# ON THE SYMPLECTIC BORDISM GROUPS OF THE SPACES Sp(n), HP(n), AND BSp(n)

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## 1. INTRODUCTION

Although the symplectic bordism ring $\Omega^{Sp} = \sum_{k=0}^{\infty} \Omega_k^{Sp}$ is largely undetermined, one may nonetheless investigate the symplectic bordism groups $\Omega_k^{Sp}(X)$ of spaces of importance in the study of symplectic bundles, manifolds, and so forth. In this paper, we carry out such a program for the symplectic groups Sp(n), their classifying spaces BSp(n), and the quaternionic projective spaces HP(n). We hope that the methods and results will lead to some measure of understanding of the symplectic bordism ring.

By a *symplectic manifold* we mean a compact smooth manifold together with a reduction of the structure group of its stable tangent bundle to the symplectic group. The *symplectic bordism group* $\Omega_k^{Sp}(X)$ of a space X consists of the bordism classes of pairs $(M^k, f)$, where $M^k$ is a closed symplectic k-manifold and $f: M^k \to X$ is a map; the bordism class of $(M^k, f)$ is denoted by $[M^k, f]$ (see [2], [3]). $\Omega^{Sp}(X) = \sum_{k=0}^{\infty} \Omega_k^{Sp}(X)$ is a graded left module over the symplectic bordism ring $\Omega^{Sp} = \Omega^{Sp}(\text{point})$; put $[V^j][M^k, f] = [V^j \times M^k, F]$ with $F(v, m) = f(m)$.

There is a "fundamental class" homomorphism

$$\mu: \Omega_k^{Sp}(X) \to H_k(X)$$

from symplectic bordism to integral homology; $\mu[M^k, f] = f_*(\sigma_M)$, where $\sigma_M \in H_k(M^k)$ is the orientation class of the manifold. As in [2, p. 49] one may establish the following lemma.

1.1. LEMMA. *Let* X *be a* CW-*complex such that* $H_*(X)$ *is free abelian and* $\mu: \Omega^{Sp}(X) \to H_*(X)$ *is an epimorphism. Then* $\Omega^{Sp}(X)$ *is a free module over* $\Omega^{Sp}$. *Moreover, if* $\{c_i\}$ *is a homogeneous basis for* $H_*(X)$ *and* $\{\gamma_i\}$ *are homogeneous elements of* $\Omega^{Sp}(X)$ *with* $\mu(\gamma_i) = c_i$, *then* $\{\gamma_i\}$ *is a basis for* $\Omega^{Sp}(X)$ *as a free* $\Omega^{Sp}$-*module.*

If X is an H-space with multiplication $m: X \times X \to X$, then $\Omega^{Sp}(X)$ has a Pontrjagin ring structure. Namely, if we put

$$[M, f][M', f'] = [M \times M', m \circ (f \times f')],$$

then $\mu: \Omega^{Sp}(X) \to H_*(X)$ is a homomorphism of Pontrjagin rings. In case X is homotopy-commutative, $\Omega^{Sp}(X)$ has an anticommutative product

$$b \cdot a = (-1)^{\dim a \cdot \dim b}\, a \cdot b.$$

For the spaces mentioned in the title, we shall show that $\mu$ is onto, so that the symplectic bordism modules are free over $\Omega^{Sp}$. Since Sp(n) and BSp($\infty$) are

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H-spaces, their symplectic bordism modules have Pontrjagin ring structures. $\Omega^{Sp}(Sp(n))$ is anticommutative and has generators $a_m$ of dimension $4m - 1$ $(1 \leq m \leq n)$, on which it is almost an exterior algebra over $\Omega^{Sp}$. $\Omega^{Sp}(BSp(\infty))$ is a polynomial algebra over $\Omega^{Sp}$ on generators in positive dimensions divisible by 4.

In Section 2, we treat the symplectic groups $Sp(n)$, relying on the description of their homology given by Steenrod in [6, Chapter 4]. Then, in Section 3, we construct symplectic manifolds $V^{4m}$ carrying $Sp(1)$-bundles $\zeta$ with $\langle p_1(\zeta)^m, \sigma_V \rangle = 1$, and thus we determine the symplectic bordism of the quaternionic projective spaces $HP(n)$ ($p_1(\zeta)$ is the first symplectic Pontrjagin class); $V^{4m}$ may be regarded as a substitute for $HP(m)$, which is not a symplectic manifold if $m > 1$. The classifying spaces $BSp(n)$ are discussed in Section 4, and in the final section we examine the "Smith homomorphism" $\Delta: \Omega_k^{Sp}(HP(\infty)) \to \Omega_{k-4}^{Sp}(HP(\infty))$.

## 2. THE SYMPLECTIC GROUPS Sp(n)

We shall begin by introducing symplectic manifolds $N^{4m-1}$ $(m > 0)$ and maps $h_m: N^{4m-1} \to Sp(m) \subset Sp(n)$ for $m \leq n$, thereby obtaining bordism classes

$$a_m = [N^{4m-1}, h_m] \in \Omega_{4m-1}^{Sp}(Sp(n))$$

for $m \leq n$. According to Steenrod [6, p. 46], the homology classes $\mu(a_m)$ generate the Pontrjagin ring $H_*(Sp(n))$ as an exterior algebra, hence by (1.1) the Pontrjagin ring is generated by the bordism classes $a_m$ $(m \leq n)$. We go on to show that $\Omega^{Sp}(Sp(n))$ is anticommutative and is almost an exterior algebra over $\Omega^{Sp}$ on these classes.

Let $S^{4m-1} \subset H^m$ denote the unit sphere in the standard right quaternionic vector space, with inner product given by $\langle x, y \rangle = \sum \bar{x}_i y_i$. The unit quaternions $Sp(1) = S^3$ act freely to the right on $S^{4m-1}$ by scalar multiplication, and the corresponding orbit space is the quaternionic projective space $HP(m - 1)$; this gives rise to the canonical $Sp(1)$-bundle $\eta \to HP(m - 1)$. The symplectic group $Sp(m)$ is the group of linear transformations of $H^m$ that preserve the inner product. We shall identify principal $Sp(m)$-bundles with the corresponding right quaternionic $m$-plane bundles.

It is well-known (see for example Hsiang and Szczarba [5]) that the tangent bundle $\tau$ of $HP(m - 1)$ satisfies the equation

$$\tau \oplus (\eta \otimes_H \eta^*) = m\eta_R,$$

where $\eta^*$ denotes the conjugate left quaternionic bundle and $\eta_R$ the underlying real bundle. Let $N^{4m-1}$ denote the sphere bundle of the real 4-plane bundle

$$\eta \otimes_H \eta^* \to HP(m - 1),$$

with projection $\pi: N^{4m-1} \to HP(m - 1)$. It follows from [7, Theorem 1.2] that $N^{4m-1}$ has as stable tangent bundle $m \cdot \pi^!(\eta_R)$; thus $N^{4m-1}$ is a symplectic manifold.

The real 4-plane bundle $\eta \otimes_H \eta^* \to HP(m - 1)$ is obtained from $\eta \to HP(m - 1)$ by applying the representation $\lambda \mapsto \lambda\nu\lambda^{-1}$ of $Sp(1)$ on $R^4 = H$. Thus $N^{4m-1}$ arises from $S^{4m-1} \times S^3$ by identification of $(x\lambda, \nu)$ with $(x, \lambda\nu\lambda^{-1})$ for $\lambda \in Sp(1)$. Following Steenrod [6, p. 38], we define a map $\phi: S^{4m-1} \times S^3 \to Sp(m)$ by letting $\phi(x, \nu)$ fix the hyperplane in $H^m$ orthogonal to the unit vector $x \in H^m$ and send $x \mapsto x\nu$. It is easily verified that $\phi$ induces maps

$$h_m: N^{4m-1} \to Sp(m) \subset Sp(n)$$

for $m \leq n$. Thus we obtain bordism classes

$$a_m = [N^{4m-1}, h_m] \in \Omega^{Sp}_{4m-1}(Sp(n)) \qquad (m = 1, 2, \cdots, n).$$

It follows from [6, p. 46] that the Pontrjagin ring $H_*(Sp(n))$ is the exterior algebra over the integers on generators $\mu(a_m)$ $(m = 1, 2, \cdots, n)$. Hence, by (1.1) and the succeeding remarks, $\Omega^{Sp}(Sp(n))$ is a free left $\Omega^{Sp}$-module on the classes

(1) $$a_{m_1} \cdot a_{m_2} \cdots a_{m_r} \qquad (1 \leq m_1 < m_2 < \cdots < m_r \leq n).$$

Thus the Pontrjagin ring $\Omega^{Sp}(Sp(n))$ is generated by the classes $a_m$ $(1 \leq m \leq n)$ as an algebra over $\Omega^{Sp}$.

2.1. THEOREM. *For* $1 \leq n \leq \infty$, $\Omega^{Sp}(Sp(n))$ *is an anticommutative algebra over* $\Omega^{Sp}$, *generated over* $\Omega^{Sp}$ *by the classes* $a_m$ $(1 \leq m \leq n)$. *These classes satisfy the relations*

$$a_m \cdot a_{m'} + a_{m'} \cdot a_m = 0, \qquad 2a_m \cdot a_m = 0.$$

*Moreover,* $\Omega^{Sp}(Sp(n))$ *is a free left* $\Omega^{Sp}$*-module on the classes* (1) *above.*

It remains to show that $\Omega^{Sp}(Sp(n))$ is anticommutative, since the relations on the generating classes then follow. It may easily be shown that $a_1 \cdot a_1 = 0$, and I conjecture that $a_m \cdot a_m = 0$ for all $m > 1$, but am unable to prove this. In any event, $\Omega^{Sp}(Sp(n))$ is almost an exterior algebra over $\Omega^{Sp}$.

Since for $n \leq n' \leq \infty$ the inclusion map $\Omega^{Sp}(Sp(n)) \to \Omega^{Sp}(Sp(n'))$ is a monomorphism of Pontrjagin rings according to what has already been established, it suffices to show that $\Omega^{Sp}(Sp(\infty))$ is anticommutative. According to the remarks following (1.1), it is enough to prove that $Sp(\infty)$ is homotopy-commutative.

2.2. LEMMA. *For* $1 \leq n < \infty$, *the mappings*

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad (A, B) \mapsto \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$$

*of* $Sp(n) \times Sp(n)$ *into* $Sp(2n)$ *are homotopic.*

2.3. COROLLARY. *For* $1 \leq n < \infty$, *the mappings*

$$(A, B) \mapsto \begin{pmatrix} AB & 0 \\ 0 & I_n \end{pmatrix}, \qquad (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

*of* $Sp(n) \times Sp(n)$ *into* $Sp(2n)$ *are homotopic.*

The corollary follows from the lemma if we note that

$$\begin{pmatrix} AB & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} \cdot \begin{pmatrix} B & 0 \\ 0 & I_n \end{pmatrix}, \qquad \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & B \end{pmatrix}.$$

Combining (2.2) and (2.3), we see that for $1 \leq n < \infty$ the mappings

$$(A, B) \mapsto \begin{pmatrix} AB & 0 \\ 0 & I_n \end{pmatrix}, \quad (A, B) \mapsto \begin{pmatrix} BA & 0 \\ 0 & I_n \end{pmatrix}$$

of $Sp(n) \times Sp(n)$ into $Sp(2n)$ are homotopic, from which it follows that $Sp(\infty)$ is homotopy-commutative. This proves the theorem.

*Proof of* (2.2). It suffices to observe that the product of the block matrices

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

lies in $Sp(2n)$ for $0 \leq t \leq \pi/2$, and that it yields $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ for $t = 0$ and $\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$ for $t = \pi/2$.

*Remark.* With somewhat less difficulty, one may show by the same methods that the complex bordism $\Omega^U(U(n))$ of the unitary group $U(n)$ maps onto the homology of $U(n)$ and that it is an exterior algebra over $\Omega^U$ on classes $a_m$ $(1 \leq m \leq n)$ of dimension $2m - 1$. Here $a_m = [CP(m-1) \times S^1, h_m]$, and the map

$$h_m\colon CP(m-1) \times S^1 \to U(m) \subset U(n)$$

is analogous to the map described above; see [6, Chapter 4].

## 3. THE QUATERNIONIC PROJECTIVE SPACES HP(n)

In this section we exhibit symplectic manifolds $V^{4m}$ that carry Sp(1)-bundles $\zeta \to V^{4m}$ such that $\langle p_1(\zeta)^m, \sigma_V \rangle = 1$. We shall at the same time describe maps

$$g_m\colon V^{4m} \to HP(m) \subset HP(n) \qquad (m \leq n)$$

such that $\zeta = g_m^!(\eta)$, where $\eta \to HP(n)$ continues to denote the canonical Sp(1)-bundle. Thus we obtain bordism classes

$$[V^{4m}, g_m] \in \Omega_{4m}^{Sp}(HP(n)) \qquad (0 \leq m \leq n)$$

whose images generate $H_{4m}(HP(n))$. Hence, by (1.1), $\Omega^{Sp}(HP(n))$ is free over $\Omega^{Sp}$ on the classes $[V^{4m}, g_m]$ $(0 \leq m \leq n)$.

Let $N^{4m+4}$ denote the sphere bundle of the real 5-plane bundle $(\eta \otimes_H \eta^*) \oplus R$ over HP(m), with projection $\pi\colon N^{4m+4} \to HP(m)$. As in the previous section, $N^{4m+4}$ has tangent bundle $(m+1)\pi^!(\eta_R)$, and it is thus a symplectic manifold.

The group Sp(1) acts to the left by scalar multiplication on $S^7 \subset H^2$; since this action commutes with the right action of Sp(1) on $S^7$, Sp(1) also acts to the left on HP(1). Thus there is a diagram of fibrations

$$\begin{tikzcd} S^{4m+3} \times_{Sp(1)} S^7 \arrow[r, "p"] \arrow[dr] & S^{4m+3} \times_{Sp(1)} HP(1), \arrow[d, "\pi'"] \\ & HP(m) \end{tikzcd}$$

and p is evidently the projection of an Sp(1)-bundle $\xi$ over $S^{4m+3} \times_{Sp(1)} HP(1)$, a "fibrewise Hopf bundle."

3.1. LEMMA. *There exists a diffeomorphism $\phi$ such that the diagram*

$$\begin{tikzcd} N^{4m+4} \arrow[r, "\phi"] \arrow[dr, "\pi"'] & S^{4m+3} \times_{Sp(1)} HP(1) \arrow[d, "\pi'"] \\ & HP(m) \end{tikzcd}$$

*is commutative.*

Once the lemma is proved, we shall regard $\phi$ as an identification, and we shall let i: $V^{4m} \subset N^{4m+4}$ be the inclusion of a symplectic manifold dual to the Sp(1)-bundle $\xi \to N^{4m+4}$.

Note that $N^{4m+4} = S^{4m+3} \times_{Sp(1)} S^4$, if we regard $S^4$ as the suspension of the unit sphere $S^3 \subset H$ and let Sp(1) act on $S^4$ by suspending the inner automorphism action $(\lambda, \nu) \mapsto \lambda\nu\lambda^{-1}$ of Sp(1) on $S^3$. To obtain $\phi$, it will therefore suffice to describe a diffeomorphism f: $S^4 \to HP(1)$ that is equivariant relative to the left actions of Sp(1). We regard $S^4 = D_1 \cup D_2$, where the $D_i$ are copies of the unit disk in H, and where Sp(1) acts by inner automorphism on the $D_i$. In terms of homogeneous coordinates for HP(1), define f by

$$f(\nu) = \begin{cases} \langle 1, \nu \rangle & (\nu \in D_1), \\ \langle \bar{\nu}, 1 \rangle & (\nu \in D_2). \end{cases}$$

Clearly, f is a diffeomorphism, and for $\nu \in D_1$ and $\lambda \in Sp(1)$, we have the relations

$$f(\lambda\nu\lambda^{-1}) = \langle 1, \lambda\nu\lambda^{-1} \rangle = \langle \lambda, \lambda\nu \rangle = \lambda \cdot \langle 1, \nu \rangle = \lambda \cdot f(\nu);$$

similarly, noting that $\bar{\lambda} = \lambda^{-1}$, we see that f is equivariant on $D_2$; thus f is an equivariant diffeomorphism, and the lemma is established.

Henceforth, we shall *identify* $N^{4m+4}$ *with* $S^{4m+3} \times_{Sp(1)} HP(1)$ *via* $\phi$. Thus $N^{4m+4}$ carries Sp(1)-bundles $\pi^!(\eta)$ and $\xi$. In order to describe the cohomology ring of $N^{4m+4}$, put

$$h = p_1(\eta) \in H^4(HP(m)) \quad \text{and} \quad a = p_1(\xi) - \pi^* p_1(\eta) \in H^4(N^{4m+4});$$

$p_1(\cdot)$ is the first symplectic Pontrjagin class (see [1, p. 488]).

3.2. LEMMA. $H^*(N^{4m+4})$ *is generated by* $\pi^* h$ *and* a, *modulo the relations*

$$(\pi^* h)^{m+1} = 0, \qquad a^2 = 0.$$

Observe that for $m' \leq m$ there is an inclusion $N^{4m'+4} \subset N^{4m+4}$; it is easily seen that the Sp(1)-bundle $\xi \to N^{4m+4}$ restricts to $\xi \to N^{4m'+4}$. In particular, $HP(1) = N^4 \subset N^{4m+4}$ is a fibre of the map $\pi$: $N^{4m+4} \to HP(m)$, and $\xi \to HP(1)$ is by construction the Hopf bundle. It then follows from Dold's theorem [4, (7.4)] that $\pi^*$: $H^*(HP(m)) \to H^*(N^{4m+4})$ is a monomorphism and that $H^*(N^{4m+4})$ is free over $H^*(HP(m))$ with basis $\{1, a\}$. Thus $H^*(N^{4m+4})$ is free abelian with basis

(2) $$(\pi^* h)^i,\ a \cdot (\pi^* h)^i \qquad (0 \leq i \leq m).$$

Since $h^{m+1} = 0$, we see that also $(\pi^* h)^{m+1} = 0$; thus, in order to complete the description of the cohomology ring, we must show that $a^2 = 0$.

To this end, consider the reduced symplectic bundle $\alpha = \xi - \pi^! \eta$ in $\widetilde{K}Sp(N^{4m+4})$. As in the proof of (3.1), we have the relation

$$N^{4m+4} = S^{4m+3} \times_{Sp(1)} D_1 \cup S^{4m+3} \times_{Sp(1)} D_2 .$$

Moreover, the restrictions of $\xi$ and $\pi^! \eta$ to each $S^{4m+3} \times_{Sp(1)} D_i$ are isomorphic bundles; hence

$$\alpha \otimes_H \alpha^* = 0 \qquad \text{in } \widetilde{KO}(N^{4m+4}).$$

W. C. Hsiang and R. H. Szczarba [5] gave formulas for the Pontrjagin classes of a quaternionic tensor product; a computation based on their Lemma 4.1 (the first line of which should be corrected to read "$p_1(\xi \otimes_H \zeta^*) = 2(p_1(\xi) + p_1(\zeta))$") shows that $p_1(\alpha)^2 = 0$. Since $p_1(\alpha) = a$, the lemma is established.

We shall now describe a classifying map f: $N^{4m+4} \to HP(2m + 1)$ for the Sp(1)-bundle, that is, a map such that $f^!(\eta) = \xi$. We first let $f''$: $S^{4m+3} \times S^7 \to S^{8m+7}$ be the map

$$(x_0, \cdots, x_m) \times (z_0, z_1) \to (x_0 z_0, \cdots, x_m z_0, x_0 z_1, \cdots, x_m z_1).$$

Notice that if $\sum \bar{x}_i x_i = 1$ and $\bar{z}_0 z_0 + \bar{z}_1 z_1 = 1$, then also

$$\sum \bar{x}_i \bar{z}_0 x_i z_0 + \sum \bar{x}_i \bar{z}_1 x_i z_1 = 1 .$$

In fact, $f''$ passes through a map $f'$: $S^{4m+3} \times_{Sp(1)} S^7 \to S^{8m+7}$ that commutes with the right actions of Sp(1), thus giving rise to the desired classifying map for $\xi$ by passage to the orbit spaces.

Let now i: $V^{4m} \subset N^{4m+4}$ be the inclusion of a submanifold dual to $\xi$. In terms of homogeneous coordinates $\langle w_0, \cdots, w_{2m} \rangle$ for $HP(2m + 1)$, the classifying map f is transverse regular to the hyperplane defined by $w_0 + w_{m+2} = 0$; thus we may take for $V^{4m}$ the inverse image of this hyperplane, so that in the notation of the preceding paragraph the submanifold $V^{4m}$ is ultimately defined by the condition $x_0 z_0 + x_1 z_1 = 0$. The normal bundle of $V^{4m}$ in $N^{4m+4}$ is $i^!(\xi)$; hence the tangent bundle of $V^{4m}$ satisfies the condition

$$\tau \oplus i^!(\xi) = (m + 1)(\pi \circ i)^!(\eta),$$

and $V^{4m}$ is a symplectic manifold.

Let $g_m: V^{4m} \to HP(m) \subset HP(n)$ be the composition $\pi \circ i$ followed by the inclusion in HP(n), for $m \leq n$; then we have bordism classes $[V^{4m}, g_m]$ in $\Omega^{Sp}_{4m}(HP(n))$. Also, let $\zeta \to V^{4m}$ denote the induced bundle $g^!_m(\eta)$.

3.3. THEOREM. *For* $1 \leq n \leq \infty$, $\Omega^{Sp}(HP(n))$ *is a free module over* $\Omega^{Sp}$ *on the classes* $[V^{4m}, g_m]$ $(0 \leq m \leq n)$. *Moreover, the* Sp(1)-*bundles* $\zeta \to V^{4m}$ *satisfy the equation* $\langle p_1(\zeta)^m, \sigma_V \rangle = 1$.

Taking account of (1.1), one sees that we need only verify the last assertion. In the notation of (3.2), we have the relation $p_1(\xi) = a + \pi^* h \in H^4(N^{4m+4})$; hence

$$\langle p_1(\xi)^m, \sigma_V \rangle = \langle i^* (\pi^* h)^m, \sigma_V \rangle = \langle (a + \pi^* h)(\pi^* h)^m, \sigma_N \rangle$$
$$= \langle a(\pi^* h)^m, \sigma_N \rangle = 1,$$

as was asserted.

*Remark.* The symplectic manifolds $N^{4m+3}$ and $N^{4m+4}$ are by definition sphere bundles over HP(m), and in fact they bound the corresponding disk bundles as symplectic manifolds. Calculations show that all integral characteristic numbers of $V^{4m}$ vanish; thus $[V^{4m}]$ is a torsion element of the symplectic bordism ring.

## 4. THE CLASSIFYING SPACES BSp(n)

Since BSp(n) classifies Sp(n)-bundles over compact spaces, the symplectic bordism groups $\Omega^{Sp}_k(BSp(n))$ may be viewed as bordism groups of pairs $(M^k, \alpha)$, where $M^k$ is a closed symplectic k-manifold and $\alpha \to M^k$ is an Sp(n)-bundle; the resulting bordism class is denoted by $[M^k, \alpha]$. The Whitney sum maps

$$\oplus: BSp(n) \times BSp(n') \to BSp(n + n')$$

make $BSp(\infty)$ a homotopy-commutative H-space; hence $\Omega^{Sp}(BSp(\infty))$ is an anticommutative algebra over $\Omega^{Sp}$. In fact, the Pontrjagin product is given by

$$[M, \alpha] \cdot [M', \alpha'] = [M \times M', \alpha \oplus \alpha']$$

in the present notation, for reduced bundles $\alpha \to M$ and $\alpha' \to M'$.

4.1. THEOREM. *For* $n < \infty$, $\Omega^{Sp}(BSp(n))$ *is free over* $\Omega^{Sp}$ *on the basis*

$$[V^{4m_1} \times \cdots \times V^{4m_r}, \zeta \oplus \cdots \oplus \zeta \oplus (n - r)H]$$

*as* $(m_1 \leq m_2 \leq \cdots \leq m_r)$ *runs through all* r-*tuples of positive integers* $(0 \leq r \leq n)$; *the* Sp(n)-*bundle is the Whitney sum of the bundles* $\zeta \to V^{4m_i}$ *over the factors and a trivial bundle. Moreover,* $\Omega^{Sp}(BSp(\infty))$ *is a polynomial algebra over* $\Omega^{Sp}$ *on generators* $[V^{4m}, \zeta - H]$ $(m > 0)$.

Omitting details of the proof, we simply observe that $BSp(1) = HP(\infty)$ and that the iterated Whitney sum map

$$BSp(1) \times \cdots \times BSp(1) \to BSp(n)$$

induces a monomorphism in cohomology and so induces an epimorphism of homology groups. An application of (3.3) and (1.1) completes the proof.

## 5. THE SMITH HOMOMORPHISM

In this section we shall describe an $\Omega^{Sp}$-module homomorphism

$$\Delta: \Omega^{Sp}(HP(\infty)) \to \Omega^{Sp}(HP(\infty))$$

of degree -4, whose real and complex analogues have been treated in [2, Section 26] and [3, Section 5]. In addition, we shall construct an $\Omega^{Sp}$-module homomorphism

$$\chi: \Omega^{Sp}(HP(\infty)) \to \Omega^{Sp}(HP(\infty))$$

of degree +4 such that $\Delta \circ \chi$ = identity.

For $[M^k, f] \in \Omega^{Sp}(HP(\infty))$, we may assume that $f(M^k) \subset HP(n)$ with $n < \infty$, and that f is transverse regular to the hyperplane $HP(n - 1)$. Then $W^{k-4} = f^{-1}(HP(n-1))$ is a submanifold of codimension 4 in $M^k$; if we put $g = f \mid W^{k-4}$, the normal bundle of the inclusion $i: W^{k-4} \to M^k$ is $g^!(\eta)_R$. Hence the tangent bundles of W and M satisfy the condition

$$\tau_W + g^!(\eta)_R = i^! \tau_M ;$$

thus $W^{k-4}$ inherits a symplectic structure. We therefore put $\Delta[M^k, f] = [W^{k-4}, g]$, which may be shown to be well-defined and a homomorphism of $\Omega^{Sp}$-modules.

The right inverse $\chi$ of $\Delta$ has the following description. Given $[M^k, f]$ in $\Omega^{Sp}(HP(\infty))$, assume that f is differentiable, so that $\alpha = f^!(\eta)$ is a differentiable Sp(1)-bundle over $M^k$. The total space of the quaternionic projective-space bundle $\pi: P(\alpha \oplus H) \to M^k$ carries a canonical Sp(1)-bundle $\beta$. We shall show that $P(\alpha \oplus H)$ is a symplectic manifold, and we shall put $\chi[M^k, f] = [P(\alpha \oplus H), g]$, where g is a classifying map for $\beta$.

5.1. LEMMA. *The tangent bundle $\tau_F$ along the fibre of $\pi$ is stably isomorphic to $\pi^!(\alpha)_R$.*

According to Hsiang and Szczarba (see [5, Theorems 1.3, 2.1] and [7]), $\tau_F$ satisfies the equation

$$\tau_F \oplus (\beta \otimes_H \beta^*) = (\pi^!(\alpha) \oplus H) \otimes_H \beta^* ,$$

hence

$$\tau_F = \beta_R + (\pi^! \alpha - \beta) \otimes_H \beta^*$$

stably. According to [4, Sections 7 and 9], there is a relation

$$[(H - \beta) \otimes_H (H - \beta)^*] - [\pi^! p_1(\alpha) \otimes_H (H - \beta)^*] + \pi^! p_2(\alpha) = 0$$

in $\widetilde{KO}(P(\alpha \oplus H))$, where $p_1(\alpha) = H - \alpha$ and $p_2(\alpha)$ are the $KO^*$-theory Pontrjagin classes of the Sp(1)-bundle $\alpha$. This relation simplifies to

$$(\pi^! \alpha - \beta) \otimes_H \beta^* = (\pi^! \alpha - \beta)_R ,$$

from which the lemma follows directly.

5.2. THEOREM. *The $\Omega^{Sp}$-module homomorphism $\chi$ is a right inverse of the Smith homomorphism $\Delta$. Thus $\Delta$ is an epimorphism.*

*Remark.* The second statement also follows from (3.3), since one may show that $\Delta[V^{4m}, g_m] = [V^{4m-4}, g_{m-1}]$ for $m > 0$.

It follows from (5.1) that the tangent bundles $\tau_P$ and $\tau_M$ of $P(\alpha \oplus H)$ and $M^k$ satisfy

(3) $$\tau_P = \pi^!(\tau_M + \alpha_R);$$

hence the (k + 4)-manifold $P(\alpha \oplus H)$ carries a symplectic structure. As with $\Delta$, one may show that $\chi$ is a well-defined homomorphism of $\Omega^{Sp}$-modules. It remains to show that $\Delta \circ \chi$ = identity.

Let $i: M^k \to P(\alpha \oplus H)$ be the section of $\pi$ with $i(M^k) = P(\alpha) \subset P(\alpha \oplus H)$. The induced bundle $i^!(\beta)$ of the canonical Sp(1)-bundle $\beta \to P(\alpha \oplus H)$ is then $\alpha$, and in view of (3), $\alpha_R$ is the normal bundle of $M^k$ in $P(\alpha \oplus H)$. Thus we can choose a classifying map g of $\beta$ such that the diagram

$$\begin{array}{ccc} P(\alpha \oplus H) & \xrightarrow{g} & HP(n) \\ \uparrow{\scriptstyle i} & & \uparrow \\ M^k & \xrightarrow{f} & HP(n-1) \end{array}$$

is commutative ($n < \infty$) and g is transverse regular to $HP(n-1)$ with $i(M^k) = g^{-1}(HP(n-1))$. The theorem follows at once.

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