STRUCTURE THEOREMS FOR REGULAR LOCAL NOETHER LATTICES

Kenneth P. Bogart

1. INTRODUCTION

The concept of a Noether lattice was introduced by R. P. Dilworth [2] as an abstration of the concept of the lattice of ideals of a Noetherian ring. A Noether lattice is a modular multiplicative lattice satisfying the ascending chain condition in which every element is a join of elements called principal elements. The principal elements are characterized by a pair of identities that are satisfied by the principal ideals of a ring. A generalization of Krull's principal-ideal theorem for Noether lattices states that the rank of a minimal prime containing a principal element is at most 1.

A Noether lattice is *local* if it has a unique proper maximal element. The definitions of dimension and rank carry over directly from Noetherian rings to Noether lattices. A local Noether lattice of dimension n is *regular* if its maximal element is a join of n principal elements. The structure of arbitrary regular local Noether lattices is closely related to a special class $\{RL_n\}$ of Noether lattices.

The elements of RL_n are those ideals of $F[x_1, \cdots, x_n]$ which are joins of products of the ideals $(x_1), (x_2), \cdots, (x_n)$. We show that RL_n is a sublattice of the lattice of ideals of $F[x_1, \cdots, x_n]$, and that it is a regular local Noether lattice. Our main results describe the relationship between $\{RL_n\}$ and arbitrary regular local Noether lattices as follows.

A local Noether lattice L *of dimension* n *is regular if and only if there exists a sublattice* L' *of* L *with the property that prime, primary, and principal elements in* L' *are, respectively, prime, primary, and principal in* L, *and* L' *is isomorphic to* RL_n.

A distributive regular local Noether lattice is isomorphic to one of the lattices RL_n.

In addition, we show that for $n \geq 2$, RL_n is not isomorphic to the lattice of ideals of any ring. In fact, an appropriate quotient sublattice of RL_2 provides an example of a Noether lattice for which the usual "converse" to Krull's principal-ideal theorem (a prime of rank 1 is a minimal prime of some principal ideal) does not hold.

2. PRELIMINARY DEFINITIONS AND RESULTS

The notation and terminology of this paper are the same as those of [2], with the exception that we use $\vee$ and $\wedge$ to denote the lattice operations, and $\leq$ to denote the lattice partical ordering, with $<$ reserved for proper inequality.

By a *multiplicative lattice* we mean a complete lattice L containing a unit element I and a null element 0, and provided with a commutative, associative, join-

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distributive multiplication for which I is an identity element. For each A, B in L, $A:B$ is the join of all X in L such that $XB \leq A$. An element $E \in L$ is *principal* if

$$(A \wedge B:E)E = AE \wedge B \qquad (\text{all } A, B \in L) \tag{2.1}$$

and

$$(A \vee BE):E = A:E \vee B \qquad (\text{all } A, B \in L). \tag{2.2}$$

We reserve the letters E, F, and H for principal elements.

Prime and primary elements are defined for Noether lattices just as prime and primary ideals are defined for rings. The usual theorems about the existence and uniqueness of primary decompositions hold for Noether lattices.

Let P be a prime element of a Noether lattice L. P has *rank* r if r is the maximum of the lengths of chains of distinct primes less than P. P has *dimension* d if d is the maximum of the lengths of chains of distinct proper primes greater than P. Let $A \in L$. Then A has *rank* r if r is the minimum of the ranks of its associated primes; A has *dimension* d if d is the maximum of the dimensions of its associated primes. If L is local, then dim (0) is finite and is called the *dimension* of L.

This paper uses an abstract version of Krull's intersection theorem [2]; restricted to a local Noether lattice with maximal element M, it states that $\bigwedge_k M^k = 0$. The following important lemma is an immediate consequence of the intersection theorem.

LEMMA 2.1. *If* L *is a local Noether lattice and* $A, B \in L$, *then* $AB = B$ *implies* $A = I$ *or* $B = 0$.

Proof. Let $A \neq I$. Then

$$B = A^k B \leq M^k \qquad (\text{all } k).$$

Thus $B \leq \bigwedge_k M^k = 0$, so that $B = 0$.

LEMMA 2.2. *Let* L *be a local Noether lattice. Then an element of* L *is principal if and only if it is join-irreducible.*

Proof. Clearly, join-irreducible elements are principal; so let E be a principal element in L ($E \neq 0$). Let $E = D_1 \vee \cdots \vee D_n$. Then, by equation (2.1),

$$D_i = (D_i:E)E \qquad (\text{all } i).$$

Thus

$$E = (D_1:E)E \vee \cdots \vee (D_n:E)E = (D_1:E \vee \cdots \vee D_n:E)E.$$

By Lemma 2.1,

$$(D_1:E) \vee \cdots \vee (D_n:E) = I.$$

Since L is local, there must exist a j such that $D_j:E = I$; but this implies that $E \leq D_j$, so that $E = D_j$.

COROLLARY 2.1. *Let* L *be the lattice of ideals of a local Noetherian ring* R. *Then the principal elements of* L *are precisely the principal ideals of* R.

Applying the Kurosh-Ore theorem to the dual lattice of a local Noether lattice, we obtain the following corollary.

COROLLARY 2.2. *Let* L *be a local Noether lattice, and let* $A \in L$. *Then any two minimal representations of* A *as a join of principal elements have the same number of principal elements.*

Of course, the usual replacement properties [1] of the Kurosh-Ore theorem follow also.

Some of the examples and proofs in this paper use computations with quotient sublattices $L/D = \{A \in L \mid A \geq D\}$. With the multiplication $A \circ B = AB \vee D$, L/D is a Noether lattice, and if F is a principal element in L, then $F \vee D$ is principal in L/D [2].

LEMMA 2.3. *If* L *is a local Noether lattice, then the principal elements of* L/D *are precisely the elements of the form* $D \vee E$, *where* E *is principal in* L.

Proof. Let E' be a principal element in L/D. There exist $E_1, \cdots, E_k$ in L such that

$$E' = D \vee E_1 \vee \cdots \vee E_k = (D \vee E_1) \vee \cdots \vee (D \vee E_k).$$

E' is principal in L/D, so that we can apply Lemma 2.2 to L/D to obtain a j such that $E' = D \vee E_j$.

If L is a local Noether lattice of dimension n, a set of n principal elements whose join is primary with respect to the maximal element of L is a *system of parameters*. A set of n principal elements whose join is the maximal element is a *regular system of parameters*, and if L has a regular system of parameters, L is a *regular local Noether lattice*.

3. EXAMPLES OF LOCAL NOETHER LATTICES

In the proof that every local ring has a system of parameters, the following lemma is often used. If $P_1, P_2, \cdots, P_n$ are prime ideals of a Noetherian ring R and A is an ideal of R not contained in any P_i, then there exists a principal ideal $(a) \leq A$ such that $(a) \not\leq P_i$ for all i [3, p. 12]. This lemma does not hold for Noether lattices, as the following example shows.

Let RL_2 be the set consisting of (0) and all the ideals of $F[x, y]$ (F a field) of the form

$$(x)^{i(1)}(y)^{j(1)} \vee \cdots \vee (x)^{i(n)}(y)^{j(n)}.$$

It is easily seen that RL_2 is closed under join and multiplication. We shall show that if

$$A = \bigvee_s (x)^{i(s)}(y)^{j(s)} \quad \text{and} \quad B = \bigvee_t (x)^{k(t)}(y)^{h(t)},$$

then

$$A \wedge B = \bigvee_{s,t} (\text{l.c.m.}(x^{i(s)}y^{j(s)}, x^{k(t)}y^{h(t)})).$$

Clearly, l. c. m. $(x^{i(s)}y^{j(s)}, x^{k(t)}y^{h(t)}) \in A \wedge B$; therefore, let $p(x, y)$ be in $A \wedge B$. Since $p(x, y) \in A$, there exist polynomials $p_s(x, y)$ such that

$$p(x, y) = \sum_s p_s(x, y)x^{i(s)}y^{j(s)}.$$

Thus each nonzero term of $p(x, y)$ is divisible by $x^{i(s)}y^{j(s)}$ for some s. Similarly, each term is divisible by $x^{k(t)}y^{h(t)}$ for some t. Therefore,

$$p(x, y) \in \bigvee_{s,t} (\text{l. c. m.}(x^{i(s)}y^{j(s)}, x^{k(t)}y^{h(t)})).$$

This shows that RL_2 is closed under meet. To show that RL_2 is closed under residuation, observe that since $A:(B \vee C) = A:B \wedge A:C$ and $A:(BC) = (A:B):C$, it is sufficient to show that $A:(x)$ is in RL_2 for all A in RL_2. We assume that $A':(x)$ is in RL_2 if A' is a join of fewer $(x)^i(y)^j$ than A. If $A = (y)^j$, then $A:(x) = (y)^j:(x) = (y)^j \in RL_2$, so that we may assume

$$A = (x)^i(y)^j \vee A' \qquad (i \geq 1).$$

Then, using equation (2.2), we find that

$$A:(x) = (x)^{i-1}(y)^j \vee A':(x);$$

the right-hand member is in RL_2, by the induction hypothesis. But now, since principal elements are defined by equations using meet, join, multiplication, and residuation, the elements $(x)^i(y)^j$ are principal in RL_2. Thus RL_2 is a Noether lattice.

We note also that RL_2 is distributive; for if

$$A = (a_1) \vee \cdots \vee (a_n), \qquad B = (b_1) \vee \cdots \vee (b_m), \qquad C = (c_1) \vee \cdots \vee (c_k)$$

are elements of RL_2, then

$$\begin{aligned} A \wedge (B \vee C) &= \bigvee \{\text{l. c. m.}(a_i, d_j) \mid d_i = b_i,\ i \leq m;\ d_i = c_{i-m},\ i > m\} \\ &= \bigvee_{i,j} (\text{l. c. m.}(a_i, b_j)) \vee \bigvee_{i,j} (\text{l. c. m.}(a_i, c_j)) = (A \wedge B) \vee (A \wedge C). \end{aligned}$$

It is clear that the only proper prime elements of RL_2 are $(x) \vee (y)$, (x), (y), and (0). However, by Lemma 2.2, the only principal elements in RL_2 are the elements $(x)^i(y)^j$. Thus every principal element of RL_2 is less than or equal to (x) or (y). Now, with $A = (x) \vee (y)$, $P_1 = (x)$, and $P_2 = (y)$, it is clear that $A \not\leq P_1$ and $A \not\leq P_2$, while every principal element contained in A is contained in either P_1 or P_2. Clearly, though, RL_2 has a system of parameters, and in fact it is regular.

Next we give an example of a local Noether lattice without a system of parameters. Let $L = RL_2/(x)(y)$. By Lemma 2.3, all principal elements of L are less than or equal to (x) or (y), since $(x)(y)$ is less than both (x) and (y). But since (x) and (y) are primes of rank 0, every principal element of L has rank 0. Thus $(x) \vee (y)$ is a prime of rank 1 containing no principal elements of rank 1, so that the "converse" to the Krull theorem does not hold for L and L has no system of parameters.

These examples show that RL_2 cannot be isomorphic to the lattice of ideals of any ring.

4. REGULAR LOCAL NOETHER LATTICES

The concept of a *Noether-lattice imbedding* is used in the main theorem of this section. Let L and L' be Noether lattices. We say that ϕ: L → L' is a *Noether-lattice imbedding* of L in L' if ϕ is an isomorphism of L into L' and the images under ϕ of prime, primary, and principal elements of L are prime, primary, and principal, respectively, in L'.

Recall that RL_n consists of all joins of products of the ideals (x_1), (x_2), $\cdots$, (x_n) in the lattice of ideals of $F[x_1, \cdots, x_n]$. As in the case of RL_2, it is easily verified that RL_n is a regular local Noether lattice that is not isomorphic to the lattice of ideals of any ring. Again, since the meet of two elements in RL_n is the join of the ideals generated by the least common multiples of their generators, we can apply the computation by which we showed that RL_2 is distributive to prove that RL_n is distributive. The main theorem states that if L is a local Noether lattice of dimension n, then L is a regular local Noether lattice if and only if there exists a Noether-lattice imbedding of RL_n in L.

In the proof of the main theorem, we shall use two theorems that are generalizations of well-known theorems [3, p. 73], [4, p. 303] about regular local rings. The following lemma and its corollary form the basis of the proof of these two theorems.

LEMMA 4.1. *Let* L *be a local Noether lattice, let* $A = E_1 \vee \cdots \vee E_r$ *be a member of* L, *and let* A *have dimension* s. *Then the dimension of* $E_1 \vee \cdots \vee E_{r-1}$ *is at most* $s + 1$.

Proof. Let $\dim(E_1 \vee \cdots \vee E_{r-1}) = s + i$. Then there exists a chain of primes

$$P_1 > \cdots > P_{s+i} > P_{s+i+1} \geq E_1 \vee \cdots \vee E_{r-1}.$$

Since E_r is a principal element, there exists (by Lemma 6.4 of [2]) a chain of primes

$$P_1 = P_1^* > P_2^* > \cdots > P_{s+i}^* > P_{s+i+1}$$

such that $P_{s+i}^* \geq E_r$. Since $P_{s+i}^* \geq A$ and A has dimension at least $s + i - 1$, we see that $s \geq s + i - 1$, which implies that $i \leq 1$.

COROLLARY 4.1. *Let* L *be a regular local Noether lattice, and let* $\{E_1, \cdots, E_n\}$ *be a regular system of parameters for* L. *Then*

$$\dim(E_1 \vee \cdots \vee E_k) = n - k.$$

Proof. Apply Lemma 4.1 $n - k$ times to show that $\dim(E_1 \vee \cdots \vee E_k)$ is at most $n - k$. Suppose $\dim(E_1 \vee \cdots \vee E_k)$ is $n - k - i$ $(i \geq 0)$. Then apply Lemma 4.1 k times to show that $\dim(0) \leq n - i$. Thus $n \leq n - i$, which implies $i = 0$.

The usual ring-theoretic proof [3, p. 75] shows that if L is a regular local Noether lattice of dimension one, then every element of L is of the form E^k, where E is the maximal element of L. Clearly, this implies that 0 is a prime in L; for if E^k were 0, then E would be contained in some prime of rank 0, contrary to the relation $\operatorname{rank}(E) = \dim(0) = 1$. The next theorem extends this remark, and its proof is a rather natural extension of the simple computation used to prove the remark.

THEOREM 4.1. *Let* L *be a regular local Noether lattice. Then any join of a subset of a regular system of parameters is a prime.*

Proof. Let $\{E_1, \cdots, E_n\}$ be a regular system of parameters for L. The proof uses induction on $n - r$ to show that 0 is a prime in $L/(E_1 \vee \cdots \vee E_r)$. By Lemma 4.1 and the remark above, 0 is a prime in $L/(E_1 \vee \cdots \vee E_{n-1})$.

Assume $E_1 \vee \cdots \vee E_r$ is a prime if $r > i$. Let

$$L' = L/(E_1 \vee \cdots \vee E_i),$$

and let X' denote $X \vee E_1 \vee \cdots \vee E_i$ for all X in L. By the induction hypothesis, E_j' is a prime in L' for all $j > i$. Now E_j' must contain a minimal prime of $0'$, for it is prime. By Corollary 4.1, L' has dimension $n - i$ and E_j' has dimension $n - i - 1$. By Lemma 6.4 in [2], there exists a chain

$$(E_1 \vee \cdots \vee E_n)' > P_1^* > \cdots > P_{n-i-1}^* > P_{n-i}'$$

in L' such that $P_{n-i-1}^* \geq E_j'$. Thus $E_j' = P_{n-i-1}^*$, and E_j' is not a minimal prime of $0'$.

Now let P' be a minimal prime of $0'$ contained in E_j'. Since $P' = (P' : E_j')E_j'$ and P' is prime, $P' : E_j' \leq P'$. Therefore, $P' = P' : E_j'$, and so $P' = P'E_j'$, which implies that $P' = 0'$, by Lemma 2.1. Therefore, $0'$ is a prime in L', and hence $E_1 \vee \cdots \vee E_i$ is a prime for all i.

THEOREM 4.2. *Let* L *be a regular local Noether lattice, and let* D *be an element of* L. *Then* L/D *is regular if and only if* D *is a join of a subset of a regular system of parameters.*

Proof. Corollary 4.1 implies that if L is a regular local Noether lattice and $\{E_1, \cdots, E_n\}$ is a regular system of parameters for L, then $L/(E_1 \vee \cdots \vee E_k)$ is regular.

Assume that L/D is regular, and let M be the maximal element of L. Then, by Lemma 2.3, $M = D \vee F_1 \vee \cdots \vee F_k$, where $\{D \vee F_1, \cdots, D \vee F_k\}$ is a regular system of parameters in L/D. Let $D = H_1 \vee \cdots \vee H_r$. Since

$$\{D \vee F_1, \cdots, D \vee F_k\}$$

is a regular system of parameters, we may assume, by renumbering the H_i and dropping superfluous ones, that

$$M = H_1 \vee \cdots \vee H_s \vee F_1 \vee \cdots \vee F_k$$

is a minimal representation of M as a join of principal elements. By Corollary 2.2, $s + k = \dim(L)$; therefore $\{H_1, \cdots, H_s\}$ is a subset of a regular system of parameters. Thus $H_1 \vee \cdots \vee H_s = D'$ is a prime. But rank $D' = s$ and

$$\dim(D) = k = \dim(L) - s,$$

so that rank $(D) \leq s$, which implies that $D = D'$.

In proving the main theorem, we shall use the fact that in a Noether lattice $A : B = A$ if and only if no associated prime of A contains B (this can be proved as for rings; see [3, p. 23]). We shall also need the following lemma (the symbol $\cong$ indicates lattice isomorphism).

LEMMA 4.2. *Suppose that* L *is a Noether lattice in which* 0 *is a prime, that* A, E $\in$ L (E *principal*), *and that* A : E = A. *Then*

$$(E \vee A)/(E^i \vee A) \cong I/(E^{i-1} \vee A).$$

Proof. In the relations

$$\begin{aligned}(E \vee A)/(E^i \vee A) &= [E \vee (E^i \vee A)]/(E^i \vee A) \cong E/[E \wedge (E^i \vee A)] \\ &= E/[E^i \vee (E \wedge A)] = E/[E^i \vee (A:E)E] = E/(E^i \vee EA) \\ &= E/[(E^{i-1} \vee A)E] \cong I/(E^{i-1} \vee A),\end{aligned}$$

the first isomorphism follows by modularity, the second by Lemma 6.3 of [2].

THEOREM 4.3. *Let* L *be a local Noether lattice of dimension* n. *Then* L *is a regular local Noether lattice if and only if there exists a Noether-lattice imbedding of* RL_n *in* L.

Proof. Clearly, if RL_n can be imbedded in L, the maximal prime of RL_n maps onto the maximal prime of L. Thus the maximal prime of L is a join of n principal elements, and L is regular.

Now assume that L is a regular local Noether lattice, and let $\{E_1, \cdots, E_n\}$ be a regular system of parameters for L. Define ϕ: $RL_n \to L$ by $\phi(0) = 0$ and

$$\phi\Big[\bigvee_j (x_1)^{i(j,1)}(x_2)^{i(j,2)} \cdots (x_n)^{i(j,n)}\Big] = \bigvee_j E_1^{i(j,1)} E_2^{i(j,2)} \cdots E_n^{i(j,n)}.$$

Note that RL_k may be considered as a subset of RL_n. We shall use induction to prove that ϕ is a Noether-lattice imbedding; in particular, we shall show that ϕ restricted to RL_k is a Noether-lattice imbedding of RL_k in L for $k \leq n$.

The restriction of ϕ to RL_1 is an isomorphism of RL_1 into L, since the image of ϕ is a regular local Noether lattice of dimension 1 and is therefore isomorphic to RL_1 (see the remark preceding Theorem 4.1). A simple inductive argument shows that E_1^i is primary; since E_1 is both principal and prime (Theorem 4.1), it follows that this isomorphism is a Noether-lattice imbedding.

Now assume that ϕ restricted to RL_j is a Noether-lattice imbedding of RL_j in L, for $j < k \leq n$. Denote the restriction of ϕ to RL_k by ϕ'. To show that ϕ' is an isomorphism, observe first that ϕ' preserves products and joins. It is evident that ϕ' preserves meets of principal elements, for

$$E_1^{i(1)} \wedge E_2^{i(2)} \wedge \cdots \wedge E_k^{i(k)} = E_1^{i(1)} E_2^{i(2)} \cdots E_k^{i(k)}.$$

Thus

$$E_1^{i(1)} E_2^{i(2)} \cdots E_k^{i(k)} \wedge E_1^{j(1)} E_2^{j(2)} \cdots E_k^{j(k)} = E_1^{m(1)} E_2^{m(2)} \cdots E_k^{m(k)},$$

where $m(t) = \max(i(t), j(t))$.

We shall use computations with residuations to show that ϕ' preserves arbitrary meets; but first we must prove that ϕ' preserves residuation in certain special cases. Let

$$A = \bigvee_h E_2^{i(h,2)} E_3^{i(h,3)} \cdots E_k^{i(h,k)}.$$

Then, by the induction hypothesis, the element A has a normal decomposition in which all the associated primes are contained in $E_2 \vee \cdots \vee E_k$, and since $E_1 \not\leq E_2 \vee \cdots \vee E_k$, $A : E_1 = A$.

Thus, in view of equation (2.2),

$$\phi'[B:(x_1)] = \phi'(B):\phi'[(x_1)].$$

Since $X:(YZ) = (X:Y):Z$, it follows that

$$\phi'(B:F) = \phi'(B):\phi'(F)$$

for all $B \in RL_k$ and all principal elements $F \in RL_k$. Also,

$$\phi'(B \wedge F) = \phi'[(B:F)F] = \phi'(B:F)\phi'(F) = [\phi'(B):\phi'(F)]\phi'(F) = \phi'(B) \wedge \phi'(F).$$

We shall now show that ϕ' preserves all meets. Since RL_k is distributive, $(A \vee B) \wedge F = (A \wedge F) \vee (B \wedge F)$ for all A, B, and principal elements F in RL_k. Then, in L,

$$\phi'[(A \vee B) \wedge F] = [\phi'(A) \wedge \phi'(F)] \vee [\phi'(B) \wedge \phi'(F)].$$

Now let $C = F_1 \vee \cdots \vee F_s \in RL_k$. Temporarily, let $\phi'(X) = X'$. Assume that

$$(F_1' \vee \cdots \vee F_r') \wedge D' = (F_1' \wedge D') \vee \cdots \vee (F_r' \wedge D'), \tag{4.1}$$

for all $r < s$ and for all D'. Then

$$\begin{aligned}
(F_1' \vee \cdots \vee F_s') \wedge D' &= (F_1' \vee \cdots \vee F_s') \wedge (F_s' \vee D') \wedge D' \\
&= \{F_s' \vee [(F_1' \vee \cdots \vee F_{s-1}') \wedge (F_s' \vee D')]\} \wedge D' \\
&= \{F_s' \vee [F_1' \wedge (F_s' \vee D')] \vee \cdots \vee [F_{s-1}' \wedge (F_s' \vee D')]\} \wedge D' \\
&= [F_s' \vee (F_1' \wedge F_s') \vee (F_1' \wedge D') \vee \cdots \vee (F_{s-1}' \wedge F_s') \vee (F_{s-1}' \wedge D')] \wedge D' \\
&= [F_s' \vee (F_1' \wedge D') \vee \cdots \vee (F_{s-1}' \wedge D')] \wedge D' \\
&= (F_s' \wedge D') \vee (F_1' \wedge D') \vee \cdots \vee (F_{s-1}' \wedge D').
\end{aligned}$$

This shows that equation (4.1) holds for all r. Now let $D = H_1 \vee \cdots \vee H_t$. Then

$$\begin{aligned}
\phi'(C) \wedge \phi'(D) &= (F_1' \vee \cdots \vee F_s') \wedge D' = (F_1' \wedge D') \vee (F_2' \wedge D') \vee \cdots \vee (F_s' \wedge D') \\
&= \bigvee_{i,j} (F_i' \wedge H_j') = \bigvee_{i,j} \phi'(F_i) \wedge \phi'(H_j) \\
&= \bigvee_{i,j} \phi'(F_i \wedge H_j) = \phi'\Big[\bigvee_{i,j} (F_i \wedge H_j)\Big] = \phi'(C \wedge D).
\end{aligned}$$

Now, since $X:(Y \vee Z) = (X:Y) \wedge (X:Z)$ and $\phi'(A:F) = \phi'(A):\phi'(F)$, ϕ' preserves residuation and is therefore a homomorphism.

But $\phi'(RL_k)$ is distributive, and therefore two elements are equal if and only if they are joins of exactly the same principal elements. But this implies that the mapping ϕ' is one-to-one, since it is clearly one-to-one on principal elements. Thus ϕ' is an isomorphism.

By Lemma 2.2, the only principal elements of RL_k are the elements $(x_1)^{i(1)}(x_2)^{i(2)}\cdots(x_k)^{i(k)}$; therefore ϕ' maps principal elements to principal elements. By Theorem 4.1, ϕ' maps primes to primes. To show that ϕ' is a Noether-lattice imbedding, we must show that it preserves primary elements. Since every meet-irreducible element is primary, since the intersection of primaries with the same associated prime is primary, and since every element is an intersection of meet-irreducible elements, it is sufficient to show that ϕ' preserves meet-irreducible elements.

Since RL_k is distributive, it is easy to see that the only meet-irreducible elements in RL_k are the elements of the form $(x_1)^{i(1)} \vee \cdots \vee (x_h)^{i(h)}$. Now, in L, the elements $E_1 \vee \cdots \vee E_s$ are meet-irreducible, since they are prime by Theorem 4.1. We shall use induction to show that the elements $E_1^{i(1)} \vee \cdots \vee E_s^{i(s)}$ are irreducible in L. Suppose that $E_1^{i(1)-1} \vee E_2^{i(2)} \vee \cdots \vee E_s^{i(s)}$ is irreducible in L. Then, by Lemma 4.2, $B = E_1^{i(1)} \vee \cdots \vee E_s^{i(s)}$ is irreducible in B_1/B, where $B_1 = E_1 \vee E_2^{i(2)} \vee \cdots \vee E_s^{i(s)}$. Now assume that $B = C_1 \wedge \cdots \wedge C_r$. Since $B = B \wedge B_1$,

$$B = (B_1 \wedge C_1) \wedge \cdots \wedge (B_1 \wedge C_r).$$

Since B is irreducible in B_1/B, there exists a j such that $B = B_1 \wedge C_j$, and therefore

$$B:E_1 = (B_1 \wedge C_j):E_1 = B_1:E_1 \wedge C_j:E_1 = C_j:E_1,$$

since $E_1 \leq B_1$. Thus

$$C_j \leq C_j:E_1 = B:E_1 = E_1^{i(1)-1} \vee E_2^{i(2)} \vee \cdots \vee E_s^{i(s)}.$$

But this implies that C_j is in B_1/B, hence $C_j = B$. Therefore B is meet-irreducible, and ϕ' preserves meet-irreducible elements. This implies that ϕ' is a Noether-lattice imbedding of RL_k in L. But now, by induction, ϕ is a Noether-lattice imbedding of RL_n in L.

5. DISTRIBUTIVE REGULAR LOCAL NOETHER LATTICES

In this section, we show that the lattices RL_n are the only distributive regular local Noether lattices.

Let L be a distributive regular local Noether lattice with regular parameters $E_1, \cdots, E_n$, and let $E \neq I$ be a nonzero principal element of L. Then

$$E = E \wedge (E_1 \vee \cdots \vee E_n) = (E \wedge E_1) \vee \cdots \vee (E \wedge E_n).$$

By Lemma 2.2, $E = E \wedge E_i$ for some i. Now let $E_i^{j(i)}$ be the highest power of E_i containing E. Then

$$E = (E : E_i^{j(1)}) E_i^{j(i)} = F_1 E_i^{j(i)} \vee \cdots \vee F_m E_i^{j(i)},$$

where $E : E_i^{j(i)} = F_1 \vee \cdots \vee F_m$. By Lemma 2.2, $E = F_k E_i^{j(i)}$ for some k. If $F_k \neq I$, then $F_k \leq E_s$ for some $s \neq i$. Thus, by iteration, E may be written

$$E = HE_1^{j(1)} E_2^{j(2)} \cdots E_n^{j(n)},$$

where H is principal and $E_h^{j(h)}$ is the highest power of E_h containing E. Thus $H \not\leq E_i$ for each i, so that $H = I$. Thus every principal element of L is a product of powers of the E_i. Define ϕ: $RL_n \to L$ as in Theorem 4.3; that is, let $\phi(x_i) = E_i$. Then ϕ is an isomorphism of RL_n into L; but since every element in L is a join of principal elements, ϕ is onto and L and RL_n are isomorphic. This proves the following theorem.

THEOREM 5.1. *A distributive regular local Noether lattice is isomorphic to one of the lattices* RL_n.

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California Institute of Technology
Pasadena, California 91109