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POSSIBILITY-ELIMINATION IN NATURAL DEDUCTION

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F. B. Fitch's extension of the subordinate-proof technique to modal logic[1] represents an interesting and valuable contribution to both study and exposition in the field. The modal introduction and elimination (intelim) rule-schemata he offers are these:

$\Box$**E**: $\begin{array}{|l} \Box p \\ \hline p \end{array}$ $\qquad$ $\Diamond$**I**: $\begin{array}{|l} p \\ \hline \Diamond p \end{array}$

$\Box$**I**: $\begin{array}{|l} \begin{array}{l|l} \Box & \vdots \\ & p \end{array} \\ \Box p \end{array}$ $\qquad$ $\Diamond$**E**: $\begin{array}{|l} \Diamond p \\ \hline \begin{array}{l|l} \Box & p \\ & \vdots \\ & q \end{array} \\ \Diamond q \end{array}$

$\sim\Box$**E**: $\begin{array}{|l} \sim\Box p \\ \hline \Diamond\sim p \end{array}$ $\qquad$ $\sim\Diamond$**I**: $\begin{array}{|l} \Box\sim p \\ \hline \sim\Diamond p \end{array}$

$\sim\Box$**I**: $\begin{array}{|l} \Diamond\sim p \\ \hline \sim\Box p \end{array}$ $\qquad$ $\sim\Diamond$**E**: $\begin{array}{|l} \sim\Diamond p \\ \hline \Box\sim p \end{array}$

If the propositional base, to which it is understood that these rules are appended, is classical, then a system similar to Lewis' S4 is obtained by permitting only propositions of the form $\Box p$ (or $\sim\Diamond p$) to be reiterated into the strict subordinate proofs of $\Box$**I** and $\Diamond$**E**. A weaker system similar to S2 is obtained by requiring such a reiterated proposition to drop its left-most modal operator.[2]

Two peculiarities, related in part to Fitch's restricted form of $\sim$**I**, emerge upon consideration of his modal rules. (1) Even on a classical base (which will be assumed throughout), the last four rules—those relating $\Box$ and $\Diamond$ —cannot be derived from the first four—the fundamental intelim rules for $\Box$ and $\Diamond$; and they are thus needed to complete the modal apparatus, (2) $\Box$**E** and $\Diamond$**I** can be derived from each other, and $\Diamond$**E** from $\Box$**I** (in the appropriate forms determined by the definition of $\Box p$ as $\sim\Diamond\sim p$ and $\Diamond p$ as $\sim\Box\sim p$). But $\Box$**I** in the form

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$$\begin{array}{|l} \Box \begin{array}{|l} \vdots \\ p \end{array} \\ \sim \Diamond \sim p \end{array}$$

cannot be derived from $\Diamond$**E**. That is to say, the fundamental intelim rules for $\Box$ and for $\Diamond$ are not deductively equivalent, those for $\Box$ being stronger than those for $\Diamond$.

A more satisfactory modal system on a classical base would require just one pair of intelim rules for $\Box$ and one for $\Diamond$, such that (a) the pairs are deductively equivalent and (b) the equivalences relating $\Box$ and $\Diamond$ are theorems or represented by derivable rules. Fitch's $\Box$**I**, $\Box$**E**, and $\Diamond$**I** are desirable for at least two reasons: their expression of "ordinary" modal intuition, and their close analogy to the corresponding intelim rules for $\forall$ and $\exists$ in the logic of quantification. This recommends the strengthening of $\Diamond$**E**. Therefore this alternative is proposed:

$$\Diamond \mathbf{E}^*: \quad \begin{array}{|l} \underline{\Diamond p} \\ \Box \begin{array}{|l} \underline{p} \\ \vdots \\ q \\ \sim q \end{array} \\ r \end{array}$$

Loosely put, if p is possible but strictly entails a contradiction, then anything goes. That this and $\Box$**I** are deductively equivalent is easily shown.

Given $\Diamond$**E*** and the hypothesis

$$\Box \begin{array}{|l} \vdots \\ p\,, \end{array}$$

to derive $\sim\Diamond\sim p$:

$$\begin{array}{r|l l}
1 & \begin{array}{|l} \underline{\Diamond \sim p} \end{array} & \\
2 & \quad \Box \begin{array}{|l} \underline{\sim p} \end{array} & \\
3 & \quad \quad\ \ p & \text{Hypothesis} \\
4 & \quad q & 1,\ 2\text{-}3\ \Diamond\mathbf{E}^* \\
5 & \quad \sim q & 1,\ 2\text{-}3\ \Diamond\mathbf{E}^* \\
6 & \sim\Diamond\sim p & 1\text{-}5\ \ \sim\mathbf{I}
\end{array}$$

And given $\Box$**I** and the hypothesis

$$\Box \begin{array}{|l} \underline{p} \\ \vdots \\ q \\ \sim q\,, \end{array}$$

to derive r from $\sim\Box\sim p$:

1	$\sim\Box\sim p$		
2	$\Box \quad \mid \quad p$		
3	$\quad\quad q$	Hypothesis	
4	$\quad\quad \sim q$	Hypothesis	
5	$\quad \sim p$	2-4	$\sim$I
6	$\Box\sim p$	2-5	$\Box$I
7	r	6, 1	$\sim$E

A few remarks about the modified rule-system are appropriate.

(1) Either $\Box$ or $\Diamond$ can with equal ease be treated as primitive, the other being introducible by definition with no further apparatus. Even on a classical base, $\Diamond$E (with $\Diamond$I) is too weak to permit the generation of all desired theorems if $\Diamond$ is considered primitive.

(2) The modified system is indifferent with respect to modal strength—i.e., it permits all of the deductions desired in "S2," or the desired deductions of any stronger system. Use of $\Diamond$E* is unrelated to the nature of the restrictions one might impose on reiteration into strict subordinate proofs. For no such reiteration takes place in the derivations above.

(3) $\Diamond$E seems somewhat more analogous that $\Diamond$E* to $\exists$E in the form

$$\begin{array}{l|l}
 & (\exists x)\phi x \\ \hline
 & y \quad \mid \ \phi y \\
 & \quad\quad \vdots \\
 & \quad\quad p \\
 & p
\end{array}$$

(with the usual qualifications).

But $\Diamond$E* is even more nearly analogous to the deductively equivalent form of $\exists$E:[3]

$$\begin{array}{l|l}
 & (\exists x)\phi x \\ \hline
 & y \quad \mid \ \phi y \\
 & \quad\quad \vdots \\
 & \quad\quad p \\
 & \quad\quad \sim p \\
 & q
\end{array}$$

(with the usual qualifications, except that y may be free in p and q).

(4) In the modified system, the four equivalences relating $\Box$ and $\Diamond$ are all theorems or can be represented as derived rules. Since some of the proofs require the reiteration of propositions of the form $\sim\Diamond p$ into strict subordinate proofs, a comment on the justification of this is in order. One might be willing to permit such reiteration simply on the grounds that, particularly when $\Diamond$ is being treated as primitive, some things—and indeed only the *right* things—must be permitted such reiteration; and propositions of the form $\sim\Diamond p$ are (or are among) those things. It is considerations of this sort, after all, which lead to permitting propositions of the form $\Box p$ to be reiterated in this way. Or, if further justification is desired, it can be noted that, in order to pass into strict subordinate proofs—themselves more

closely connected with the operations involving $\Box$—propositions involving $\Diamond$ may be related to propositions involving $\Box$ in accordance with the definition of $\Diamond p$ as $\sim \Box \sim p$. A proposition of the form $\sim\Diamond p$ is then equivalent, by the appropriate definition, to one of the form $\sim\sim \Box \sim p$, or $\Box{\sim}p$, which *can* be reiterated.

(5) $\Diamond$**E*** is subject to a weakness or undesirability displayed by none of Fitch's four fundamental intelim rules for $\Box$ and $\Diamond$: it makes reference to a propositional connective. On the one hand, this seems relatively inoffensive inasmuch as the modal rules are understood to be appended to a propositional base anyway. And on the other, the effected elimination of the need for the four rules relating $\Box$ and $\Diamond$, which all refer to the propositional connective, seems to make the slight sacrifice worthwhile.[4]

NOTES

1. *Symbolic Logic* (New York, 1952), pp. 64-80, 164-66.

2. Robert Price has pointed out that a system similar to S5 can be obtained by permitting the reiteration, into strict subordinate proofs, of propositions of the form $\Diamond p$ (or, equivalently, of any "fully modalized" propositions, with no modally "free" constituents), along with propositions of the form $\Box p$.

3. The deductive equivalence of these two forms of $\exists$**E** was also suggested by Price.

4. In correspondence, Fitch has proposed a deductively equivalent form of $\Diamond$**E*** which "is 'pure' in the sense of involving no operators other that $\Diamond$":

$$
\begin{array}{|l}
\underline{\Diamond p} \\
\begin{array}{c|l}
\Box & \underline{p} \\
q & \cdot \\
 & \cdot \\
 & \cdot \\
 & q
\end{array} \\
r
\end{array}
$$

"where the q to the left of the vertical line indicates that the subproof is 'general' with respect to q, that is, correct no matter how the proposition q is chosen." This proposal, which involves the equivalence of $\sim p$ and $p \supset (\forall q)q$, is very interesting but too provocative and possibly far-reaching in its implications to examine here. In any event, the "purity" achieved is far outweighed by the introduction of a notion alien to elementary modal logic.

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