PERIODIC SOLUTIONS OF THE TRICOMI PROBLEM

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INTRODUCTION

Consider the system of differential equations

$$(0.1)\qquad \begin{cases} u_x(x, y) = F(x, y, u, v), \\ v_y(x, y) = G(x, y, u, v) \end{cases}$$

and the boundary conditions

$$(0.2)\qquad u(0, y) = \tau(y), \quad v(x, 0) = \sigma(x).$$

Here $F(x, y, u, v)$ and $G(x, y, u, v)$ are continuous vector functions (possibly of different dimensions) defined in a region

$$|x| \leq a_1 \leq \infty, \quad |y| \leq a_2 \leq \infty, \quad |u| \leq u_1, \quad |v| < v_1,$$

while $\sigma(x)$ and $\tau(y)$ are prescribed continuous vector functions defined for $|x| \leq a_1 \leq \infty$ and $|y| \leq a_2 \leq \infty$. *Tricomi's problem* (see [10]) is to find two vector functions $u(x, y)$ and $v(x, y)$ that, together with u_x and u_y, are continuous in $|x| \leq a_1$, $|y| \leq a_2$, and that satisfy the system (0.1) and the boundary conditions (0.2) on $|x| < a_1$, $|y| < a_2$. We emphasize that a solution of (0.1) - (0.2) does not generally have a continuous second-order mixed partial derivative, so that the system (0.1) - (0.2) cannot generally be reduced to a system of the form $w_{xy} = H(x, y, w, w_x, w_y)$.

Under some regularity conditions on F and G (see F. Tricomi [8], G. Villari [10], G. Santagati [7]), there exists at least one solution of problem (0.1) - (0.2); under other more restrictive conditions, there exists a unique solution of (0.1) - (0.2). In this paper we investigate the existence of periodic solutions for the Tricomi problem. We use a method of L. Cesari [5], which consists in treating first a slightly modified (relaxed) problem. In particular, L. Cesari used the method in [2], [3], and [4] to obtain the periodic solutions of the Darboux problem

$$u_{xy} = f(x, y, u, u_x, u_y), \quad u(x, 0) = \nu(x), \quad u(0, y) = \mu(y).$$

We remark that the problem (0.1) - (0.2) is equivalent to the problem of finding continuous solutions of the system

$$u(x, y) = \tau(y) + \int_0^x F[\xi, y, u(\xi, y), v(\xi, y)]d\xi,$$

$$v(x, y) = \sigma(x) + \int_0^y G[x, \eta, u(x, \eta), v(x, \eta)]d\eta.$$

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If F, G, and σ are periodic of period T in x, a continuous solution $u(x, y)$, $v(x, y)$ may nevertheless fail to be periodic in x, for $y \neq 0$. One sees immediately that a necessary condition for a solution $u(x, y)$, $v(x, y)$ to be T-periodic in x ($|y| \leq a$) is that

$$\int_0^T F[\xi, y, u(\xi, y), v(\xi, y)]\,d\xi = 0 \qquad (|y| \leq a).$$

This leads us to consider (see [3]) the modified problem

$$(0.3)\qquad \begin{cases} u_x = F(x, y, u, v) - m(y), & v_y = G(x, y, u, v), \\ u(x+T, y) = u(x, y), & v(x+T, y) = v(x, y), \\ u(0, y) = \tau(y), & u(x, 0) = \sigma(x), \\ m(y) = T^{-1} \displaystyle\int_0^T F[\xi, y, u(\xi, y), v(\xi, y)]\,d\xi, & \end{cases}$$

where $|x| < \infty$ and $|y| \leq a$, where $\sigma(x)$ is a prescribed continuous vector function, T-periodic in x, where $\tau(y)$ is a prescribed continuous vector function, and where F and G are T-periodic in x. Suppose we prove that under certain conditions (see for example Theorem 1 below) the modified problem (0.3) has a solution; then a second problem arises, namely to find additional conditions that allow us to determine the data $\sigma(x)$ and $\tau(y)$ in such a manner that $m(y) = 0$. In this paper, we shall discuss this second problem not in its generality, but only in a simple case. On the other hand, if we try to find criteria for the existence of periodic solutions that are T-periodic in x and y, as in [4], we must consider another modified problem (see Section 2). In Sections 1 and 2, the reader will observe a close analogy with Cesari's arguments and results in [2], [3], [4]. In Section 3, we extend the existence theorems of Sections 1 and 2 to the modified problem. The extension is analogous to extensions of Cesari's theorems that A. K. Aziz [1] obtained by using more stringent estimates in Cesari's treatment of the relaxed problem. With obvious modifications, our results can probably be extended to more general systems, as is indicated in Villari's paper [10].

1. BASIC THEOREMS

In what follows, $|z|$ denotes the Euclidean norm of the vector z.

THEOREM 1. *Let* a *and* $T > C$ *denote positive constants, and let* $N_0, N_1, S_1, S_2, L, M, P_1, P_2$ *be nonnegative constants such that*

$$(1.1)\qquad N_1 + S_1 T \leq P_1, \qquad N_0 + S_2 a \leq P_2,$$

and denote by A *and* R *the intervals*

$$A = [0 \leq x \leq T,\ |y| \leq a], \qquad R = [0 \leq x \leq T,\ |y| \leq a,\ |u| \leq P_1,\ |v| \leq P_2].$$

Suppose $\sigma(x)$ *and* $\tau(y)$ *are* n- *and* m-*dimensional, real-valued vector functions, continuous in* $0 \leq x \leq T$ *and* $|y| \leq a$, *and such that*

$$(1.2) \qquad \sigma(T) = \sigma(0), \quad |\sigma(x)| \leq N_0 \quad (0 \leq x \leq T),$$

$$(1.3) \qquad |\tau(y)| \leq N_1 \quad (|y| \leq a).$$

Suppose $F(x, y, u, v)$ *and* $G(x, y, u, v)$ *are* m- *and* n-*dimensional, real-valued vector functions, continuous in* R *and satisfying the conditions*

$$(1.4) \qquad F(T, y, u, v) = F(0, y, u, v), \quad G(T, y, u, v) = G(0, y, u, v),$$

$$(1.5) \qquad |F(x, y, u, v)| \leq S_1, \quad |G(x, y, u, v)| \leq S_2$$

and

$$(1.6) \qquad \begin{cases} |F(x, y, \bar{u}, v) - F(x, y, u, v)| \leq L|\bar{u} - u|, \\ |G(x, y, u, \bar{v}) - G(x, y, u, v)| \leq M|\bar{v} - v| \end{cases}$$

for $0 \leq x \leq T$; $|y| \leq a$; $|u|, |\bar{u}| \leq P_1$; $|v|, |\bar{v}| \leq P_2$. *Then, for*

$$(1.7) \qquad 2LT < 1 \quad \textit{and} \quad Ma < 1,$$

there exist two vector functions $\Phi(x, y)$ *and* $\Psi(x, y)$ (*the first* m-*dimensional, the second* n-*dimensional*) *defined and continuous* (*together with* Φ_x *and* Ψ_y) *in* A, *and an* m-*dimensional vector function* m(y), *continuous in* $|y| \leq a$, *such that*

$$(1.8) \qquad \Phi(0, y) = \Phi(T, y) = \tau(y),$$

$$(1.9) \qquad \Psi(x, 0) = \sigma(x), \quad \Psi(0, y) = \Psi(T, y),$$

$$(1.10) \qquad m(y) = T^{-1} \int_0^T F(\xi, y, \Phi(\xi, y), \Psi(\xi, y))\,d\xi,$$

$$(1.11) \qquad \Phi_x(x, y) = F(x, y, \Phi(x, y), \Psi(x, y)) - m(y),$$

$$(1.12) \qquad \Psi_y(x, y) = G(x, y, \Phi(x, y), \Psi(x, y)),$$

for all $(x, y) \in A$. *Thus, if we extend* $\Phi(x, y)$, $\Psi(x, y)$, $F(x, y, u, v)$, *and* $G(x, y, u, v)$ *to all of* $|x| < \infty$, $|y| \leq a$, $|u| \leq P_1$, $|v| \leq P_2$, *by means of* T-*periodicity in* x, *then equations* (1.11) *and* (1.12) *are satisfied in the strip* $|x| < \infty$, $|y| \leq a$.

Proof. Let $\phi(x, y)$ and $\psi(x, y)$ be m- and n-dimensional vector functions.

Remark 1. If $\phi(x, y)$ and $\psi(x, y)$ satisfy the conditions

$$(1.13) \qquad \begin{cases} \phi(0, y) = \phi(T, y) = \tau(y), \quad \psi(x, 0) = \sigma(x), \quad \psi(T, y) = \psi(0, y), \\ |\phi(x_1, y) - \phi(x_2, y)| \leq 2S_1|x_1 - x_2|, \quad |\psi(x, y_1) - \psi(x, y_2)| \leq S_2|y_1 - y_2| \end{cases}$$

for all $0 \leq x, x_1, x_2 \leq T$ and $|y|, |y_1|, |y_2| \leq a$, then

$$(1.14) \qquad |\phi(x, y)| \leq P_1, \quad |\psi(x, y)| \leq P_2 \quad ((x, y) \in A).$$

Indeed,

$$|\phi(x, y)| \leq |\phi(0, y)| + |\phi(x, y) - \phi(0, y)| \leq |\tau(y)| + 2S_1 x,$$

$$|\phi(x, y)| \leq |\phi(T, y)| + |\phi(T, y) - \phi(x, y)| \leq |\tau(y)| + 2S_1 (T - x);$$

thus

$$|\phi(x, y)| \leq N_1 + 2S_1 \min(x, T - x) \leq N_1 + S_1 T \leq P_1 .$$

On the other hand,

$$|\psi(x, y)| \leq |\psi(x, 0)| + |\psi(x, y) - \psi(x, 0)| \leq |\sigma(x)| + S_2 |y| \leq N_0 + S_2 a \leq P_2 .$$

Note also that (1.13) implies the existence a.e. of ϕ_x and ψ_y, together with the condition $|\phi_x| \leq 2S_1$, $|\psi_y| \leq S_2$ a.e. in A.

Now $F(x, y, u, v)$ and $G(x, y, u, v)$ are continuous on R, and $\sigma(x)$ and $\tau(y)$ are also continuous functions. Thus there exist scalar functions $\omega_1(\alpha)$, $\omega_2(\beta)$, $\omega_3(\delta)$, $\omega_4(\alpha)$, $\pi_1(\alpha)$, $\pi_2(\beta)$, $\pi_3(\gamma)$, $\pi_4(\beta)$, continuous and nondecreasing in $[0, +\infty)$, such that

$$\omega_1(0) = \omega_2(0) = \omega_3(0) = \omega_4(0) = \pi_1(0) = \pi_2(0) = \pi_3(0) = \pi_4(0) = 0$$

and

$$(1.15) \quad \begin{cases} |F(x_1, y, u, v) - F(x_2, y, u, v)| \leq \omega_1(|x_1 - x_2|), \\ |F(x, y_1, u, v) - F(x, y_2, u, v)| \leq \omega_2(|y_1 - y_2|), \\ |F(x, y, u, v_1) - F(x, y, u, v_2)| \leq \omega_3(|v_1 - v_2|), \\ |G(x_1, y, u, v) - G(x_2, y, u, v)| \leq \pi_1(|x_1 - x_2|), \\ |G(x, y_1, u, v) - G(x, y_2, u, v)| \leq \pi_2(|y_1 - y_2|), \\ |G(x, y, u_1, v) - G(x, y, u_2, v)| \leq \pi_3(|u_1 - u_2|), \\ |\sigma(x_1) - \sigma(x_2)| \leq \omega_4(|x_1 - x_2|), \qquad |\tau(y_1) - \tau(y_2)| \leq \pi_4(|y_1 - y_2|), \end{cases}$$

for $0 \leq x, x_1, x_2 \leq T$; $|y|, |y_1|, |y_2| \leq a$; $|u|, |u_1|, |u_2| \leq P_1$; $|v|, |v_1|, |v_2| \leq P_2$. Let

$$(1.16) \qquad \eta_1(\beta) = (1 - 2LT)^{-1}[\pi_4(\beta) + 2T\omega_2(\beta) + 2T\omega_3(S_2\beta)],$$

$$(1.17) \qquad \eta_2(\alpha) = (1 - aM)^{-1}[\omega_4(\alpha) + a\pi_1(\alpha) + a\pi_3(2S_1\alpha)].$$

Evidently, $\eta_1(\beta)$ and $\eta_2(\alpha)$ are continuous and nondecreasing in $[0, +\infty)$, and $\eta_1(0) = \eta_2(0) = 0$.

Now let E be the linear space of the $(m + n)$-dimensional vector functions that are continuous in A. Denote by $\phi(x, y)$ the vector formed by the first m components of an element of E and by $\psi(x, y)$ the vector of its last n components. Thus, an element of E will be denoted by

$$z(x, y) = \begin{bmatrix} \phi(x, y) \\ \psi(x, y) \end{bmatrix}.$$

Let $\|\cdot\|$ denote the norm

$$\|z\| = \sup|\phi(x, y)| + \sup|\psi(x, y)| \tag{1.18}$$

in E, where the supremum is taken in A. The convergence in this norm is the uniform convergence on A for each component of the vector $z \in E$.

Let K consist of all elements of E satisfying (1.13) together with the inequalities

$$\begin{cases} |\phi(x, y_1) - \phi(x, y_2)| \leq \eta_1(|y_1 - y_2|), \\ |\psi(x_1, y) - \psi(x_2, y)| \leq \eta_2(|x_1 - x_2|). \end{cases} \tag{1.19}$$

For any $z = \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in K$, (1.13) is satisfied, and thus, by Remark 1, the inequalities (1.14) are satisfied. Hence, $F(x, y, \phi(x, y), \psi(x, y))$ and $G(x, y, \phi(x, y), \psi(x, y))$ are defined and continuous in A. Thus

$$m(y) = T^{-1} \int_0^T F(\xi, y, \phi(\xi, y), \psi(\xi, y))\,d\xi \tag{1.20}$$

is defined and continuous in $|y| \leq a$. Now let

$$\tau\colon z(x, y) = \begin{bmatrix} \phi(x, y) \\ \psi(x, y) \end{bmatrix} \to Z(x, y) = \begin{bmatrix} \Phi(x, y) \\ \Psi(x, y) \end{bmatrix}$$

be the map $Z(x, y) = (\tau z)(x, y)$ defined by

$$\begin{cases} \Phi(x, y) = \tau(y) + \int_0^x \{F(\xi, y, \phi(\xi, y), \psi(\xi, y)) - m(y)\}\,d\xi, \\ \\ \Psi(x, y) = \sigma(x) + \int_0^y G(x, \eta, \phi(x, \eta), \psi(x, \eta))\,d\eta \end{cases} \tag{1.21}$$

for every $z(x, y) \in K$.

We shall prove that τ maps K into K. Observe first that

$$|m(y)| \leq S_1 \qquad (|y| \leq a). \tag{1.22}$$

We see from (1.20) and (1.21) that

$$\Phi(T, y) = \tau(y) = \Phi(0, y), \qquad \Psi(x, 0) = \sigma(x), \qquad \Psi(T, y) = \Psi(0, y).$$

On the other hand, (1.21), (1.22), and (1.5) imply that

$$|\Phi(x_1, y) - \Phi(x_2, y)| \leq 2S_1|x_1 - x_2|,$$

$$|\Psi(x, y_1) - \Psi(x, y_2)| \leq S_2|y_1 - y_2|.$$

Thus $\Phi(x, y)$ and $\Psi(x, y)$ satisfy (1.13). From (1.20), (1.15), (1.6), (1.13), (1.19), we now obtain the inequality

$$|m(y_1) - m(y_2)| \leq \omega_2(|y_1 - y_2|) + \omega_3(S_2|y_1 - y_2|) + L\eta_1(|y_1 - y_2|),$$

and hence, from (1.21), (1.3), (1.16), we find that

$$\begin{aligned}|\Phi(x, y_1) - \Phi(x, y_2)| \\ &\leq \pi_4(|y_1 - y_2|) + 2T\omega_2(|y_1 - y_2|) + 2T\omega_3(S_2|y_1 - y_2|) + 2TL\eta_1(|y_1 - y_2|) \\ &= \eta_1(|y_1 - y_2|).\end{aligned}$$

Similarly, we deduce from (1.21), (1.15), (1.6), (1.13), (1.19), (1.17) that

$$\begin{aligned}|\Psi(x_1, y) - \Psi(x_2, y)| \\ &\leq \omega_4(|x_1 - x_2|) + a\pi_1(|x_1 - x_2|) + a\pi_3(2S_1|x_1 - x_2|) + aM\eta_2(|x_1 - x_2|) \\ &= \eta_2(|x_1 - x_2|).\end{aligned}$$

Thus τ maps K into K.

It is obvious that

$$\|\tau z_1 - \tau z_2\| \leq (2TL + aM)\|z_1 - z_2\| + 2T\omega_3(\|z_1 - z_2\|) + \pi_3(\|z_1 - z_2\|)$$

for $z_1, z_2 \in K$, and hence τ is a continuous map from K to K.

Finally, it is clear from (1.13) and (1.19) that K is convex, closed, and (by the theorem of Arzelà and Ascoli) compact with respect to the norm of E. By Schauder's fixed-point theorem, it follows that there exists an element $Z(x, y) = \begin{bmatrix} \Phi(x, y) \\ \Psi(x, y) \end{bmatrix} \in K$ such that

$$\Phi(x, y) = \tau(y) + \int_0^x \{F(\xi, y, \Phi(\xi, y), \Psi(\xi, y)) - m(y)\}\, d\xi,$$

$$\Psi(x, y) = \sigma(x) + \int_0^y G(x, \eta, \Phi(x, \eta), \Psi(x, \eta))\, d\eta,$$

$$m(y) = m_{\Phi,\Psi}(y) = T^{-1}\int_0^T F(\xi, y, \Phi(\xi, y), \Psi(\xi, y))\, d\xi,$$

for $0 \leq x \leq T$, $|y| \leq a$. Obviously, Φ_x and Ψ_y exist and are continuous everywhere in A, and we have the relations

$$\Phi_x(x, y) = F(x, y, \Phi(x, y), \Psi(x, y)) - m(y),$$

$$\Psi_y(x, y) = G(x, y, \Phi(x, y), \Psi(x, y)).$$

This completes the proof of Theorem 1.

Remark 2. The conditions of Theorem 1 are not enough for uniqueness, as the following example shows: Take $T = 1$, $a = 1$, $\sigma(x) \equiv 0$, $\tau(y) \equiv 0$,

$$F(x, y, u, v) = 3\pi v \sin 2\pi x, \qquad G(x, y, u, v) = 2|u|^{1/2} \sin \pi x,$$

for $0 \leq x \leq 1$ and for all y, u, v. Then the system

$$u_x = 3\pi v \sin 2\pi x, \qquad v_y = 2|u|^{1/2} \sin \pi x$$

has the trivial solution $u \equiv 0$, $v \equiv 0$, and also the solution

$$u(x, y) = |y| y \sin^6 \pi x, \qquad v(x, y) = |y| y \sin^4 \pi x.$$

Both solutions satisfy the condition $m(y) = 0$. We can take $N_0 = N_1 = M = L = 0$, $T = 1$, $a = 1$, $S_1 = 3\pi$, $S_2 = 2$, $P_1 = 4\pi$, $P_2 = 3$, and we observe that although all conditions of Theorem 1 hold, there are two distinct solutions of (1.8) - (1.12).

THEOREM 2 (uniqueness). *Let* $\omega_3(\delta) = L_1 \delta$, $\pi_3(\gamma) = M_1 \gamma$, *where* L_1 *and* M_1 *are nonnegative constants. Then, under the hypotheses of Theorem* 1, *there exist a unique vector function*

$$Z(x, y) = \begin{bmatrix} \Phi(x, y) \\ \Psi(x, y) \end{bmatrix},$$

continuous (together with Φ_x *and* Ψ_y*) in* A, *and a unique vector function* m(y), *continuous in* [-a, a], *such that conditions* (1.8) *to* (1.12) *are satisfied.*

Proof. We employ a standard technique used by Cesari in [2]. Let

$$Z_1(x, y) = \begin{bmatrix} \Phi_1(x, y) \\ \Psi_1(x, y) \end{bmatrix}, \qquad Z_2(x, y) = \begin{bmatrix} \Phi_2(x, y) \\ \Psi_2(x, y) \end{bmatrix},$$

and let $m_i(y)$ $(i = 1, 2)$ be given by (1.10). Suppose that (1.8) to (1.12) are satisfied by both $Z_1(x, y)$ and $Z_2(x, y)$. Since $Z_1(x, y) \equiv Z_2(x, y)$ implies $m_1(y) \equiv m_2(y)$, it suffices to prove that $Z_1(x, y) \equiv Z_2(x, y)$, in other words, that $\Phi_1(x, y) \equiv \Phi_2(x, y)$ and $\Psi_1(x, y) \equiv \Psi_2(x, y)$. Suppose this is false. Then

$$\chi(x, y) = |\Phi_1(x, y) - \Phi_2(x, y)| + |\Psi_1(x, y) - \Psi_2(x, y)| \not\equiv 0 \tag{1.23}$$

in A. Without loss of generality, we can suppose that there exists a minimal number s $(0 \leq s < a)$ such that $\chi(x, y) \not\equiv 0$ in each strip

$$0 \leq x \leq T, \qquad s \leq y \leq s + c \qquad (c > 0,\ s + c \leq a).$$

Let $\alpha = \sup |\Phi_1(x, y) - \Phi_2(x, y)|$ and $\beta = \sup |\Psi_1(x, y) - \Psi_2(x, y)|$, where the supremum is taken over the strip $0 \leq x \leq T$, $s \leq y \leq s + c$ $(s + c \leq a)$. Then

$$|m_1(y) - m_2(y)| \leq L\alpha + L_1 \beta,$$

and hence

$$|\Phi_1(x, y) - \Phi_2(x, y)| \leq 2TL\alpha + 2TL_1 \beta$$

for $0 \leq x \leq T$, $s \leq y \leq s+c$. Also,

$$|\Psi_1(x, y) - \Psi_2(x, y)| \leq cM\beta + cM_1\alpha$$

for $0 \leq x \leq T$, $s \leq y \leq s+c$, and thus

$$\alpha \leq 2TL\alpha + 2TL_1\beta, \qquad \beta \leq cM\beta + cM_1\alpha. \tag{1.24}$$

It follows from (1.23) that

$$\alpha + \beta > 0 \quad \text{and} \quad \alpha \geq 0,\ \beta \geq 0. \tag{1.25}$$

If $\alpha = 0$, the second relation of (1.24) gives $\beta \leq cM\beta \leq aM\beta$, that is, $\beta = 0$ (because $Ma < 1$ in (1.7)). Thus (1.25) does not hold. If $\alpha > 0$, then we deduce from (1.24) that

$$\alpha \leq \frac{2TL_1}{1 - 2TL}\beta,$$

and choosing c small so that

$$c_1 = c\left[M + \frac{2TL_1M_1}{1 - 2TL} \right] < 1,$$

we obtain the inequality $0 \leq \beta \leq c_1\beta$, with $c_1 < 1$; hence $\beta = 0$. Putting $\beta = 0$ in the first of relations (1.24), we find that $\alpha \leq 2TL\alpha$, with $2TL < 1$, by (1.7). This contradicts our assumption that $\alpha > 0$. Theorem 2 is thereby proved.

THEOREM 3 (stability). *Under the conditions of Theorems* 1 *and* 2 (that is, with $\omega_3(\delta) = L_1\delta$, $\pi_3(\gamma) = M_1\gamma$, and $L_1, M_1 \geq 0$) *the unique solution* $\Phi(x, y)$, $\Psi(x, y)$, $m(y)$ *of* (1.8) - (1.12) *depends continuously on* $\sigma(x)$ *and* $\tau(y)$.

Proof. We prove the theorem for the strip $A' = [0 \leq x \leq T,\ 0 \leq y \leq c]$, with $c = a/k$ sufficiently small (k is an integer). By repeating the argument for the strips $[0 \leq x \leq T,\ nc \leq y \leq (n+1)c]$ ($n = 1, 2, \cdots, k-1$, $n = -1, -2, \cdots, -k$), one shows the continuous dependence of $Z(x, y) = \begin{bmatrix} \Phi(x, y) \\ \Psi(x, y) \end{bmatrix}$ on $\sigma(x)$ and $\tau(y)$.

Let $\sigma_1(x)$, $\tau_1(y)$ and $\sigma_2(x)$, $\tau_2(y)$ be two pairs of functions as in Theorems 1 and 2, and let $\Phi_1(x, y)$, $\Psi_1(x, y)$, $\Phi_2(x, y)$, $\Psi_2(x, y)$, $m_1(y)$, $m_2(y)$ be the corresponding solutions of (1.8) - (1.12). Let

$$\varepsilon = \sup|\sigma_1(x) - \sigma_2(x)| + \sup|\tau_1(y) - \tau_2(y)|,$$

$$\alpha = \sup|\Phi_1(x, y) - \Phi_2(x, y)|,$$

$$\beta = \sup|\Psi_1(x, y) - \Psi_2(x, y)|,$$

where the supremum is taken over $0 \leq x \leq T$, $0 \leq y \leq c$. We obtain the relations

$$|m_1(y) - m_2(y)| \leq L\alpha + L_1\beta,$$

$$|\Phi_1(x, y) - \Phi_2(x, y)| \leq 2TL\alpha + 2TL_1\beta + \varepsilon,$$

$$|\Psi_1(x, y) - \Psi_2(x, y)| \leq cM\beta + cM_1\alpha + \varepsilon.$$

Thus

$$\alpha \leq 2TL\alpha + 2TL_1\beta + \varepsilon, \qquad \beta \leq cM\beta + cM_1\alpha + \varepsilon,$$

and hence, by (1.25), (1.7), and the inequality $c < a$,

$$\beta \leq \frac{cM_1}{1 - aM}\alpha + \frac{1}{1 - aM}\varepsilon. \tag{1.26}$$

The last relation and the first condition in (1.25) imply that

$$\alpha \leq \left[2LT + \frac{2TL_1M_1}{1 - aM} c \right]\alpha + \left[1 + \frac{2TL_1}{1 - aM} \right]\varepsilon. \tag{1.27}$$

Taking

$$\frac{a}{k} = c < \frac{(1 - aM)(1 - 2TL)}{2TL_1M_1} = M_2,$$

that is, $k > aM_2^{-1}$, we obtain the estimate

$$2TL + \frac{2TL_1M_1}{1 - aM} c = M_3 < 1,$$

and thus, by (1.27),

$$\alpha \leq \frac{1}{1 - M_3}\left(1 + \frac{2TL_1}{1 - aM}\right)\varepsilon = M_4\varepsilon. \tag{1.28}$$

Inequality (1.26) implies $\beta \leq M_5\varepsilon$, and from (1.28) it follows that

$$\alpha + \beta \leq (M_4 + M_5)\varepsilon.$$

This completes the proof of Theorem 3.

As in [2], we now specialize to the case where $m(y) \equiv 0$.

PROPOSITION 1. *Suppose that all the hypotheses of Theorem* 1 *hold, and that* $F(x, y, u, v)$, $G(x, y, u, v)$, $\sigma(x)$, $\tau(y)$ *are defined for* $|x| < \infty$, $|y| \leq a$, $|u| \leq P_1$, $|v| \leq P_2$. *Suppose that the relations*

$$F(x + T, y, u, v) = F(x, y, u, v), \qquad G(x + T, y, u, v) = G(x, y, u, v),$$

and

$$F(-x, y, u, v) = -F(x, y, u, v), \qquad \sigma(-x) = \sigma(x), \qquad G(-x, y, u, v) = G(x, y, u, v) \tag{1.29}$$

hold for $|x| < \infty$, $|y| \leq a$, $|u| \leq P_1$, $|v| \leq P_2$. *Then the conclusion of Theorem* 1 *holds with* $m(y) \equiv 0$.

Proof. Let K_1 consist of all vector functions in K that satisfy (1.13), (1.19), and the relations

$$\phi(-x, y) = \phi(x, y), \qquad \psi(-x, y) = \psi(x, y). \tag{1.30}$$

By virtue of the relations (1.29) and (1.30), the function

$$f(x, y) = F(x, y, \phi(x, y), \psi(x, y))$$

satisfies the relation $f(-x, y) = -f(x, y)$, and it follows that

$$m(y) = T^{-1} \int_0^T f(\xi, y) d\xi = 0 .$$

Furthermore, from (1.21) and (1.30) we see that $\Phi(-x, y) = \Phi(x, y)$ and $\Psi(-x, y) = \Psi(x, y)$. Thus τ maps K_1 into K_1, and the remainder of the proof of Theorem 1 remains unchanged.

2. PERIODIC SOLUTIONS OF THE TRICOMI PROBLEM

THEOREM 1*. *Let* T *be a positive constant, let* N_0, N_1, S_1, S_2, L, M, P_1, P_2 *be nonnegative constants such that*

$$N_1 + S_1 T \leq P , \qquad N_0 + S_2 T \leq P_2 ,$$

and denote by A^* *and* R^* *the intervals*

$$A^* = [0 \leq x \leq T,\ 0 \leq y \leq T], \qquad R^* = [0 \leq x \leq T,\ 0 \leq y \leq T,\ |u| \leq P_1,\ |v| \leq P_2].$$

Suppose $\sigma(x)$ *and* $\tau(y)$ *are* n- *and* m-*dimensional vector functions, continuous in* $0 \leq x \leq T$, $0 \leq y \leq T$, *and such that*

$$\sigma(T) = \sigma(0), \quad |\sigma(x)| \leq N_0 \quad (0 \leq x \leq T),$$

$$\tau(T) = \tau(0), \quad |\tau(y)| \leq N_1 \quad (0 \leq y \leq T).$$

Suppose $F(x, y, u, v)$ *and* $G(x, y, u, v)$ *are* m- *and* n-*dimensional vector functions, continuous in* R^*, *and satisfying the relations*

$$\text{(2.1)} \qquad \begin{cases} F(T, y, u, v) = F(0, y, u, v), & F(x, T, u, v) = F(x, 0, u, v), \\ G(T, y, u, v) = G(0, y, u, v), & G(x, T, u, v) = G(x, 0, u, v), \end{cases}$$

$$\text{(2.2)} \qquad |F(x, y, u, v)| \leq S_1, \qquad |G(x, y, u, v)| \leq S_2,$$

and

$$\text{(2.3)} \qquad \begin{cases} |F(x, y, \bar{u}, v) - F(x, y, u, v)| \leq L|\bar{u} - u|, \\ |G(x, y, u, \bar{v}) - G(x, y, u, v)| \leq M|\bar{v} - v| \end{cases}$$

for all $0 \leq x \leq T$, $0 \leq y \leq T$; $|u|$, $|\bar{u}| \leq P_1$; $|v|$, $|\bar{v}| \leq P_2$. *If* $2LT < 1$ *and* $2MT < 1$, *then for* $0 \leq x,\ y \leq T$ *there exist two vector functions* $\Phi(x, y)$ *and* $\Psi(x, y)$ *that are continuous, together with* Φ_x *and* Ψ_y, *and two continuous vector functions* $m(y)$ *and* $n(x)$, *such that*

$$\text{(2.4)} \qquad \Phi(0, y) = \Phi(T, y) = \tau(y), \qquad \Phi(x, 0) = \Phi(x, T),$$

$$\Psi(0, y) = \Psi(T, y), \quad \Psi(x, 0) = \Psi(x, T) = \sigma(x), \tag{2.5}$$

$$m(y) = T^{-1} \int_0^T F(\xi, y, \Phi(\xi, y), \Psi(\xi, y))\,d\xi, \tag{2.6}$$

$$n(x) = T^{-1} \int_0^T G(x, \eta, \Phi(x, \eta), \Psi(x, \eta))\,d\eta, \tag{2.7}$$

$$\Phi_x(x, y) = F(x, y, \Phi(x, y), \Psi(x, y)) - m(y), \tag{2.8}$$

$$\Psi_y(x, y) = G(x, y, \Phi(x, y), \Psi(x, y)) - n(x) \tag{2.9}$$

for all $(x, y) \in A^*$. *Thus, if we extend all functions* $\Phi(x, y)$, $\Psi(x, y)$, $m(y)$, $n(x)$, $F(x, y, u, v)$, $G(x, y, u, v)$ *to all of* $|x| < \infty$, $|y| < \infty$, $|u| \leq P_1$, $|v| \leq P_2$ *by means of* T*-periodicity in* x *and* y, *then the system* (2.8) - (2.9) *is satisfied in the whole* x,y-plane.

Proof. Note first that the relations (2.4), (2.5), (2.8), and (2.9) imply (2.6) and (2.7). Note also that (2.1), (2.4), (2.5), (2.6), and (2.7) imply that

$$m(0) = m(T), \quad n(0) = n(T).$$

(Thus the extension of $m(y)$ and $n(x)$ by T-periodicity in x and y is justified after we have proved the theorem.)

Remark 1*. If $\phi(x, y)$ and $\psi(x, y)$ satisfy the conditions

$$\begin{cases} \phi(0, y) = \phi(T, y) = \tau(y), \quad \phi(x, 0) = \phi(x, T), \\ \psi(0, y) = \psi(T, y), \quad \psi(x, 0) = \psi(x, T) = \sigma(x), \\ |\phi(x_1, y) - \phi(x_2, y)| \leq 2S_1 |x_1 - x_2|, \quad |\psi(x, y_1) - \psi(x, y_2)| \leq 2S_2 |y_1 - y_2| \end{cases} \tag{2.10}$$

for $0 \leq x, x_1, x_2, y, y_1, y_2 \leq T$, then one can prove, as in Section 1, that

$$|\phi(x, y)| \leq P_1, \quad |\psi(x, y)| \leq P_2$$

for all x and y $(0 \leq x, y \leq T)$.

On the other hand, there exist scalar functions

$$\omega_1(\alpha),\ \omega_2(\beta),\ \omega_3(\gamma),\ \omega_4(\alpha),\ \pi_1(\alpha),\ \pi_2(\beta),\ \pi_3(\gamma),\ \pi_4(\beta),$$

continuous and nondecreasing in $[0, \infty)$, such that

$$\omega_1(0) = \omega_2(0) = \omega_3(0) = \omega_4(0) = \pi_1(0) = \pi_2(0) = \pi_3(0) = \pi_4(0) = 0$$

and

$$(2.11)\quad \begin{cases} |F(x_1, y, u, v) - F(x_2, y, u, v)| \leq \omega_1(|x_1 - x_2|), \\ |F(x, y_1, u, v) - F(x, y_2, u, v)| \leq \omega_2(|y_1 - y_2|), \\ |F(x, y, u, v_1) - F(x, y, u, v_2)| \leq \omega_3(|v_1 - v_2|), \\ |G(x_1, y, u, v) - G(x_2, y, u, v)| \leq \pi_1(|x_1 - x_2|), \\ |G(x, y_1, u, v) - G(x, y_2, u, v)| \leq \pi_2(|y_1 - y_2|), \\ |G(x, y, u_1, v) - G(x, y, u_2, v)| \leq \pi_3(|u_1 - u_2|), \\ |\sigma(x_1) - \sigma(x_2)| \leq \omega_4(|x_1 - x_2|), \quad |\tau(y_1) - \tau(y_2)| \leq \pi_4(|y_1 - y_2|). \end{cases}$$

Let now

$$(2.12)\quad \begin{cases} \eta_1^*(\beta) = (1 - 2LT)^{-1}[\pi_4(\beta) + 2T\omega_2(\beta) + 2T\omega_3(2S_2\beta)], \\ \eta_2^*(\alpha) = (1 - 2MT)^{-1}[\omega_4(\alpha) + 2T\pi_1(\alpha) + 2T\pi_3(2S_1\alpha)]. \end{cases}$$

Then η_1^* and η_2^* are nondecreasing and continuous, and $\eta_1^*(0) = \eta_2^*(0) = 0$. Let E^* be the space described in Section 1, except that the interval A is replaced by the interval A^*. Let the norm in E be defined as in (1.18), with the difference that the supremum is taken on A^*. Let K^* consist of all elements of E^* satisfying (2.10) and

$$(2.13)\quad \begin{cases} |\phi(x, y_1) - \phi(x, y_2)| \leq \eta_1^*(|y_1 - y_2|), \\ |\psi(x_1, y) - \psi(x_2, y)| \leq \eta_2^*(|x_1 - x_2|). \end{cases}$$

Consider the map τ^*: $z(x, y) \to Z(x, y)$, where

$$z(x, y) = \begin{bmatrix} \phi(x, y) \\ \psi(x, y) \end{bmatrix} \quad \text{and} \quad Z(x, y) = \begin{bmatrix} \Phi(x, y) \\ \Psi(x, y) \end{bmatrix}$$

are defined by

$$(2.14)\quad \begin{cases} \Phi(x, y) = \tau(y) + \int_0^x \{F(\xi, y, \phi(\xi, y), \psi(\xi, y)) - m(y)\}\, d\xi, \\ \Psi(x, y) = \sigma(x) + \int_0^y \{G(x, \eta, \phi(x, \eta), \psi(x, \eta)) - n(x)\}\, d\eta, \end{cases}$$

with

$$m(y) = T^{-1} \int_0^T F(\xi, y, \phi(\xi, y), \psi(\xi, y))\, d\xi,$$

$$n(x) = T^{-1} \int_0^T G(x, \eta, \phi(x, \eta), \psi(x, \eta))\, d\eta.$$

Almost exactly as in the proof of Theorem 1, one can show that τ^* maps K^* into K^* and that τ^* is a continuous map in the norm of K^*. Since K^* is convex, closed, and compact, Theorem 1^* is a consequence of Schauder's fixed-point theorem.

Appropriate changes produce the following analogue to Proposition 1.

PROPOSITION 1^*. *Suppose that all the hypotheses of Theorem* 1^* *hold, and that* $F(x, y, u, v)$, $G(x, y, u, v)$, $\sigma(x)$, $\tau(y)$ *are defined for* $|x| < \infty$, $|y| < \infty$, $|u| \leq P_1$, $|v| \leq P_2$. *Suppose*

$$F(x+T, y, u, v) = F(x, y, u, v) = F(x, y+T, u, v),$$

$$G(x+T, y, u, v) = G(x, y, u, v) = G(x, y+T, u, v),$$

and

$$F(-x, y, u, v) = -F(x, y, u, v) = -F(x, -y, u, v),$$

$$G(-x, y, u, v) = G(x, y, u, v) = -G(x, -y, u, v),$$

$$\sigma(-x) = \sigma(x), \qquad \tau(-y) = \tau(y).$$

Then the statement of Theorem 1^* *holds with* $m(y) \equiv 0 \equiv n(x)$.

We leave the proof to the reader. It is similar to that of Proposition 1, with the difference that one considers the transformation τ^* on the subset K_1^* consisting of all pairs of functions $(\phi(x, y), \psi(x, y))$ that satisfy the conditions

$$\phi(-x, y) = \phi(x, y) = \phi(x, -y), \quad \psi(-x, y) = \psi(x, y) = \psi(x, -y).$$

We can now prove a uniqueness and a stability theorem similar to Theorems 2 and 3. Notice that by means of the principle of contractive maps we can easily prove the following existence and uniqueness theorem.

THEOREM 4^*. *Under the hypotheses of Theorem* 1^*, *suppose that* $\omega_3(\gamma) = L_1\gamma$ *and* $\pi_3(\delta) = M_1\delta$, *and let* $L^* = \max(L, L_1)$, $M^* = \max(M, M_1)$. *Then, if*

$$0 < 2T(L^* + M^*) < 1,$$

there is a unique solution $(\Phi(x, y), \Psi(x, y))$ *of* (2.4) - (2.9). *This solution is the (uniform) limit of the successive approximations given by*

$$\phi_p(x, y) = \tau(y) + \int_0^x \{F(\xi, y, \phi_{p-1}(\xi, y), \psi_{p-1}(\xi, y)) - m_{p-1}(y)\}\, d\xi,$$

$$\psi_p(x, y) = \sigma(x) + \int_0^y \{G(x, \eta, \phi_{p-1}(x, \eta), \psi_{p-1}(x, \eta)) - n_{p-1}(x)\}\, d\eta$$

$(p = 1, 2, \cdots)$, *where* $\phi_0(x, y) = \tau(y)$, $\psi_0(x, y) = \sigma(x)$, *and where*

$$m_p(y) = T^{-1}\int_0^T F(\xi, y, \phi_p(\xi, y), \psi_p(\xi, y))\, d\xi,$$

$$n_p(x) = T^{-1} \int_0^T G(x, \eta, \phi_p(x, \eta), \psi_p(x, \eta))\, d\eta .$$

3. EXTENSIONS

Let a and T be positive constants, and let S_1, S_2, M_1, M_2, P_1, P_2 be nonnegative constants such that

$$M_2 + 2S_1 T \leq P_1, \qquad M_1 + S_2 a \leq P_2. \tag{3.1}$$

Let $A = [0 \leq x \leq T,\ |y| \leq a]$, $R = [0 \leq x \leq T,\ |y| \leq a,\ |z| \leq P_1,\ |w| \leq P_2]$. Denote by (H_1) and (H_2) the following hypotheses.

(H_1) *The vector functions* $F(x, y, z, w)$ *and* $G(x, y, z, w)$ *are continuous in* R *and satisfy the conditions*

$$\begin{cases} |F(x, y, z_1, w_1) - F(x, y, z_2, w_2)| \leq \omega_1(|z_1 - z_2|) + \omega_2(|w_1 - w_2|), \\ |G(x, y, z_1, w_1) - G(x, y, z_2, w_2)| \leq \omega_3(|z_1 - z_2|) + \omega_4(|w_1 - w_2|), \end{cases} \tag{3.2}$$

and

$$|F(x, y, z, w)| \leq S_1, \qquad |G(x, y, z, w)| \leq S_2 \tag{3.3}$$

for $0 \leq x \leq T$, $|y| \leq a$; $|z|, |z_1|, |z_2| \leq P_1$; $|w|, |w_1|, |w_2| \leq P_2$, *where* $\omega_1(\alpha)$, $\omega_2(\beta)$, $\omega_3(\alpha)$, $\omega_4(\beta)$ *are continuous and nondecreasing scalar functions in* $[0, \infty)$, *with* $\omega_1(0) = \omega_2(0) = \omega_3(0) = \omega_4(0) = 0$. *Suppose also that*

$$F(T, y, z, w) = F(0, y, z, w), \qquad G(T, y, z, w) = G(0, y, z, w)$$

for $|y| \leq a$, $|z| \leq P_1$, $|w| \leq P_2$.

(H_2) *The vector function* $\sigma(x)$ *is continuous in* $0 \leq x \leq T$, *and* $\sigma(0) = \sigma(T)$. *The vector function* $\tau(y)$ *is continuous in* $|y| \leq a$, *and*

$$|\sigma(x)| \leq M_1, \qquad |\tau(y)| \leq M_2 \tag{3.4}$$

for $0 \leq x \leq T$, $|y| \leq a$.

Consider the problem

$$u_x = F(x, y, u, v) - m(y), \qquad v_y = G(x, y, u, v), \tag{3.5}$$

$$m(y) = T^{-1} \int_0^T F(\xi, y, u(\xi, y), v(\xi, y))\, d\xi, \tag{3.6}$$

$$u(0, y) = u(T, y) = \tau(y), \qquad v(x, 0) = \sigma(x), \qquad v(0, y) = v(T, y), \tag{3.7}$$

for $(x, y, u, v) \in R$.

Remark 1. Set $u_x(x, y) = p(x, y)$, $v_y(x, y) = q(x, y)$. Then two vector functions $u(x, y)$, $v(x, y)$, continuous in A, with partial derivatives u_x and v_y continuous in A, and satisfying (3.5), (3.6), and (3.7), have the property that $p(x, y)$, $q(x, y)$ satisfy

$$
(3.8)\quad \begin{cases} p(x, y) \\ \quad = F(x, y, (B_1 p)(x, y), (B_2 q)(x, y)) - T^{-1} \displaystyle\int_0^T F(\xi, y, (B_1 p)(\xi, y), (B_2 q)(\xi, y))\, d\xi, \\ q(x, y) = G(x, y, (B_1 p)(x, y), (B_2 q)(x, y)), \end{cases}
$$

where

$$
(3.9)\qquad (B_1 \theta)(x, y) = \tau(y) + \int_0^x \theta(\xi, y)\, d\xi,
$$

$$
(3.10)\qquad (B_2 \theta)(x, y) = \sigma(x) + \int_0^y \theta(x, \eta)\, d\eta .
$$

Conversely, if $(p(x, y), q(x, y))$ is a continuous solution of (3.8) such that $q(T, y) = q(0, y)$, and if we set $u(x, y) = (B_1 p)(x, y)$ and $v(x, y) = (B_2 q)(x, y)$, then $(u(x, y), v(x, y))$ is a solution of (3.5), (3.6), (3.7).

Remark 2. Taking into account the continuity of F and G under the hypothesis (H_1), we conclude that there are two scalar functions $\pi_1(\alpha)$, $\pi_2(\beta)$, continuous and nondecreasing in $[0, \infty)$, with $\pi_1(0) = \pi_2(0) = 0$, such that

$$
(3.11)\quad \begin{cases} |F(x_1, y, z, w) - F(x_2, y, z, w)| \le \pi_1(|x_1 - x_2|), \\ |F(x, y_1, z, w) - F(x, y_2, z, w)| \le \pi_2(|y_1 - y_2|), \\ |G(x_1, y, z, w) - G(x_2, y, z, w)| \le \pi_1(|x_1 - x_2|), \\ |G(x, y_1, z, w) - G(x, y_2, z, w)| \le \pi_2(|y_1 - y_2|), \\ |\sigma(x_1) - \sigma(x_2)| \le \pi_1(|x_1 - x_2|), \quad |\tau(y_1) - \tau(y_2)| \le \pi_2(|y_1 - y_2|) \end{cases}
$$

for $0 \le x, x_1, x_2 \le T$; $|y|, |y_1|, |y_2| \le a$; $|z| \le P_1$; $|w| \le P_2$.

Let

$$
(3.12)\quad \begin{cases} \Omega_1(\delta) = \pi_1(\delta) + \omega_1(2S_1\delta), \qquad \Omega_2(\delta) = 2[\pi_2(\delta) + \omega_2(S_2\delta)], \\ \Omega_3(\delta) = \pi_1(\delta) + \omega_3(2S_1\delta), \qquad \Omega_4(\delta) = \pi_2(\delta) + \omega_4(S_2\delta). \end{cases}
$$

THEOREM 5 (existence). *Suppose that hypotheses* (H_1) *and* (H_2) *hold and that, with the notation* (3.12), *the continuous solutions of the equations*

$$
\rho_1(t; \delta) = \Omega_1(\delta) + \omega_2\left[\pi_1(\delta) + \int_0^t \rho_3(\tau; \delta)\, d\tau\right] \qquad (t \in [0, a]),
$$

$$\rho_2(t;\,\delta) = \Omega_2(\delta) + \omega_1\left[\pi_2(\delta) + \int_0^t \rho_2(\tau;\,\delta)\,d\tau\right]$$

$$+\, T^{-1}\int_0^T \omega_1\left[\pi_2(\delta) + \int_0^\tau \rho_2(\tau_1;\,\delta)\,d\tau_1\right]d\tau \qquad (t \in [0,\, T]),$$

$$\rho_3(t;\,\delta) = \Omega_3(\delta) + \omega_4\left[\pi_1(\delta) + \int_0^t \rho_3(\tau;\,\delta)\,d\tau\right] \qquad (t \in [0,\, a]),$$

$$\rho_4(t;\,\delta) = \Omega_4(\delta) + \omega_3\left[\pi_2(\delta) + \int_0^t \rho_2(\tau;\,\delta)\,d\tau\right] \qquad (t \in [0,\, T])$$

satisfy the conditions $\lim_{\delta\to 0} \rho_i(t;\,\delta) = 0$ $(i = 1, 2, 3, 4)$, *uniformly with respect to* t.

Then there exists a solution $(u(x, y), v(x, y))$ *of* (3.5), (3.6), *and* (3.7), *continuous in* A *together with* u_x *and* v_y. (*Thus, extending this solution by* T-*periodicity in* x, *together with* F, G, *and* σ, *we obtain a periodic solution in the strip* $|x| < \infty$, $|y| \leq a$.)

Proof. Let E be the linear space of the continuous vector functions $\begin{bmatrix} p(x, y) \\ q(x, y) \end{bmatrix}$, where p and q are m- and n-dimensional continuous vector functions in A.

Let K be the subset of the elements of E satisfying

$$(3.13) \qquad |p(x, y)| \leq 2S_1, \quad |q(x, y)| \leq S_2, \quad q(T, y) = q(0, y), \quad \int_0^T p(\xi, y)\,d\xi \equiv 0$$

for $(x, y) \in A$, and

$$(3.14) \qquad \begin{cases} |p(x_1, y_1) - p(x_2, y_2)| \leq \rho_1(y_1,\, |x_1 - x_2|) + \rho_2(x_2,\, |y_1 - y_2|), \\ |q(x_1, y_1) - q(x_2, y_2)| \leq \rho_3(y_1,\, |x_1 - x_2|) + \rho_4(x_2,\, |y_1 - y_2|). \end{cases}$$

Let $\|\cdot\|$ be the norm of E, defined by

$$\left\|\begin{matrix} p(x, y) \\ q(x, y) \end{matrix}\right\| = \sup|p(x, y)| + \sup|q(x, y)|,$$

where the suprema are taken over $(x, y) \in A$. Define the map $\tau\colon \begin{bmatrix} p \\ q \end{bmatrix} \to \begin{bmatrix} P \\ Q \end{bmatrix}$ by

$$P(x, y) = F(x, y, (B_1 p)(x, y), (B_2 q)(x, y)) - \frac{1}{T}\int_0^T F(\xi, y, (B_1 p)(\xi, y), (B_2 q)(\xi, y))\,d\xi,$$

$$Q(x, y) = G(x, y, (B_1 p)(x, y), (B_2 q)(x, y)).$$

By (3.9), (3.10), (3.4), (3.13), and (3.1), we see that $|B_1 p| \leq M_2 + 2S_1 T \leq P_1$, $|B_2 q| \leq M_1 + aS_2 \leq P_2$, that is, $P(x, y)$ and $Q(x, y)$ are defined and continuous for

$\begin{bmatrix} p \\ q \end{bmatrix} \in K$. On the other hand, the set K is obviously convex and closed. By the theorem of Arzelà and Ascoli, it is also compact in the norm of E. Thus, if we prove that τ maps K into K and that τ is continuous in the norm of K, then Theorem 5 is a consequence of Schauder's fixed-point theorem applied to the map τ and the set K.

We prove first that τ maps K into K. For $\begin{bmatrix} p \\ q \end{bmatrix} \in K$, it follows from (3.9), (3.10), (3.11), (3.13), (H_1), and (H_2), that

$$\begin{aligned}
|P(x_1, y_1) - P(x_2, y_1)| &\leq \pi_1(|x_1 - x_2|) + \omega_1(2S_1|x_1 - x_2|) \\
&\quad + \omega_2\left[\pi_1(|x_1 - x_2|) + \left|\int_0^{y_1} \rho_3(\eta; |x_1 - x_2|)d\eta\right|\right] \\
&= \Omega_1(|x_1 - x_2|) + \omega_2\left[\pi_1(|x_1 - x_2|) + \left|\int_0^{y_1} \rho_3(\eta; |x_1 - x_2|)d\eta\right|\right] \\
&= \rho_1(y_1; |x_1 - x_2|),
\end{aligned}$$

$$\begin{aligned}
|P(x_2, y_1) - P(x_2, y_2)| &\leq \Omega_2(|y_1 - y_2|) \\
&\quad + \omega_1\left[\pi_2(|y_1 - y_2|) + \int_0^{x_2} \rho_2(\xi; |y_1 - y_2|)d\xi\right] \\
&\quad + T^{-1}\int_0^T \omega_1\left[\pi_2(|y_1 - y_2|) + \int_0^{\xi} \rho_2(\xi_1; |y_1 - y_2|)d\xi_1\right]d\xi \\
&= \rho_2(x_2; |y_1 - y_2|).
\end{aligned}$$

$$\begin{aligned}
|Q(x_1, y_1) - Q(x_2, y_1)| &\leq \Omega_3(|x_1 - x_2|) \\
&\quad + \omega_4\left[\pi_1(|x_1 - x_2|) + \left|\int_0^{y_1} \rho_3(\eta; |x_1 - x_2|)d\eta\right|\right] = \rho_3(y_1; |x_1 - x_2|),
\end{aligned}$$

$$\begin{aligned}
|Q(x_2, y_1) - Q(x_2, y_2)| &\leq \Omega_4(|y_1 - y_2|) \\
&\quad + \omega_3\left[\pi_2(|y_1 - y_2|) + \int_0^{x_2} \rho_2(\xi; |y_1 - y_2|)d\xi\right] = \rho_4(x_2; |y_1 - y_2|).
\end{aligned}$$

Thus,

$$|P(x_1, y_1) - P(x_2, y_2)| \leq \rho_1(y_1; |x_1 - x_2|) + \rho_2(x_2; |y_1 - y_2|),$$

$$|Q(x_1, y_1) - Q(x_2, y_2)| \leq \rho_3(y_1; |x_1 - x_2|) + \rho_4(x_2; |y_1 - y_2|).$$

On the other hand,

$$|P(x, y)| \leq 2S_1, \qquad |Q(x, y)| \leq S_2, \qquad Q(T, y) = Q(0, y), \qquad \int_0^T P(\xi, y)d\xi \equiv 0$$

for $\begin{bmatrix} p \\ q \end{bmatrix} \in K$, and hence τ maps K into K. Now, if $Z_i = \begin{bmatrix} P_i \\ Q_i \end{bmatrix}$, $z_i = \begin{bmatrix} p_i \\ q_i \end{bmatrix}$ $(i = 1, 2)$, then

$$\|Z_1 - Z_2\|$$
$$\leq 2\omega_1(T\|z_1 - z_2\|) + 2\omega_2(a\|z_1 - z_2\|) + \omega_3(T\|z_1 - z_2\|) + \omega_4(a\|z_1 - z_2\|).$$

This shows that τ is continuous.

COROLLARY. *Suppose that* $\omega_1(\delta) = L\delta$, $\omega_4(\delta) = M\delta$, *where* L, M *are positive constants, and that*

$$e^{LT} < 1 + 2LT. \tag{3.15}$$

Then the second hypothesis of Theorem 5 *holds, and thus there exists a solution of* (3.5), (3.6), (3.7) *in the strip* $|x| < \infty$, $|y| \leq a$.

We do not give the proof here, because it is a simple transposition of the proof given by A. K. Aziz [1, Corollary 2.1, page 564]. We merely remark that while the single restriction on the constant M is that it be positive, condition (3.15) on L is less restrictive than condition (1.7).

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