t,*

$$\frac{DA}{dt}(x) = \frac{DA_x}{dt},$$

where the left-hand side refers to the covariant derivative on $\mathfrak{M}$ and the right-hand side to the covariant derivative on N_x.

There are two simple corollaries of this formula, which state that geodesics and curvature on $\mathfrak{M}$ can be treated fiberwise on S.

COROLLARY A.4. *A curve $\phi(t)$ is a geodesic if and only if $\phi_x(t)$ is a geodesic in N_x for each x.*

COROLLARY A.5. *The curvature operator on $\mathfrak{M}$ is given pointwise: $R(A,B)(x) = R_x(A_x, B_x)$.*

Our basic space $\mathrm{Met}(M)$ of this paper is the space of sections of the bundle $GL^+(TM)/SO(TM, g_0)$, where $GL^+(TM)$ is the bundle of frames consistent with the orientation and g_0 is any metric on M. The fiber N_x is the space of endomorphisms of T_xM which are self-adjoint with respect to g_0, and is diffeomorphic to $GL^+(n)/SO(n)$. For any other metric h there is a unique (as well as self-adjoint and positive-definite) $h \in \Gamma(\mathrm{End}(TM))$ satisfying $h(A,B) = g_0(h(A), B)$ [cf. the discussion following the proof of Proposition (1.5)]. The metric (1.1) can be written

$$\langle A, B\rangle_h = \int_M \mathrm{tr}_h(AB)\det(h)^{1/2}\mu_0,$$

where $\mu_0 = \mu(g_0)$. Define a vertical metric on the bundle of positive-definite quadratic forms over M by

$$h(A,B) = \mathrm{tr}_h(AB)\det(h)^{1/2}. \tag{A.6}$$

Then formula (A.2) determines an L^2 metric on $\mathrm{Met}(M)$, and Corollaries (A.4) and (A.5) apply. The computation in Section 1 may be viewed as giving the curvature of $GL^+(n)/SO(n)$ in the metric (A.6), where A, B, h are to be viewed simply as symmetric matrices and $\boldsymbol{h} = h$ (i.e., $(g_0)_{ij} = \delta_{ij}$).

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