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Around Silver's Theorem

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Abstract We state some results related to the Silver Theorem.

The following statement is a modern formulation of Silver's Theorem.

Theorem 1 (Shelah [3], II, 2.4(1), p. 59) *Let κ be a singular cardinal of uncountable cofinality. If $pp(\kappa) > \kappa^+$ then the set $\{\delta < \kappa \mid pp(\delta) > \delta^+\}$ contains a closed unbounded subset.*

What happens if we drop the assumption $pp(\kappa) > \kappa^+$? Consider the following principle:

$(*)_\kappa$ There exists an increasing continuous sequence $\langle \kappa_i \mid i < \mathrm{cf}(\kappa)\rangle$ with limit κ such that for each limit $i < \mathrm{cf}(\kappa)$ we have $\max(\mathrm{PCF}(\{\kappa_j^+ \mid j < i\})) = \kappa_i^+$.

Note that once i has uncountable cofinality, then $\kappa_i^+ = \max(\mathrm{PCF}(\{\kappa_j^+ \mid j < i\}))$ always holds by [3], Claim 2.1, p. 55.

Schindler showed ([1], 1.3) that if $(*)_\kappa$ fails then Projective Determinancy holds. The exact strength of $\neg(*)_\kappa$ is unknown but results below give a supercompact as an upperbound.

Theorem 2 ([1]) *Let κ be a singular cardinal of uncountable cofinality. Assume $(*)_\kappa$. Then either*

1. $\{\delta < \kappa \mid pp(\delta) = \delta^+\} \supseteq$ *club, or*
2. $\{\delta < \kappa \mid pp(\delta) > \delta^+\} \supseteq$ *club.*

A similar result holds if we replace $pp(\delta)$ by 2^δ.

Our aim is to give the consistency of a situation when the sets $\{\delta < \kappa \mid 2^\delta = \delta^+\}$ and $\{\delta < \kappa \mid 2^\delta = \delta^{++}\}$ are both stationary. It turns out that it is easier to deal first with gaps bigger than 1.

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Thus fix an increasing continuous sequence $\langle \kappa_i \mid i < \mathrm{cf}(\kappa)\rangle$ with limit κ. Set $S_n = \{\kappa_i^{+n} \mid i < \mathrm{cf}(\kappa)\}$ for each $n \geq 1$. Consider the following principle:

$(*)_{\kappa,n}$ There is a club $C \subseteq \mathrm{cf}(\kappa)$ such that $\kappa \cap \mathrm{PCF}(\{\kappa_i^{+n} \mid i \in C\}) \subseteq S_n$.

Clearly $(*)_\kappa$ is equivalent to $(*)_{\kappa,1}$.

Building on the ideas of Gitik and Mitchell [2], it is possible to show the following.

Theorem 3 (¬(∃ inner model with a strong cardinal)) *Suppose that for every $i < cf(\kappa)$ we have $\kappa_i^{+\omega} = (\kappa_i^{+\omega})^K$. Then for each n, $1 \leq n < \omega$, the following holds: if for all $i < cf(\kappa)$, $2^{\kappa_i} \leq \kappa_i^{+n}$, then $(*)_{\kappa,n}$ holds.*

This means that in order to make $(*)_{\kappa,n}$ false we need to get above $(\kappa_i^{+\omega})^K$. Thus the first reasonable candidate is $\omega + 1$. The next result shows that (given reasonable assumptions) $\omega + 1$ cannot work.

Theorem 4 ([1]) *Assume that for each $i < \mathrm{cf}(\kappa)$, $pp(\kappa_i) \geq \kappa_i^{+\omega+1}$ and $pp(\kappa_i^{+\omega}) = \kappa_i^{+\omega+1}$. If $(*)_{\kappa,n}$ holds for all $n < \omega$, then $(*)_{\kappa,\omega+1}$ also holds.*

The next candidate is $\omega + 2$. The following shows that it is already a good one.

Theorem 5 *Assume that there is a coherent sequence of $(\kappa, \kappa^{+\omega+3})$—extenders of length ω_1. Then in a generic extension it is possible to have the following:*

1. $\mathrm{cf}(\kappa) = \aleph_1$,
2. *sets $\{\delta < \kappa \mid 2^\delta = \delta^{++}\}$ and $\{\delta < \kappa \mid 2^\delta = \delta^{+3}\}$ are both stationary.*

The construction uses a combination of Magidor forcing on extenders with short extender forcings.

Using supercompacts in the previous construction to collapse successors of δs, it is possible to obtain the following.

Theorem 6 *Assume that κ is a supercompact cardinal. Then in a generic extension it is possible to have the following:*

1. $\mathrm{cf}(\kappa) = \aleph_1$,
2. *sets $\{\delta < \kappa | 2^\delta = \delta^+\}$ and $\{\delta < \kappa \mid 2^\delta = \delta^{++}\}$ are both stationary.*

References

[1] Gitik, M., R. Schindler, and S. Shelah, "PCF theory and Woodin cardinals," in preparation. 323, 324

[2] Gitik, M., and W. J. Mitchell, "Indiscernible sequences for extenders and the singular cardinal hypothesis," *Annals of Pure and Applied Logic*, vol. 82 (1996), pp. 273–316. Zbl 0871.03041. MR 98a:03080. 324

[3] Shelah, S., *Cardinal Arithmetic*, vol. 29 of *Oxford Logic Guides*, The Clarendon Press, New York, 1994. Zbl 0848.03025. MR 96e:03001. 323

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