SPECIFIED RELATIONS IN THE IDEAL GROUP

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Introduction. Let $J = \sum Z x_i$ be the free abelian group based on the set $\{x_i\}$. We shall say that a subset I of elements of J satisfies condition α provided (i) all coefficients occurring on elements of I are nonnegative, (ii) to each finite subset $x_1, \cdots, x_k$ of $\{x_i\}$ and each finite set of nonnegative integers $n_1, \cdots, n_k$ there corresponds an element of I whose coefficient on x_i is n_i.

If A is a Dedekind domain and J is the divisor group of A (the free abelian group based on the primes of A), then the set I of integral principal divisors satisfies (i) by definition, and the weak-approximation theorem says that I satisfies (ii). Thus I satisfies condition α.

The main result of this paper provides a converse of the weak-approximation theorem (at least for the case where J has a countably infinite base). We shall prove a slight refinement (see Theorem 2.1) of the following assertion: If $J = \sum Z x_i$ is the free abelian group based on a countably infinite set $\{x_i\}$ and I is a subset of J that satisfies condition α, then there exists a Dedekind domain A such that the primes of A are in correspondence with the x_i in such a way that the principal divisors of A correspond to the elements of the subgroup generated by I.

This result fails for free groups of larger cardinality (we need a stronger hypothesis on I than condition α, and the proofs require transfinite techniques in almost every phase).

Section 1 of the present paper is devoted to some lemmas that are basically refinements of a technique, due to Goldman [3], for producing discrete valuations of specified types. In Section 2 we use these lemmas to give a proof of the theorem indicated above.

In Section 3 we give applications of the main theorem; we produce examples of 1) a Dedekind domain whose class group is cyclic of order n and all of whose prime ideals fall into one class, 2) a Dedekind domain A whose class group is isomorphic to Z, and with the property that each proper overring of A is a principal ideal domain, and 3) a Dedekind domain that is not an overring of the integral closure of a principal ideal domain (see [1, Example 1-9 and Remark 1-10 on page 61]).

Finally, in Section 4, we show that we can realize any finitely or countably infinitely generated class group by a Dedekind domain with finite residue class fields and unit group ± 1 (that is, in Goldman's sense, by a "special" Dedekind domain).

1. Throughout this section, A denotes a principal ideal domain subject to the four conditions

(1) A is countable,

(2) A has an infinite number of prime ideals,

(3) A/P is finite for all prime ideals $P \neq (0)$,

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(4) the characteristic of A is 0.

We denote the quotient field of A by F; if $P = \pi A$ is a prime ideal of A, we denote the P-adic completion of F by F_π, and the integral elements of F_π by A_π. By v_π we denote the P-adic valuation of F_π (or F).

LEMMA 1-1. *Let* A *be a principal ideal domain satisfying* (1), (2), (3), *and* (4). *Let* π *be a prime element of* A, *and let* n *be a nonnegative integer. Then there are infinitely many maximal ideals* M *of* A[X] *such that* M *is the center on* A[X] *of a valuation* w *with valuation ring* W *satisfying the three conditions* a) $W \supseteq A[X]$, b) *the residue field of* W *is finite, and* c) $w(\pi) = n$.

Proof. We treat first the case $n > 0$. Choose any monic irreducible element $\bar{g}(X)$ of $A/\pi[X]$, and let $g(X)$ be a monic pre-image of $\bar{g}(X)$. Set $M = (\pi, g(X))$.

Since A is countable, the usual cardinality argument shows that there exist elements of F_π that are transcendental over F. Multiplying by an appropriate integral power of π, we can produce an element t in F_π such that t is transcendental over F and $v_\pi(t) = 1$. Adjoin to F_π a root y of $g(X)^n - t$, and set $K = F(y)$. We can determine an isomorphism α: $F(X) \to K$ by setting $\alpha(r(X)) = r(y)$ for $r(X)$ in $F(X)$. Let v denote the (unique) extension of v_π to K, and define a valuation w on $F(X)$ by setting $w(r(X)) = v(\alpha(r(X)))$ for $r(X)$ in $F(X)$.

We shall show that w satisfies the required conditions.

Since $g(X)$ is monic, $g(X)^n - t$ is monic in $F_\pi[X]$, and so y is integral over A_π. This shows that $w(X) = v(y) \geq 0$, and clearly $w(A) = v(A) \geq 0$, so that $W \supseteq A[X]$. The center of w on A[X] contains π and $g(X)$; since $(\pi, g(X))$ is a maximal ideal, it must be the center of w on A[X]. The residue field of v is finite, and therefore the residue field of w is certainly finite.

To see that $w(\pi) = n$, let e denote the reduced ramification index, and f the relative degree of v over v_π. Certainly, $[K: F_\pi] \leq n \deg g(X)$. Since $ef \leq [K: F_\pi]$, we get the inequality $ef \leq n \deg g(X)$. If $\bar{y}$ denotes the image of y in the residue field of v, then $\bar{g}(\bar{y}) = 0$. Since $\bar{g}(X)$ is irreducible, we see that

$$f \geq \deg \bar{g}(X) = \deg g(X).$$

From the relation $g(y)^n = t$ (in K) we deduce that $nv(g(y)) = v(t)$. Thus $e \geq n$. We conclude that $f = \deg g(X)$ and $e = n$. Therefore $w(\pi) = v(\pi) = e\, v_\pi(\pi) = e = n$.

To handle the case where $n = 0$, we proceed as follows. Choose any maximal ideal $N = (\sigma, f(X))$ with σ in A relatively prime to π. Apply the above procedure to produce a valuation w with center N on A[X] such that $w(\sigma) = 1$. Then certainly $w(\pi) = 0$.

LEMMA 1-2. *Let* A *be a principal ideal domain satisfying* (1), (2), (3), *and* (4). *Let* $h(X)$ *be a nonconstant irreducible polynomial of* A[X], *and* n *a nonnegative integer. Then there are infinitely many maximal ideals* M *of* A[X] *such that* M *is the center of a valuation* w *with valuation ring* W *satisfying the conditions* a) $W \supseteq A[X]$, b) *the residue field of* w *is finite, and* c) $w(h(X)) = n$.

Proof. Choose any prime element σ of A that divides neither the leading coefficient of $h(X)$ nor the discriminant of $h(X)$. Let $\bar{h}(X)$ denote the image of $h(X)$ in $A/\sigma A[X]$. Then $\bar{h}(X)$ factors into distinct irreducible factors, say $\bar{h}(X) = \bar{q}_1(X) \cdots \bar{q}_k(X)$. Set

$$\bar{g}_1(X) = \bar{q}_1(X) \quad \text{and} \quad \bar{g}_2(X) = \bar{q}_2(X) \cdots \bar{q}_k(X).$$

Then $\bar{h}(X) = \bar{g}_1(X)\bar{g}_2(X)$, with $\bar{g}_1(X)$ and $\bar{g}_2(X)$ relatively prime. Using Hensel's lemma, we factor $h(X)$ over $A_\sigma[X]$ as $h(X) = G_1(X)G_2(X)$, where

$$\deg G_i = \deg g_i \quad \text{and} \quad \bar{G}_i(X) = \bar{g}_i(X) \quad (i = 1, 2).$$

Assume for the moment that $n > 0$. Let $g_1(X)$ and $g_2(X)$ denote elements of $A[X]$ such that $\deg g_i = \deg G_i$ for $i = 1, 2$ and $g_i(X) \equiv G_i(X)$ modulo σ^{n+1} for $i = 1, 2$. Let t be an element of F_σ such that t is transcendental over F and $v_\sigma(t) = n$. Adjoin to F_σ a root y of $g_1(X) - t$, and let $K = F_\sigma(y)$. As in Lemma 1-1, let v denote the extension of v_σ to K. We have an isomorphism α: $F(X) \rightarrow K$ given by $\alpha(r(X)) = r(y)$ for $r(X)$ in $F(X)$, and we define a valuation w on $F(X)$ by setting $w(r(X)) = v(\alpha(r(X)))$. Set $M = (\sigma, g_1(X))$.

We shall show that w satisfies the conditions of the lemma. The element $g_1(X)$ need not be monic; but since its leading coefficient is a unit in A_σ, y is integral over A_σ. Thus $w(X) = v(y) \geq 0$, and again it is clear that $w(A) = v(A) \geq 0$; therefore $W \supseteq A[X]$. The center of w on $A[X]$ contains σ and $g_1(X)$; since $(\sigma, g_1(X))$ is a maximal ideal, it must be the center of w on $A[X]$. The residue class field of w is again finite.

Finally we show that $w(h(X)) = n$. Let e denote the reduced ramification index, and f the relative degree of v over v_σ. We have the inequality $[K: F_\sigma] \leq \deg g_1(X)$, and hence $ef \leq \deg g_1(X)$. If $\bar{y}$ denotes the image of y in the residue field of v, then $\bar{g}_1(\bar{y}) = 0$. Since $\bar{g}_1(X)$ is irreducible, $f \geq \deg \bar{g}_1(X) = \deg g_1(X)$. Thus $e = 1$ and $f = \deg g_1(X)$. Since $g_1(y) = t$ in K, we find that

$$w(g_1(X)) = v(g_1(y)) = v(t) = ev_\sigma(t) = n.$$

Since $\bar{g}_1(X)$ and $\bar{g}_2(X)$ are relatively prime, $w(g_2(X)) = 0$. From the relation $h(X) \equiv g_1(X)g_2(X)$ modulo σ^{n+1} we deduce that $w(h(X)) = n$.

The case where $n = 0$ can be treated as in the proof of Lemma 1-1.

Remark. If w is a valuation produced by Lemma 1-1, then there exists a monic polynomial $f(X)$ in $A[X]$ such that $M = (\pi, f(x))$ and $w(f(X)) = 1$. In fact, in the notation of the proof of that lemma, the polynomial $f(X) = g(X)$ will do.

Also, if w is a valuation produced by Lemma 1-2, then there exists a monic polynomial $f(X)$ in $A[X]$ such that $M = (\sigma, f(X))$ and $w(f(X)) = 1$. In the notation of the proof of that lemma, let $g_1(X) = a_k X^k + \cdots + a_0$. Since σ does not divide a_k, we can choose b in A so that $ba_k = 1 + c\sigma$ for some c in A. Let

$$g(X) = bg_1(X) - c\sigma X^k.$$

Then $M = (\sigma, g(X))$. Set

$$f(X) = \begin{cases} g(X) & \text{if } w(g(X)) = 1, \\ g(X) + \sigma & \text{if } w(g(X)) > 1. \end{cases}$$

LEMMA 1-3. *Let* $w_1, \cdots, w_g$ *be valuations with distinct centers, each of which is produced as in* Lemma 1-1 *or* Lemma 1-2. *Then, for any nonnegative integers* $n_1, \cdots, n_g$, *there exists a monic irreducible polynomial* $p(X)$ *of* $A[X]$ *such that* $w_i(p(X)) = n_i$ *for* $i = 1, \cdots, g$.

Proof. Let w be one of the valuations, M its center on $A[X]$, and τ its center on A. By the remark, we can choose an $f(X)$ such that $w(f(X)) = 1$ and $M = (\tau, f(X))$.

Label the remaining centers so that $f(X)$ is in $M_1, \cdots, M_a$ while $f(X)$ is not in $M_{a+1}, \cdots, M_b$. Notice that τ cannot be in $M_1, \cdots, M_a$. Let $t(X)$ be in $\bigcap_{j=a+1}^{b} M_j$ but not in $\bigcup_{i=1}^{a} M_i$. Let $c = \deg t(X)$, and let $d(X)$ be a monic polynomial of degree greater than c that is not in the center of any w_i $(i = 1, \cdots, g)$. Consider the polynomial $q(X) = d(X)f(X) + \tau^2 t(X)$. By our choices, $w(q(X)) = 1$, but $q(X)$ has value 0 for any of the remaining valuations. Clearly, $q(X)$ is monic.

This shows that we can produce monic polynomials $q_1(X), \cdots, q_g(X)$ such that $w_i(q_j(X)) = \delta_{ij}$. Set $P(X) = \prod_{i=1}^{g} q_i(X)^{n_i}$. Then $P(X)$ is a monic polynomial such that $w_i(P(X)) = n_i$ for $i = 1, \cdots, g$.

Now let $\pi_1, \cdots, \pi_g$ be the centers on A of $w_1, \cdots, w_g$ (the π_i may not be distinct), let δ be the product $\pi_1 \cdots \pi_g$, and let $n = \max_{1 \leq i \leq g} n_i$. Choose a prime element ε of A outside the set $\{\pi_1, \cdots, \pi_g\}$, and let $\bar{g}(X)$ be a monic irreducible polynomial over $A/\varepsilon A[X]$ of the same degree as $P(X)$. Now choose a monic $p(X)$ in $A[X]$ such that

$$p(X) \equiv P(X) \ (\delta^{n+1} A[X]), \qquad p(X) \equiv g(X) \ (\varepsilon A[X]).$$

Then $p(X)$ is irreducible, by the second congruence, and it has the same values for $w_1, \cdots, w_g$ as $P(X)$, by the first congruence.

2. We now give a precise formulation of the main result.

THEOREM 2.1. *Let* $J = \sum_1^{\infty} Z x_i$ *be a free, countably generated abelian group, and let* I *be a subset of* J *satisfying condition* α. *Then there exists a Dedekind domain* B *whose prime ideals are in correspondence with the generators* x_i *in such a way that the principal divisors of* A *correspond to the elements of the subgroup of* J *generated by* I. B *may be chosen so that it is countable and of characteristic* 0, *and so that all its residue class fields are finite.*

Proof. Let A be a principal ideal domain satisfying conditions (1), (2), (3), and (4) of Section 2. Choose one representative from each set of associated irreducible elements of $A[X]$, and list these in some order $t_1, t_2, \cdots$.

For convenience, let J_n denote the subgroup of J that is generated by the elements of I whose coefficients on x_j are 0 for $j > n$. Note that J_n is finitely generated; in fact, J_n can be generated by a finite number of elements of I.

To begin the construction, we use the appropriate one of Lemma 1-1 or 1-2 to produce a valuation w_1 such that $w_1(t_1) = m_1 > 0$. There will be an element $i_0 = m_1 x_1 + \cdots + m_u x_u$ $(m_u > 0)$ of I. Again using either Lemma 1-1 or 1-2, we produce valuations $w_2, \cdots, w_u$ with distinct centers on $A[X]$ such that $w_i(t_1) = m_i$ for $i = 1, 2, \cdots, u$. There exist elements $i_1, \cdots, i_k$ of I that, together with i_0, generate J_u. By Lemma 1-3 we can choose irreducibles $t_{n_1}, \cdots, t_{n_k}$ from the list in such a way that the value of t_{n_j} under w_g gives the coefficient on x_g of i_j, for $j = 1, \cdots, k$ and $g = 1, \cdots, u$.

Now choose the first element in the list that is not in the set $\{t_1, t_{n_1}, \cdots, t_{n_k}\}$, and call it t_2'. Let the value of t_2' for w_i be n_i $(i = 1, \cdots, u)$. The set I contains an element

$$i_0' = \sum_{i=1}^{u} n_i x_i + \sum_{j=u+1}^{v} n_j x_j \qquad (n_v > 0).$$

Again, using Lemma 1-1 or 1-2, produce valuations $w_{u+1}, \cdots, w_v$ such that

(1) $w_j(t_2') = n_j$ for $j = u+1, \cdots, v$,

(2) the centers of $w_{u+1}, \cdots, w_v$ are distinct from each other and the centers of $w_1, \cdots, w_u$,

(3) none of the centers of $w_{u+1}, \cdots, w_v$ contains any of the elements $t_1, t_{n_1}, \cdots, t_{n_k}$ (this is possible, since $A[X]$ is a Hilbert ring; see [3]).

Now choose elements $i_{k+1}, \cdots, i_h$ of I that, together with $i_0, i_1, \cdots, i_k, i_0'$, generate J_v. By Lemma 1-3, we can select irreducibles $t_{n_{k+1}}, \cdots, t_{n_h}$ from our list whose values under the w_j give the coefficients on x_j of $i_{k+1}, \cdots, i_h$, for $j = 1, \cdots, v$.

We continue this process, by induction, to obtain an infinite sequence $\{w_i\}_1^\infty$ of valuations; let W_i be the valuation ring of w_i. Set $B = \bigcap_{i=1}^\infty W_i$. Then B is the required Dedekind domain. If t_e is any irreducible in our list, then by the construction there exists an $e' \geq e$ such that $w_i(t_e) = 0$ if $i \geq e'$; this shows that each element of $A[X]$ has positive value for only a finite number of the w_i. B is therefore a Krull domain. Since the centers of distinct w's are distinct maximal ideals of $A[X]$, the prime ideals of height 1 of B are relatively prime to each other; therefore B is a Dedekind domain.

Let F be the quotient field of A. Each element f(X) of F(X) can be written as $f(X) = u \prod_{i=1}^\infty t_i^{n_i}$, where u is a unit in A and the n_i are integers, almost all 0. By the construction, each t_i gives a divisor corresponding to an element of I; this shows that the divisor of f(X) corresponds to an element from the subgroup of J generated by I. Also, the construction provides that every element of I is realizable from some f(X) in $F[X]$.

The remaining assertions concerning B are clear.

3. We apply Theorem 2-1 to obtain three examples.

Example 3-1. There exists a Dedekind domain B, having cyclic class group of order n, such that all the prime ideals of B are in one class.

Construction. Let $J = \sum_{i=1}^\infty Zx_i$, and let I consist of all elements $\sum m_i x_i$ of J such that $m_i \geq 0$ for all i and n divides $\sum m_i$. The set I satisfies condition α. By Theorem 2-1, we can produce B, using this J and I. Let P be some prime ideal of B. If Q is another prime ideal of B, then P^n and $P^{n-1}Q$ are both principal ideals. This shows that P and Q are in the same class.

Example 3-2. There exists a Dedekind domain B such that the class group of B is infinite cyclic, and such that if C denotes a generator of the class group of B, then every prime ideal is either in C or in C^{-1}.

Construction. Let $J = \sum Zx_i$, and let J_1 be the subgroup of J generated by all elements $\{x_n + x_{n+1}\}_1^\infty$. Let I consist of the elements of J_1 with nonnegative coefficients. I satisfies condition α. Let B realize J and I in accordance with Theorem 2-1. Let P_i be the prime ideal of B corresponding to x_i, for $i = 1, 2, \cdots$. Then $P_i P_{i+1}$ is a principal ideal for all $i \geq 1$; therefore, if C denotes the class containing P_1, then P_n is in C for odd n, and in C^{-1} for even n.

Remark. If B is a Dedekind domain produced as in Example 3-1 or 3-2, and B' is a proper overring of B, then B' is a principal ideal domain.

Example 3-3. There exists a Dedekind domain B such that if a is a nonunit in B, then (a) is not a semiprime ideal.

Construction. Let $J = \sum Z x_i$, and let J_1 be the subgroup of J generated by all elements $\{x_n + 2x_{n+1}\}_1^\infty$. Let I consist of the elements of J_1 with nonnegative coefficients. I satisfies condition α. Let B realize J and I in accordance with Theorem 2-1. Since the last nonzero coefficient of every nonzero element of J_1 is even, no nonunit of B generates a semiprime ideal.

Remark. Since we can choose B of characteristic 0, B cannot be an overring of the integral closure of a principal ideal domain A in a finite algebraic extension of the quotient field of A [2, Proposition 2-8, p. 803].

4. The construction involved in Theorem 2-1 gives us no control over the units; we now rectify this situation.

THEOREM 4-1. *Let* A *be a principal ideal domain satisfying conditions* (1), (2), (3), *and* (4) *of Section* 2. *Let* C *be a finitely or countably infinitely generated abelian group. Then there exists a Dedekind domain* B *whose class group is isomorphic to* C *and whose units are the same as those of* A.

Proof. We can write $C \simeq R_1/K_1$, where R_1 is a free group with a countably infinite number of generators. We can choose K_1 free on a basis S_1; by changing the basis for R_1, if necessary, we may assume that all coefficients occurring on elements of S_1 are nonnegative. (To see this, let $R_1 = \sum_1^\infty Z x_i$; we know that we may choose a basis $\{w_i\}$ for K, where $w_i = \sum_{j=1}^{i} m_j x_j$.) We shall show by induction that, changing x_i to $-x_i$ if necessary, and making elementary transformations on the w's, we can obtain the result. Suppose that by such alterations $x_1, \cdots, x_k$ have been changed to $Y_1, \cdots, Y_k$, and that $w_1, \cdots, w_k$ have been changed to $W_1, \cdots, W_k$ in such a way that in each W_i $(1 \leq i \leq k)$ each coefficient occurring on X_i $(1 \leq i \leq k)$ is nonnegative. Let

$$w_{k+1} = n_{k+1} x_{k+1} + \sum_{j \leq k} n_j x_j .$$

If $n_{k+1} < 0$, set $X_{k+1} = -x_{k+1}$; otherwise, set $X_{k+1} = x_{k+1}$. If n_j $(1 \leq j \leq k)$ is negative and Y_j has coefficient 0 in all W_i for $1 \leq i \leq k$, set $X_j = -Y_j$. If there is a W_a $(1 \leq a \leq k)$ in which the coefficient on Y_j is positive, then add a suitably large multiple of W_a to w_{k+1}.

Order all the elements of R_1 with nonnegative coefficients as $r_1, r_2, \cdots$. Include R_1 in the larger free group $R_2 = R_1 \oplus \sum_1^\infty Z y_i$. Augment S_1 to S_2 by adding to S_1 all elements of the form $y_i + r_i$. Let K_2 be the subgroup of R_2 generated by all elements of S_2. Then clearly $R_2/K_2 \simeq R_1/K_1$. Continue this process to produce R_n and S_n for each $n > 0$. Set

$$J = \bigcup_{n=1}^{\infty} R_n \quad \text{and} \quad I = \bigcup_{n=1}^{\infty} S_n .$$

Then I satisfies condition α and is free, and if K is the subgroup of J generated by I, then $J/K \simeq C$.

Now apply the construction of Theorem 2-1, using this J and I. It is clear that since the elements of I are free, we may make the irreducible elements of A[X] correspond to the elements of I.

Let F be the quotient field of A, and let f(X) be in F(X). Write $f(X) = u \prod_{i=1}^{\infty} t_i(x)^{n_i}$, where u is a unit of A and $t_i(X)$ is an irreducible element of A[X]. If s_i is the element of I to which $t_i(X)$ corresponds, then f(X) corresponds to $\sum_{i=1}^{\infty} n_i s_i$. If f(X) is a unit in B, then $n_i = 0$ for all i, since the s_i are free. We conclude that f(X) is a unit of A.

COROLLARY 4-2. *Let* C *be a finitely (or countably infinitely) generated abelian group. Then there exists a Dedekind domain* B *all of whose residue class fields are finite, whose unit group is* ± 1, *and whose class group is isomorphic to* C.

Proof. Apply Theorem 4-1 with A = Z.

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