HAUSDORFF DIMENSION AND APPROXIMATION OF SMOOTH FUNCTIONS

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We prove two theorems about differentiable transformations of sets of a specified Hausdorff dimension; the first theorem concerns the dimension of certain intersections, and it complements a theorem of J. M. Marstrand [7, Theorem III, p. 275], while the second extends results of J.-P. Kahane and R. Salem (see [2, Chapter 15], [3, Chapter 8], and [9]) on the behavior at infinity of certain Fourier-Stieltjes transforms. In both cases, we demonstrate the existence of an extremal set in a specified class, by a combination of probability theory and quantitative approximation theory.

This paragraph contains the estimates necessary for approximation; the field is largely the creation of A. N. Kolmogorov, and the material is found in [6, Chapter 10] under the name "entropy". Let S be some collection of real-valued functions on an interval $[a, b]$ whose derivatives of order $0, 1, \cdots, k$ are uniformly bounded on $[a, b]$, for a positive integer k. For each $\varepsilon > 0$, we choose a set $S^* \subseteq S$ so that for each f in S there is an f^* in S with $|f(x) - f^*(x)| \leq \varepsilon$ throughout $a \leq x \leq b$. For small $\varepsilon > 0$, we can choose S^* so that its size $|S^*|$ satisfies an inequality of the form $\log|S^*| \leq C\varepsilon^{-1/k}$, where C depends on S but not on ε. This estimate is valid for fractional values of k, for which the analogue of C^k is defined as follows. We let $k_1 = [k]$, and we admit classes S bounded above in $C^{k_1}[a, b]$, imposing a Lipschitz condition with exponent $\alpha = k - k_1$ on the k_1-st derivative:

$$|f^{(k_1)}(x) - f^{(n_1)}(y)| \leq C|x - y|^\alpha \qquad (f \in S,\ a \leq x,\ y \leq b).$$

Before turning to the theorems, we point out two technical details that should be of interest to specialists. The first theorem involves not only probability and approximation, but also a function-space argument borrowed from Fourier analysis. To prove the second theorem, we need a somewhat difficult estimate of exponential integrals; but we use only elementary inequalities from probability, in contrast with [2, Chapter 15].

1. To explain the significance of the first theorem, we denote by F a closed linear set, and by μ a probability measure in F satisfying a Lipschitz condition $\mu(a, a+h) \leq C_\beta h^\beta$ for each interval $(a, a+h)$ and each exponent $\beta < \alpha \leq 1$. Then the planar set $F \times F$ carries the measure $\mu \times \mu$, which fulfills a Lipschitz condition for each exponent $2\beta < 2\alpha$. We can apply the method of Marstrand [7, Lemmas 10 to 19] to the set $F \times F$ (using $\mu \times \mu$ in place of $\Lambda^{2\alpha}$) to prove the following result: There exists a line $y = mx + b$ $(m \neq 1)$ whose intersection with F has dimension at least $2\alpha - 1$. This means that there is an affine map $T \neq 1$ of the line — hence an infinitely differentiable map with exactly one fixed point — such that $T(F) \cap F$ has dimension at least $2\alpha - 1$. In Theorem 1, we prove that the constant $2\alpha - 1$ is best

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possible, even if we admit a larger class of transformations T. The sets B and E = f(B) occurring in Theorem 1' carry measures μ described above; for further details on the relation between dimension and measure, see [3, Chapters 1, 2, 3], and for a view of the theory of Hausdorff measures and its subtleties, see [1], [8], [10].

THEOREM 1. *Let* $1/2 < \alpha < 1$ *and* $r = (2\alpha - 1)^{-1}$. *Then there exists a compact set* E *of Hausdorff dimension* α *such that for each function* F *of class* $C^r(-\infty, \infty)$,

$$\dim E \cap F(E \setminus S) \leq 2\alpha - 1, \qquad \textit{where } S \equiv S(F) = \{F(x) = x\}.$$

To prove Theorem 1, we take some strictly increasing sequence of positive integers n_k such that

(a) $n_k \leq \alpha^{-1} k + k^{1/2}$,

(b) $n_{k+1} \geq 1 + \alpha^{-1} n_k$ for each k in an infinite set T.

Then B is the set of all sums $\sum \varepsilon_k 2^{-n_k}$ ($\varepsilon_k = 0, 1$), and B has dimension $\liminf k n_k^{-1} = \alpha$. Let W be the open subset of $C^1[0, 1]$ defined by the inequalities $1 < f' < 2$ and $-2 < f < 2$.

THEOREM 1'. *For all functions* f *in* W, *excepting a set of the first category* (*in the* C^1-*metric*), f(B) *fulfills the requirements of Theorem* 1.

The Banach space $C^r[-2, 2]$ can be expressed as an increasing union of bounded subsets U_j, for example, balls of radius j. Defining

$$S_j \equiv S_j(F) = \{x\colon |F(x) - x| \leq j^{-1}\},$$

we construct a dense G_δ-set $W_j \subseteq W$, effective for all the sets $f(B) \cap F(f(B) \setminus S_j)$ determined by functions F in U_j. Then plainly $\bigcap W_j$ is a dense G_δ-set in W, and it contains only functions of the type prescribed.

A. Corresponding to each integer k in the infinite set T, there is a covering of B by intervals I_p $(1 \leq p \leq 2^k)$ of length $2 \cdot 2^{-n_{k+1}}$ and mutual distances at least $2^{-1} 2^{-n_k}$. For a fixed F in U_j and a fixed f in W, we estimate the size of $f(B) \cap F(f(B))$ by counting the integers p for which there is a q satisfying the condition $f(I_p) \cap F(f(I_q)) \neq \emptyset$. Both sets in the relation have length $O(2^{-n_{k+1}}) \|F\|$, and because we are interested only in intervals I_q not contained entirely in $S_j(F)$, we can assume that $p \neq q$ for large k. Let x_p be some number in I_p, and choose $\tilde{F}$ in U_j so that $|\tilde{F}(x) - F(x)| \leq 2^{-n_{k+1}}$ for all x in [-2, 2]. It is enough now to count solutions of the inequality $|f(x_p) - \tilde{F}(f(x_q))| \leq K_j 2^{-n_{k+1}}$, with some K_j depending on U_j. We shall accomplish this enumeration for each $\tilde{F}$ in a finite subset $U_j^* \subseteq U_j$, affording $2^{-n_{k+1}}$-uniform approximation to every element of U_j. By the bound cited in the introduction,

$$|U_j^*| \leq \exp L_j 2^{r^{-1} n_{k+1}} = \exp L_j 2^{(2\alpha - 1) n_{k+1}},$$

for large k.

To replace a preassigned f with a "good" function -- in a sense to be specified in a moment -- we use an increment f_0:

$$f_0 = c_p \text{ on } I_p \quad (\max |c_p| \leq k^{-1} 2^{-n_k}).$$

Although f_0 is defined only on $\bigcup I_p$, it is the restriction of a function in $C^1[0, 1]$ of norm $O(k^{-1})$, according to the relative position of the intervals I_p. An acceptable number of "hits" -- counting the integers p, rather than the pairs p, q -- is $N_k = k^2 2^{(2\alpha - 1)n_{k+1}}$, because $N_k 2^{-cn_{k+1}} \to 0$ for each $c > 2\alpha - 1$. In the remaining calculations, f is fixed in W and F in U_j^*.

The inequality $|f(x_p) + c_p - F(f(x_q))| \leq K_j 2^{-n_{k+1}}$ implies that

$$|f(x_p) - F(f(x_q))| \leq K_j 2^{-n_{k+1}} + K_j' \cdot k^{-1} \cdot 2^{-n_k} = O(k^{-1} 2^{-n_k}),$$

whence, for large k, at most one index p can be paired with each q; here we used the inequality $1 < f' < 2$. We shall choose the parameters c_p of f_0 as independent random variables X_p, uniformly distributed on the interval $|X| \leq k^{-1} 2^{-n_k}$. Moreover, we shall prove, in Part B, that the (X_p) yielding an acceptable function $f + f_0$ have probability (product measure) at least $1 - \exp \delta k^2 \cdot 2^{(2\alpha - 1)n_k}$, for some $\delta > 0$. There is an exceptional set for each f and F, but our estimate on $|U_j^*|$ shows that most (X_p) are effective for this f and *all* of U_j^*. Thus, the asserted estimate of probabilities assures the existence of a good function $f + f_0$ in C^1, with $\|f_0\| = O(k^{-1})$, and therefore W_j is a dense G_δ-set in W, and Theorem 1' follows.

B. We write Y(p, q) for the event

$$|f(x_p) + X_p - F(f(x_q) + X_q)| \leq K_j 2^{-n_{k+1}};$$

we have seen that, among events actually possible, p is determined by q. These events we now arrange into two chains, beginning with an arbitrary p_1, and enumerating all $Y(p_1, q_\nu)$ $(1 \leq \nu \leq Q_1)$. Next we enumerate $Y(p_2, q_\nu)$ $(Q_1 < \nu \leq Q_2)$, subject only to the rule that p_m shall not have occurred previously as first or second coordinate. This process is extended as far as possible, and then applied anew to the events Y(p, q) not selected in the first chain; plainly, each p_0 occurs only once in the events $Y(p, p_0)$, and hence the second application of our process exhausts all possible events Y(p, q).

Let $Y(p_m) = \bigcup Y(p_m, q_\nu)$ $(Q_{m-1} < \nu \leq Q_m)$ in the first chain. The variable X_p occurs in none of the events $Y(p_\ell)$ $(1 \leq \ell < m)$, so that the conditional probability of $Y(p_m)$, relative to the field generated by $Y(p_\ell)$ $(1 \leq \ell < m)$, does not exceed $\pi_m \leq (Q_m - Q_{m-1}) \cdot K_j 2^{-n_{k+1}} \cdot k 2^{n_k}$. Thus, if Z_m is the indicator of the event $Y(p_m)$ and $Z = \sum Z_m$ (summed on the first chain), the calculus of conditional probabilities yields a bound for the expectation

$$E(2^Z) \leq \prod_m (1 + \pi_m) \leq \exp \sum \pi_m \leq \exp k K_j \cdot 2^k \cdot 2^{n_k} 2^{-n_{k+1}}.$$

Thus

$$P\left\{Z > \tfrac{1}{2}N_k\right\} \leq \exp kK_j \cdot 2^k \cdot 2^{n_k} 2^{-n_{k+1}} \exp\left(-\tfrac{1}{2}(\log 2)\cdot N_k\right) \leq \exp(-\delta N_k),$$

because $N_k = k^2 2^{(2\alpha-1)n_{k+1}}$ while $k + n_k - n_{k+1} \leq 2n_k - n_{k+1} < (2\alpha - 1)n_{k+1}$ for k in T. A similar estimate holds for the events in the second chain, and the bound $2\exp(-\delta N_k)$ is within the limits set for the construction of f_0. This concludes the proof of Theorem 1'.

We used the inequality $1 < f' < 2$ only to obtain an inequality

$$|f(x) - f(y)| \geq c\,|x - y|$$

for points x and y in B; for a C^1-function f, this means precisely that f is one-to-one on B and $f' \neq 0$ on B. Because B is totally disconnected, the set $\{f: f \in C^1, f' \neq 0 \text{ on } B\}$ is open and dense in $C^1[0, 1]$; using the covering of B by intervals I_p, we can easily prove that the functions that are one-to-one on B form a dense G_δ-set in $C^1[0, 1]$. Thus the inequalities $1 < f' < 2$ and (even more plainly) $-2 < f < 2$ can be omitted (compare this with the argument in [5]).

2. A compact set E of Hausdorff dimension $\alpha < 1$ is a *Salem set* if E carries a probability measure μ whose Fourier-Stieltjes transform $\hat{\mu}$ fulfills the condition

$$|\hat{\mu}(u)| = O(|u|^{-c}) \qquad \text{for each } c < \alpha/2 .$$

The condition $|\hat{\mu}(u)| = O(|u|^{-c})$ in any case implies that $\dim E \geq 2c$, so that Salem sets are extremal ([2], [3], [9]).

THEOREM 2. *For each* α *in* (0, 1), *there exist a set* E *of dimension* α *and a probability measure* λ *in* E *such that* $\int_E e^{iu\phi(x)}\lambda(dx) = O(|u|^{-c})$ *for every* $c < \alpha/2$ *and for every* C^∞*-function* ϕ *with positive derivative. Consequently, each transform* $\phi(E)$ *is a Salem set.*

No example is known of a Salem set lacking the stronger property claimed here, and probabilistic techniques seem unsuited for constructing such a set. It is known that even quadratic polynomials ϕ can change the behavior of Fourier transforms, so that Theorem 2 could not be proved by studying only $\hat{\lambda}$.

A. In the following estimations, h is a function of class $C^\infty(-\infty, \infty)$, vanishing outside of $(-\infty, \infty)$, while S is a bounded subset of $C^k[0, 1]$ for some integer $k \geq 1$, whose members satisfy an inequality $f' \geq c > 0$ on [0, 1]. It is convenient to write $e(t) \equiv e^{it}$, for real numbers t.

LEMMA 1. *The estimate* $\int_0^1 e(uf(rt))\,h(t)\,dt = O((ru)^{1-k})$ *holds uniformly for* f *in* S, $0 < r \leq 1$, $-\infty < u < \infty$.

Proof. Let F be the function inverse to an assigned function f in S, so that F is defined on [f(0), f(1)]. The derivatives of all functions F, up to the kth derivative, are uniformly bounded by virtue of the hypotheses on S. The integral can be written as

$$r^{-1}\int_0^r e(uf(s))\,h(r^{-1}s)\,ds = r^{-1}\int_{f(0)}^{f(r)} e(uy)\,h(r^{-1}F(y))\,F'(y)\,dy .$$

Now $h(r^{-1}F(y))$ is of class C^k and vanishes with its derivatives up to order $k-1$ at the extremes $y = f(0)$ and $y = f(r)$. Its derivatives of these orders are $O(r^{1-k})$, uniformly for f in S, and the bound $O(1)$ holds for the derivatives of F'. Integrating by parts $k-1$ times, and using the vanishing of the derivatives (and Leibniz's rule), we obtain a bound $r^{-1}(f(r) - f(0))O((ru)^{1-k}) = O((ru)^{1-k})$.

From now on, write $N_k = 2^{k!}$ for $k = 1, 2, 3, \cdots$, and define r_k so that $r_k^{\alpha}(N_1 \cdots N_{k-1}) = 1$ and $r_1 = 1$. The set E will be a vector sum $S_1 + \cdots + S_k + \cdots$ in which S_k has N_k elements and $S_k \subseteq [0, r_k]$. Then

$$E \subseteq S_1 + \cdots + S_k + [0, 2r_{k+1}],$$

and thus E has dimension at most α, since $(2r_{k+1})^{\alpha} N_1 \cdots N_k \leq 2$. To choose the sets S_k, we take a double array (Y_k, m) of independent random variables with density $h(t)\,dt$ $(1 \leq k < \infty,\ 1 \leq m \leq N_k)$, choosing h so that $h \geq 0$ and $\int h\,dt = 1$. Then $x_{k,m} = r_k Y_{k,m}$, λ_k is the probability uniformly distributed over the N_k points $x_{k,m}$, and E is the closed support of the measure $\prod_1^{\infty *} \lambda_k$.

LEMMA 2. *Let* $0 < b < \beta < \alpha$ *and* $p \geq 2(\beta - b)^{-1}$, *and let* T *be a bounded subset of* $C^p[0, 1]$. *Let* $k = k(u)$ *be defined for large numbers* u *by the inequality* $N_k \geq u^{\beta} > N_{k-1}$. *Then it is almost certain that*

$$\left| \int e(uf(t))\,\lambda_k(dt) - \int e(uf(r_k t))\,h(t)\,dt \right| \leq u^{-b/2}$$

for all f *in* T *and all* $u \geq u_0$ *(a random number).*

Proof. Corresponding to a specified $u > 0$, let T^* be a finite subset of T large enough to afford approximation, uniform within $\frac{1}{2}u^{-1}u^{-b/2}$. The size of T^* is then bounded by $\exp C u^s$, with $s = p^{-1}\left(1 + \frac{1}{2}b\right)$. We shall obtain an inequality $\left|\int - \int\right| \leq \frac{1}{2}u^{-b/2}$ valid for all f in T^*, and this will yield the weaker inequality for all f in T.

Let I_1 be the real part of the integral $\int e(uf)\,d\lambda_k$, and m_1 the real part of its expectation, written above. Standard estimates from probability theory yield the inequality

$$E(y\,|m_1 - I_1|) \leq 2\left(\exp \tfrac{1}{2}y^2 N_k^{-1}\right)(\exp B y^3 N_k^{-2}),$$

uniformly for real numbers y in $[0, N_k]$. Choosing $y = \frac{1}{2}u^{-b/2}N_k$, we obtain a bound

$$P\left(|m_1 - I_1| \geq \tfrac{1}{2}u^{-b/2}\right) \leq 2\exp(-\delta u^{-b} N_k),$$

for some δ near 1/8. Applying the same estimate to the imaginary parts, and observing that $u^{-b}N_k \geq u^{\beta-b}$ while $s = p^{-1}\left(1+\frac{1}{2}b\right) < 2p^{-1} \leq \beta - b$, we obtain our lemma for integers u, and indeed for the sequence $u = n^{1/2}$. But it is now clear that we can extend our inequality over the intervals $[n^{1/2}, (n+1)^{1/2}]$, whose length is less than $n^{-1/2}$, because $b < 1$.

B. For the proof of Theorem 2, suppose that f is of class C^∞ and $f' > 0$ on an interval containing E. Let $b < \beta < \alpha$, and let N_k be chosen as in Lemma 2, while μ_k is the cofactor of λ_k in the product $\prod \lambda_k$. Now

$$\left|\int e(uf(t))\lambda(dt)\right| < \sup\left|\int e(uf(t+s))\lambda_k(dt)\right|,$$

where s belongs to the support of μ_k, hence to E. Applying Lemma 2 to the functions $t \to f(t+s)$ ($s \in E$, $0 \leq t \leq 1$), we obtain an estimate

$$u^{-b/2} + \sup\left|\int e(uf(r_k t + s))h(t)\,dt\right| .$$

To complete the proof, we need only a suitable upper bound for the supremum on the right; because $f \in C^\infty$ and $f' > 0$, it is sufficient (by Lemma 1) to have $\log u - \log r_k \geq \delta \log u$ for some $\delta > 0$.

In fact, $\log r_k = -\alpha^{-1}(1+2+\cdots+(k-1)!)\log 2$, and $(k-1)!\log 2 < \beta \log u$. Hence $\log r_k \geq -\alpha^{-1}\beta(1+O(k^{-1}))\log u$, while $0 < \beta < \alpha$.

C. The set E has the special property that there exists a sequence $\{R_k\}$ tending to 0 such that E is contained in $O(R_k^{-\alpha})$ intervals of length R_k. However, the sequence $\{R_k\}$ must be very sparse, while for the Salem sets constructed in [2, Chapter 15], [3, Chapter 8], and [9] the condition holds uniformly for $0 < R < 1$. A final observation: in the formula $r_k^{-\alpha} = N_1 \cdots N_{k-1}$, the omission of the factor N_k is decisive; in most constructions concerned with Hausdorff dimension, N_k is constant, and this point can be ignored.

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