ON THE MODULUS OF SMOOTHNESS AND DIVERGENT SERIES IN BANACH SPACES

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1. INTRODUCTION

The notions of uniform smoothness and of the modulus of smoothness were introduced by Day [4]. He proved that a Banach space X is uniformly smooth if and only if X^* is uniformly convex and gave estimations of the modulus of smoothness of X in terms of the modulus of convexity of X^*. Later other, but equivalent, moduli of smoothness were introduced; see, for example, Köthe [13, pp. 366-367]. In Section 2 we evaluate the exact value of such a modulus of smoothness of a Banach space in terms of the modulus of convexity of its conjugate, and we apply this result to show that inner product spaces are characterized by being the "smoothest" spaces. In Section 3 we prove a result on divergent series in uniformly smooth Banach spaces, which is in a certain sense dual to a result of Kadec [11].

In Section 4 we apply the result of Section 3 to prove that if the moduli of convexity and smoothness of X behave asymptotically like those of ℓ_2 and if X has an unconditional basis, then X is isomorphic but, in general, not isometric to ℓ_2. Geometrically, this result has the following, rather surprising, formulation (ignoring for the moment the requirement concerning the basis): If the unit cell S of a Banach space is sufficiently smooth and convex (smoothness and convexity are measured by the respective moduli), then there is in the space a convex body S_1, whose smoothness and convexity are the greatest possible, such that $S \subset S_1 \subset kS$ for some $k \geq 1$. However, this S_1 cannot in general be chosen to be very close to S (for example, the best possible k may be arbitrarily large for suitable S).

In the last section we introduce two indices which classify Banach spaces in terms of convergent or divergent series in them.

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Notations. We consider only Banach spaces of dimension at least 2. Let X be a Banach space. Its modulus of convexity is defined by

$$\delta_X(\varepsilon) = \frac{1}{2} \inf_{\substack{\|x\| = \|y\| = 1 \\ \|x-y\| = \varepsilon}} (2 - \|x+y\|) \qquad (0 \leq \varepsilon \leq 2).$$

X is called uniformly convex if $\delta_X(\varepsilon) \geq 0$ for every $\varepsilon > 0$. The modulus of smoothness of a space X is defined by

$$\rho_X(\tau) = \frac{1}{2} \sup_{\|x\|=1,\ \|y\|=\tau} (\|x+y\| + \|x-y\| - 2) \qquad (\tau \geq 0).$$

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X is called uniformly smooth if $\rho_X(\tau) = o(\tau)$ as $\tau \to 0$. That this notion coincides with the one introduced by Day is shown, for example, by Köthe [13, p. 366]. It is clear that $\rho_X(\tau)$ is, for every X, a convex, increasing function of τ which satisfies the inequality $\rho_X(\tau) \leq \tau$. If $\delta_X(\varepsilon) \geq \delta_Y(\varepsilon)$ for every ε, we say that X is more convex than Y, and if $\rho_X(\tau) \leq \rho_Y(\tau)$ for every τ, we say that X is smoother than Y.

2. EVALUATION OF THE MODULUS OF SMOOTHNESS

We pass to our first theorem, whose proof is a quantitative version of the usual proof to the fact that X is uniformly convex if and only if X* is uniformly smooth.

THEOREM 1. *For every Banach space* X

(1) $$\rho_{X^*}(\tau) = \sup_{0 \leq \varepsilon \leq 2} (\tau\varepsilon/2 - \delta_X(\varepsilon)) \qquad (\tau > 0).$$

Proof. We show first that for every positive ε and τ

(2) $$\delta_X(\varepsilon) + \rho_{X^*}(\tau) \geq \tau\varepsilon/2.$$

Let $x, y \in X$ with $\|x\| = \|y\| = 1$, $\|x - y\| = \varepsilon$, and let $f, g \in X^*$ satisfy the conditions

$$\|f\| = \|g\| = 1,\ f(x+y) = \|x+y\|,\ g(x-y) = \|x-y\|.$$

Then

$$\begin{aligned} 2\rho_{X^*}(\tau) &\geq \|f + \tau g\| + \|f - \tau g\| - 2 \\ &\geq f(x) + \tau g(x) + f(y) - \tau g(y) - 2 \\ &= f(x+y) + \tau g(x-y) - 2 = \|x+y\| + \tau\varepsilon - 2, \end{aligned}$$

that is,

$$2 - \|x+y\| \geq \tau\varepsilon - 2\rho_{X^*}(\tau),$$

and (2) follows.

Let now $f, g \in X^*$ satisfy the conditions $\|f\| = 1$, $\|g\| = \tau$, and let $\eta > 0$. There exist $x, y \in X$ such that

$$\|x\| = \|y\| = 1,\ f(x) + g(x) \geq \|f+g\| - \eta,\ f(y) - g(y) \geq \|f - g\| - \eta.$$

Therefore,

$$\begin{aligned} \|f+g\| + \|f-g\| &\leq f(x) + g(x) + f(y) - g(y) + 2\eta \\ &= f(x+y) + g(x-y) + 2\eta \leq \|x+y\| + \tau\|x-y\| + 2\eta \\ &\leq 2 + 2\sup_{0 \leq \varepsilon \leq 2} (\varepsilon\tau/2 - \delta_X(\varepsilon)) + 2\eta. \end{aligned}$$

Since η may be any positive number, (1) follows from (2) and the last inequality.

Remarks. 1. Theorem 1 remains meaningful for spaces that are not uniformly convex or smooth. Therefore we did not assume that X is reflexive.

2. By arguing as in the proof of Theorem 1 it can be shown that

$$\rho_X(\tau) = \sup_{0 \le \varepsilon \le 2} (\tau\varepsilon/2 - \delta_{X^*}(\varepsilon)).$$

It follows that, even if X is not reflexive, $\rho_X(\tau) = \rho_{X^{**}}(\tau)$ for every τ. This latter result and a similar one concerning δ also follow immediately from the fact that X is w*-dense in X**.

COROLLARY. *For every Banach space* X *and every* $\tau \ge 0$,

$$\rho_X(\tau) \ge (1 + \tau^2)^{1/2} - 1. \tag{3}$$

If the equality sign holds in (3) *for every* τ $(0 \le \tau \le 1)$, *then* X *is an inner product space.*

Proof. Let H be an inner product space. From the parallelogram equality it follows that $\rho_H(\tau) = (1 + \tau^2)^{1/2} - 1$ for every $\tau \ge 0$. It is known that the inner product spaces are the most convex, that is, that for every Banach space X and every $\varepsilon \ge 0$, $\delta_X(\varepsilon) \le \delta_H(\varepsilon)$ (see Day [5], Nordlander [14]). Hence (see Remark 2)

$$\rho_X(\tau) = \sup(\varepsilon\tau/2 - \delta_{X^*}(\varepsilon)) \ge \sup(\varepsilon\tau/2 - \delta_H(\varepsilon)) = \rho_H(\tau).$$

If equality holds in (3) for every $\tau \le 1$, then it follows easily that for every $x, y \in X$

$$\|x + y\| + \|x - y\| \le 2(\|x\|^2 + \|y\|^2)^{1/2}.$$

It is well known that this inequality implies that X is an inner product space (see Day [6, p. 116]).

For the $L_p(\mu)$ spaces the modulus of smoothness can be easily evaluated by using the same inequalities which were used by Hanner [10, Theorem 1] for the evaluation of the modulus of convexity; the theorem and its proof are valid for general measure spaces. We obtain for $X = L_p(\mu)$

$$\rho_X(\tau) = \begin{cases} (((1+\tau)^p + |1-\tau|^p)/2)^{1/p} - 1 = (p-1)\tau^2/2 + O(\tau^4), & \text{if } 2 \le p < \infty \\ (1 + \tau^p)^{1/p} - 1 = \tau^p/p + O(\tau^{2p}), & \text{if } 1 \le p \le 2. \end{cases}$$

3. A THEOREM ON SERIES THAT DIVERGE FOR EVERY CHOICE OF SIGNS

Kadec [11] has proved that if X is uniformly convex and if

$$\sum x_i \quad (x_i \in X,\ i = 1, 2, \cdots)$$

converges unconditionally, that is $\Sigma \pm x_i$ converges for every choice of signs (see Day [6, p. 89]), then $\Sigma\delta_X(\|x_i\|) < \infty$. Our result is dual to that of Kadec.

THEOREM 2. *Let* X *be a uniformly smooth Banach space, and suppose that*

(4) $$\limsup_{\tau \to 0} \rho_X(2\tau)/\rho_X(\tau) < \infty.$$

Let $\{x_i\}_{i=1}^{\infty} \subset X$ *be a sequence such that* $\Sigma \pm x_i$ *diverges for every choice of signs. Then*

(5) $$\sum_{i=1}^{\infty} \rho_X(\|x_i\|) = \infty.$$

We prove a lemma first.

LEMMA 1. *Let* X *be a Banach space, and suppose* $\{x_i\}_{i=1}^{n} \subset X$. *Let* λ *and* η *be positive numbers with* η *such that* $\eta \geq \|x_i\|$ $(i = 1, \cdots, n)$. *Then there are signs* ε_i, *that is,* $\varepsilon_i = \pm 1$, $i = 1, \cdots, n$, *such that*

(6) $$\left\| \sum_{i=1}^{k} \varepsilon_i x_i \right\| \leq (\lambda^{-1} + \eta) \prod_{i=1}^{k} (1 + \rho_X(\lambda \|x_i\|))$$

for $k = 1, \cdots, n$.

Proof. For $k = 1$ (6) holds for every choice of ε_1. Suppose we have chosen ε_i for $i \leq h$ such that (6) holds for $k \leq h$. Put $S_h = \Sigma_{i=1}^{h} \varepsilon_i x_i$. If $\|S_h\| \leq \lambda^{-1}$, then for every choice of ε_{h+1} (6) holds for $k = h + 1$ since

$$\|S_h + \varepsilon_{h+1} x_{h+1}\| \leq \lambda^{-1} + \eta.$$

Suppose now that $\|S_h\| \geq \lambda^{-1}$, then

$$\|S_h\|S_h\|^{-1} + \lambda x_{h+1}\| + \|S_h\|S_h\|^{-1} - \lambda x_{h+1}\| \leq 2(1 + \rho_X(\lambda\|x_{h+1}\|)),$$

and hence there is a sign ε_{h+1} such that

$$\|S_h\|S_h\|^{-1} + \varepsilon_{h+1} \lambda x_{h+1}\| \leq 1 + \rho_X(\lambda\|x_{h+1}\|),$$

that is,

$$\|S_h + \varepsilon_{h+1} \lambda \|S_h\| x_{h+1}\| \leq \|S_h\|(1 + \rho_X(\lambda\|x_{h+1}\|)).$$

Since $S_h + \varepsilon_{h+1} x_{h+1}$ is on the segment joining S_h with $S_h + \varepsilon_{h+1} \lambda \|S_h\| x_{h+1}$, we obtain the conclusion

$$\|S_{h+1}\| = \|S_h + \varepsilon_{h+1} x_{h+1}\| \leq \|S_h\|(1 + \rho_X(\lambda\|x_{h+1}\|)),$$

and hence (6) holds in this case also for $k = h + 1$. This concludes the proof of the lemma.

Proof of Theorem 2. We prove that if $\Sigma_{i=1}^{\infty} \rho_X(\|x_i\|) < \infty$, then there are signs ε_i such that $\Sigma_{i=1}^{\infty} \varepsilon_i x_i$ converges. Let $\Sigma_{i=1}^{\infty} \rho_X(\|x_i\|) < \infty$. Then $\rho_X(\|x_i\|) \to 0$, and hence, by the corollary to Theorem 1, $\|x_i\| \to 0$. Also, by (4), $\Sigma_{i=1}^{\infty} \rho_X(2^k \|x_i\|) < \infty$ for every integer k. Consequently, there is an increasing sequence of integers n_k such that

$$\prod_{i=n_k}^{\infty} (1 + \rho_X(2^k \|x_i\|)) < 2$$

and $\|x_i\| \leq 2^{-k}$ for $i \geq n_k$. By Lemma 1, it is possible to choose signs ε_i such that for every k and h with $n_k \leq h \leq n_{k+1}$,

$$\left\| \sum_{i=n_k}^{h} \varepsilon_i x_i \right\| \leq 2^{2-k}.$$

The series $\Sigma_{i=1}^{\infty} \varepsilon_i x_i$, with these ε_i, converges.

Remarks. 1. We do not know whether the assumption (4) is really necessary. Without assuming (4) we only have been able to prove that if, for every choice of signs ε_i, the partial sums $S_k = \Sigma_{i=1}^{k} \varepsilon_i x_i$ $(k = 1, 2, \cdots)$ form an unbounded set, then (5) holds. Indeed, we may restrict ourselves to the case for which $\|x_i\| \leq 1$ for every i, and for this case the assertion above follows from Lemma 1 by taking $\lambda = \eta = 1$.

2. It is clear that we may replace the function $\rho(\tau)$ in Theorem 2 by any function $\rho'(\tau)$ which satisfies (4) and for which $\rho'(\tau) \geq \rho(\tau)$ for every $\tau \leq 1$. We shall use this fact in the sequel, taking as $\rho'(\tau)$ functions of the form $\eta\tau^p$.

4. A CHARACTERIZATION OF SPACES ISOMORPHIC TO INNER PRODUCT SPACES

As mentioned in Section 2, the inner product spaces are the "most convex" spaces. For these, and only these, spaces $\delta(\varepsilon) = 1 - (1 - \varepsilon^2/4)^{1/2}$ [5], [14]. But the inner product spaces are not the only ones (even up to isomorphism) for which $\delta(\varepsilon) = \lambda\varepsilon^2 + o(\varepsilon^2)$ with some $\lambda > 0$. Indeed, the $L_p(\mu)$ spaces with $1 < p < 2$ also have such a modulus of convexity (see Hanner [10] and the references given there). Similar remarks hold with respect to the modulus of smoothness. However, if we assume that both $\delta(t)$ and $\rho(t)$ behave for small t as a multiple of t^2, or even only that, for some $\lambda > 0$ and every $\varepsilon > 0$,

$$\delta_X(\varepsilon) \geq \lambda\varepsilon^2 \tag{7}$$

and, for some η and every $0 \leq \tau \leq 1$,

$$\rho_X(\tau) \leq \eta\tau^2, \tag{8}$$

then the situation is different. We do not know whether (7) and (8) alone are sufficient to imply that X is isomorphic to an inner product space, but it is an almost immediate consequence of the results of Section 3 that if, in addition, X is separable and has an unconditional basis, then X is indeed isomorphic to ℓ_2.

LEMMA 2. *Let* X *be a separable Banach space that has an unconditional basis and for which* (7) *and* (8) *are satisfied. Then* X *is isomorphic to* ℓ_2.

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be an unconditional basis of X with $\|e_i\| = 1$ for every i. By the result of Kadec cited in the beginning of Section 3, it follows that if $\Sigma_{i=1}^{\infty} \lambda_i e_i$ converges (and hence converges unconditionally), then $\Sigma_{i=1}^{\infty} \lambda_i^2 < \infty$. Conversely, if $\Sigma_{i=1}^{\infty} \lambda_i^2 < \infty$, it follows from Theorem 2 that there is a choice of signs ε_i

such that $\Sigma_{i=1}^{\infty} \varepsilon_i \lambda_i e_i$ converges. Using again the fact that the basis is unconditional, we conclude that $\Sigma_{i=1}^{\infty} \lambda_i e_i$ converges. Thus $\Sigma_{i=1}^{\infty} \lambda_i e_i$ converges if and only if $\Sigma_{i=1}^{\infty} \lambda_i^2 < \infty$. Hence, by a standard argument, the mapping

$$\sum \lambda_i e_i \to (\lambda_1, \lambda_2, \cdots)$$

of X onto ℓ_2 is an isomorphism.

Remark. From the proof of Lemma 2 it follows also that all the unconditional bases $\{e_i\}_{i=1}^{\infty}$ in ℓ_2, with $\|e_i\| = 1$ for every i, are equivalent. This is a result of Bari [2] and Gelfand [9]. A proof, similar to our proof, of this result was given recently in a different context by Kadec and Pelczynski [12].

The result of Lemma 2 can be easily extended to nonseparable spaces by using the next lemma.

LEMMA 3. *Let* X *be a Banach space all whose separable subspaces are isomorphic to an inner product space. Then* X *is also isomorphic to an inner product space.*

Proof. Let $\|\cdot\|$ be the norm in X. For every separable subspace Y of X, let k(Y) be defined by

$$k(Y) = \inf\{k;\ \text{there is an inner product norm } \|\cdot\|_o \text{ in } Y \text{ with}$$

$$\|y\| \leq \|y\|_o \leq k\|y\|,\ y \in Y\}.$$

There exists a finite K such that $k(Y) \leq K$ for every separable $Y \subset X$. Indeed, suppose that $k(Y_n) \to \infty$; then the separable subspace Y of X generated by $\bigcup Y_n$ is not isomorphic to an inner product space.

If X is separable there is nothing to prove. Assume now that X has a dense subset of cardinality $\aleph_1$. Then $X = \bigcup_{\alpha \in \Omega} X_\alpha$, where Ω is the set of all the ordinal numbers smaller than the first uncountable ordinal, $X_\alpha \subset X_\beta$ for $\alpha < \beta$, and X_α is a separable subspace of X for every α. Let $\|\cdot\|_\alpha$ be an inner product norm in X_α which satisfies the inequalities $\|x\| \leq \|x\|_\alpha \leq K\|x\|$ $(x \in X)$. Let Σ be the Stone-Čech compactification of the discrete space Ω, and let

$$\sigma \in \bigcap_{\alpha \in \Omega} C\ell\{\beta;\ \beta \geq \alpha\},$$

where the closures are taken in Σ. For each $x \in X$ define a function $f_x(\alpha)$ on Ω by

$$f_x(\alpha) = \begin{cases} 0 & \text{if } x \notin X_\alpha, \\ \|x\|_\alpha & \text{if } x \in X_\alpha. \end{cases}$$

Extend f_x to a continuous function $\hat{f}_x$ on Σ, and let $|||x||| = \hat{f}_x(\sigma)$ for $x \in X$. The norm $|||\cdot|||$ is an inner product norm on X since for every $x, y \in X$

$$f_{x+y}^2(\alpha) + f_{x-y}^2(\alpha) = 2(f_x^2(\alpha) + f_y^2(\alpha))$$

and

$$f_{x+y}(\alpha) \leq f_x(\alpha) + f_y(\alpha)$$

for sufficiently large α. Further $\|x\| \leq |||x||| \leq K\|x\|$ for every x, and this proves the lemma if X has a dense set of cardinality $\aleph_1$. For a general X, the lemma is proved in the same manner by using transfinite induction.

In Lemma 2 we assumed that the norm in X satisfies (7) and (8). It is clear that these conditions on the norm are not in general satisfied by spaces isomorphic to ℓ_2. Our next aim is to modify slightly these requirements on the norm so that they will be satisfied by any space isomorphic to an inner product space. To this end we prove first two lemmas.

LEMMA 4. *A Banach space* X *satisfies* (7) *if and only if* X^* *satisfies* (8). *Hence,* X *satisfies* (7) *and* (8) *if and only if* X^* *satisfies* (7) *and* (8).

Proof. It is clear that if X or X^* satisfy either (7) or (8) then they are reflexive. Further it is clear by Theorem 1 that if, X satisfies (7), then X^* satisfies (8). Also, by Theorem 1,

$$\delta_{X^*}(\varepsilon) \geq \sup_{0 \leq \tau \leq 1} (\varepsilon\tau/2 - \rho_X(\tau)) \qquad (\varepsilon > 0),$$

and hence, if X satisfies (8), then X^* satisfies (7); and this concludes the proof.

In the next lemma we consider several norms in the same space X. We say that a norm $\|\cdot\|$ in X satisfies (7) [respectively, (8)] if X with this norm satisfies (7) [respectively, (8)].

LEMMA 5. *Let* X *be a linear space, and let* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *be two equivalent norms in it. Suppose that* $\|\cdot\|_1$ *satisfies* (8) *and that* $\|\cdot\|_2$ *satisfies* (7) *and* (8). *Then* $|||\cdot||| = \|\cdot\|_1 + \|\cdot\|_2$ *is a norm which satisfies* (7) *and* (8).

Proof. Let a and b be positive numbers such that

$$\|x\|_1 \leq a\|x\|_2, \quad \|x\|_2 \leq b\|x\|_1 \qquad (x \in X).$$

We show first that $|||\cdot|||$ satisfies (7). Let $|||x||| = |||y||| = 1$ and $|||x - y||| = \varepsilon$. We may assume that $\|x\|_2 \geq \|y\|_2$. Put $\alpha = \|y\|_2/\|x\|_2$ and $z = \alpha x$. Then $|||z - y||| \geq \varepsilon/2$. Indeed, if $\alpha \leq 1 - \varepsilon/2$, then

$$|||z - y||| \geq |||y||| - |||z||| = 1 - \alpha \geq \varepsilon/2;$$

and if $\alpha \geq 1 - \varepsilon/2$, then

$$|||z - y||| \geq |||y - x||| - |||x - z||| = \varepsilon - 1 + \alpha \geq \varepsilon/2.$$

Hence, $\|z - y\|_2 \geq \varepsilon/(2a + 2)$. Since $\|z\|_2 = \|y\|_2$ and $\|\cdot\|_2$ satisfies (7),

$$\begin{aligned}\|x + y\|_2 &\leq \|x - z\|_2 + \|z + y\|_2 \\ &\leq (1 - \alpha)\|x\|_2 + 2\|y\|_2 (1 - \lambda\varepsilon^2 (2(a + 1)\|y\|_2)^{-2}) \\ &\leq \|x\|_2 + \|y\|_2 - \lambda\varepsilon^2/2(a + 1)^2.\end{aligned}$$

Therefore

$$|||x + y||| \leq 2 - \lambda\varepsilon^2/2(a + 1)^2,$$

and this proves (7). We show now that also (8) holds. Let $|||x||| = 1$, $|||y||| = \tau$. Then $\|x\|_1 \geq 1/(b+1)$ and $\|x\|_2 \geq 1/(a+1)$; and hence,

$$\begin{aligned}|||x+y||| + |||x-y||| &= \|x+y\|_1 + \|x-y\|_1 + \|x+y\|_2 + \|x-y\|_2 \\ &\leq 2\|x\|_1 (1 + \eta_1 \|y\|_1^2 \|x\|_1^{-2}) + 2\|x\|_2 (1 + \eta_2 \|y\|_2^2 \|x\|_2^{-2}) \\ &\leq 2 + 2((b+1)\eta_1 + (a+1)\eta_2)\tau^2,\end{aligned}$$

and this concludes the proof.

Remark. From the first part of the proof it follows also that if a norm $\|\cdot\|_2$ satisfies (7), then for every equivalent norm $\|\cdot\|_1$, the norm $|||\cdot||| = \|\cdot\|_1 + \|\cdot\|_2$ also satisfies (7).

COROLLARY. *Let* X *be a Banach space with norm* $\|\cdot\|_o$ *that satisfies* (7) *and* (8). *Then arbitrarily close to any norm* $\|\cdot\|$ *in* X *that is equivalent to* $\|\cdot\|_o$ *there are norms satisfying* (7) *and* (8).

By "arbitrarily close" we mean that for every $\varepsilon > 0$ there is a suitable norm $|||\cdot|||$ for which $\|\cdot\| \leq |||\cdot||| \leq (1+\varepsilon)\|\cdot\|$.

Proof. By the remark after Lemma 5, the norms $\|\cdot\| + \varepsilon\|\cdot\|_o$ $(\varepsilon > 0)$, all satisfy (7). Hence arbitrarily close to $\|\cdot\|$ there are norms satisfying (7), and thus, by duality (Lemma 4), there are also norms that satisfy (8) and are arbitrarily close to $\|\cdot\|$. By Lemma 5, there are norms satisfying (7) and (8) arbitrarily close to any norm in X that satisfies (8) and that is equivalent to $\|\cdot\|_o$. This concludes the proof.

We remark that another method for obtaining spaces satisfying (7) and (8) is to take direct sums. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces, all satisfying (7) and (8) with common (that is, independent from n) λ and η; then the space $X = (X_1 \oplus X_2 \oplus \cdots)\ell_2$ of all the sequences $x = (x_1, x_2, \cdots)$ with $x_n \in X_n$ and $\|x\| = (\Sigma \|x_n\|^2)^{1/2} < \infty$ also satisfies (7) and (8). We shall not need this result in the sequel and so we omit its proof, which is similar to the proof of the main result in Day [3].

We now summarize the results of this section in

THEOREM 3. *Let* X *be an infinite-dimensional Banach space.* X *is isomorphic to an inner product space if and only if it has the following two properties*

(i) *Every separable infinite-dimensional subspace of* X *has an unconditional basis.*

(ii) *Arbitrarily close to the norm given in* X *there exist norms satisfying* (7) *and* (8).

Condition (i) may be clearly replaced by

(i') *For every separable infinite-dimensional subspace* Y *of* X *there exists a* Z *with an unconditional basis satisfying* $Y \subset Z \subset X$.

5. INDICES OF CONVERGENCE AND DIVERGENCE

In view of the results of Section 3, it is perhaps interesting to study the following two indices of a given Banach space X.

$$\alpha_X = \inf\{p \,\|\, \Sigma \pm x_i \ (x_i \in X,\ i = 1, 2, \cdots) \text{ converges for every choice of signs } \Rightarrow \Sigma \|x_i\|^p < \infty\}$$

$$\beta_X = \sup\{p \,\|\, \Sigma \pm x_i \ (x_i \in X,\ i = 1, 2, \cdots) \text{ diverges for every choice of signs } \Rightarrow \Sigma \|x_i\|^p = \infty\}$$

The following results concerning α_X and β_X are immediate consequences of known results.

(a) For a finite-dimensional X, $\alpha_X = 1$ and $\beta_X = \infty$.

(b) For an infinite-dimensional X, $\infty \geq \alpha_X \geq 2 \geq \beta_X \geq 1$.

Indeed, (a) follows from the fact that in finite-dimensional spaces unconditional convergence implies absolute convergence, and that for every sequence x_i converging to 0 there are signs ε_i such that $\Sigma_{i=1}^{\infty} \varepsilon_i x_i$ converges (Dvoretzky and Hanani [8]). Assertion (b) is contained in the following theorem due to Dvoretzky [7, Theorem 8]:

Let X *be an infinite-dimensional Banach space, and let* $\{c_i\}_{i=1}^{\infty}$ *and* $\{d_i\}_{i=1}^{\infty}$ *be two sequences of positive numbers such that* $\Sigma_{i=1}^{\infty} c_i^2 < \infty$ *and* $\Sigma_{i=1}^{\infty} d_i^2 = \infty$. *Then there exists a sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *such that* $\|x_i\| = 1$ *for every* i, *and* $\Sigma \pm c_i x_i$ *converges for every choice of signs while* $\Sigma \pm d_i x_i$ *diverges for every choice of signs.*

From the result of Kadec [11] cited in Section 3 it follows that

$$\text{(c)} \qquad \alpha_X \leq \limsup_{\varepsilon \to 0} \log \delta_X(\varepsilon)/\log \varepsilon .$$

Similarly, by Theorem 2,

$$\text{(d)} \qquad \beta_X \geq \liminf_{\tau \to 0} \log \rho_X(\tau)/\log \tau .$$

For the evaluation of the indices the following simple remarks are useful.

(e) If Y is a subspace of X, then $\alpha_X \geq \alpha_Y$, $\beta_X \leq \beta_Y$.

(f) If Y is a quotient space of X, then $\beta_X \leq \beta_Y$.

For $X = L_p(\mu)$ of infinite dimension it is known (Orlicz [15]) that

$$\alpha_X = \begin{cases} 2, & \text{if } 1 \leq p \leq 2 \\ p, & \text{if } 2 \leq p < \infty. \end{cases}$$

Clearly $\alpha_{c_0} = \infty$, and hence, by (e), if $X = L_\infty(\mu)$, $\alpha_X = \infty$. Since every separable Banach space is a quotient space of ℓ_1 (Banach and Mazur [1]), it follows that in assertion (f) above we cannot claim that $\alpha_Y \leq \alpha_X$. If, however, $X = L_p(\mu)$ for $1 < p < \infty$, then the inequality $\alpha_Y \leq \alpha_X$ holds for every quotient space Y of X. This follows from assertion (c) and the easily verified fact that $\delta_Y(\varepsilon) \geq \delta_X(\varepsilon)$ for every ε.

Again, for X an infinite-dimensional $L_p(\mu)$-space,

$$\beta_X = \begin{cases} 2, & \text{if } 2 \le p < \infty \\ p, & \text{if } 1 \le p \le 2 . \end{cases}$$

For $2 \le p < \infty$ this result follows from assertions (b) and (d) ($\rho_X(\tau)$ is given in Section 2). For $1 \le p \le 2$ it is clear that $\beta_{\ell_p} \le p$ and that $\beta_{\ell_1} = 1$. Since every infinite-dimensional $L_p(\mu)$-space has a subspace isomorphic to ℓ_p it follows (by using assertions (d) and (e)) that $\beta_{L_p} = p$ for $1 \le p \le 2$.

Finally we show that $\beta_{c_0} = 1$. To this end, let X be the space

$$(\ell_1^1 \oplus \ell_1^2 \oplus \cdots \oplus \ell_1^n \oplus \cdots)_{c_0},$$

that is, the space of the sequences

$$x = (x_1, x_2, \cdots, x_n, \cdots)$$

with $x_n \in \ell_1^n$ and $\|x_n\| \to 0$, where $\|x\| = \max_n \|x_n\|$. The space X is isometric to a subspace of c_0. For any sequence of positive numbers $\{d_i\}_{i=1}^\infty$ tending to 0 and such that $\Sigma d_i = \infty$, there is a sequence $\{y_i\}_{i=1}^\infty \subset X$ such that $\|y_i\| = d_i$ and $\Sigma \pm y_i$ diverges for every choice of signs. Indeed, let i_k be a sequence of integers such that

$$\sum_{i=i_k+1}^{i_{k+1}} d_i \ge k .$$

Put $n_k = i_{k+1} - i_k$, and let

$$y_i = (y_{i,1}, y_{i,2}, \cdots, y_{i,n}, \cdots) \in X$$

be as follows: $y_{i,n} \ne 0$ only if $i_k \le i \le i_{k+1}$ and $n = n_k$. For these i and n, $y_{i,n} = d_i e_{i-i_k}$, where $e_1, e_2, \cdots, e_{n_k}$ denote the canonical basis of $\ell_1^{n_k}$. We see that $\|y_i\| = d_i$ and

$$\left\| \sum_{i=i_k+1}^{i_{k+1}} \pm y_i \right\| \ge k$$

for every choice of signs.

By (e), it follows that $\beta_{c_0} = 1$, and hence also for every infinite-dimensional C(K)-space (in particular, $L_\infty(\mu)$-space) X, $\beta_X = 1$.

Problem. Let $Z = X \oplus Y$. Is it true that $\beta_Z = \min(\beta_X, \beta_Y)$?

It is trivial that $\alpha_Z = \max(\alpha_X, \alpha_Y)$ and that $\beta_Z \le \min(\beta_X, \beta_Y)$.

Added in proof. Since this paper was written there has appeared a paper by G. Nordlander, *On sign-independent and almost sign-independent convergence in normed linear spaces*, Ark. Mat. 4 (1962), 287-296. In this paper a stronger version of Theorem 2 is proved for the particular case $X = L_p$. Nordlander's methods are different from ours.

Added in proof. We are now able to show that hypothesis (4) in the statement of Theorem 2 is satisfied by every Banach space.

LEMMA. *Let* X *be a Banach space, and let* $\rho_X(\tau)$ *be its modulus of smoothness. Put*

$$\gamma_X = \limsup_{\tau \to 0} \rho_X(2\tau)/\rho_X(\tau).$$

Then $2 \leq \gamma_X \leq 4$. *For every* γ *with* $2 \leq \gamma \leq 4$ *there exists a Banach space* X *with* $\gamma_X = \gamma$.

Proof. By convexity of ρ, $\rho_X(\tau) \leq \rho_X(2\tau)/2$ for every τ; and thus $\gamma_X \geq 2$.

Let $x, y \in X$ with $\|x\| = 1$, $\|y\| = \tau < 1/2$, and set $a = \|x + y\|$, $b = \|x - y\|$. Then

$$\begin{aligned} \|x + 2y\| - 1 &= \|x + y + y\| + \|x + y - y\| - 2 \\ &= a\left(\left\| \frac{x+y}{a} + \frac{y}{a} \right\| + \left\| \frac{x+y}{a} - \frac{y}{a} \right\| - 2 \right) + 2a - 2 \\ &\leq 2a\rho_X(\tau/a) + 2a - 2. \end{aligned}$$

Similarly, $\|x - 2y\| - 1 \leq 2b\rho_X(\tau/b) + 2b - 2$. Hence,

$$(\|x + 2y\| + \|x - 2y\| - 2)/2 \leq a\rho_X(\tau/a) + b\rho_X(\tau/b) + 2\rho_X(\tau). \tag{9}$$

Clearly, $a, b \leq 1 + \tau$ and $a^{-1}, b^{-1} \leq 1 + 2\tau$. Taking the supremum of the left hand side of (1) and using the convexity of ρ, we see that

$$\begin{aligned} \rho_X(2\tau) &\leq (4 + 2\tau)\rho_X(\tau(1 + 2\tau)) \\ &\leq (4 + 2\tau)((1 - 2\tau)\rho_X(\tau) + 2\tau\rho_X(2\tau)). \end{aligned}$$

It follows that $\rho_X(2\tau) \leq (4 + O(\tau))\rho_X(\tau)$, and thus $\gamma_X \leq 4$.

For $X = L_p(\mu)$ with $1 \leq p \leq 2$, $\gamma_X = 2^p$ (compare this with the value of $\rho_X(\tau)$ for these X, given at the end of Section 2). This concludes the proof of the lemma.

We can now state Theorem 2 as follows.

THEOREM. *Let* X *be a Banach space and let* $\{x\}_{i=1}^\infty \subset X$. *If* $\Sigma_i \pm x_i$ *diverges for every choice of signs, then* $\Sigma_i\, \rho_X(\|x_i\|) = \infty$.

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