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# A NOTE ON CONSTRUCTIBLE SETS OF INTEGERS

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Is there always a constructible[1] set of integers of order $\alpha + 1$ which is not of order $\alpha$, when $\alpha$ is less than $\omega_1$?[2] I shall answer this question in the negative. It is clear that there is always a constructible set of order $\alpha + 1$ which is not of order $\alpha$; for $M_\alpha$ itself (the set of *all* sets of order $\alpha$) is a member of $M_{\alpha+1}$, and certainly not a member of itself by the axiom of foundation. But there is not always a set *of integers* in $M_{\alpha+1}$ which is not already in $M_\alpha$. In fact, I shall show that it can happen that "for a long time" (a $\Delta_2^1$ ordinal) we get no new sets of integers in the hierarchy of constructible sets, and then "pop!" a new set of integers appears.

The methods used in this paper are new, and not due to me but to Paul Cohen, who has used them to show[3] that there is an $\alpha < \omega_1$ such that $< M_\alpha, \varepsilon >$ is a *model for von Neumann-Bernays set theory* (VB), if there is any well-founded model[4] for that theory at all. I am indebted to Georg Kreisel for calling my attention to this method.

THEOREM 1. *There is an ordinal $\alpha$ less than constructible $\omega_1$ such that there is no set of integers in $M_{\alpha+1} - M_\alpha$.*

*Proof*: Let $T$ be the set of sentences of set theory which are true in the model $\mathbf{V} = \mathbf{L}$ (i.e., in the model $< \mathbf{L}, \varepsilon >$. By the Löwenheim-Skolem theorem there is a countable model $< M_1, \varepsilon >$ such that $< \mathbf{L}, \varepsilon >$ is an extension of $< M_1, \varepsilon >$ and such that the same sentences $T$ are true in $< M_1, \varepsilon >$ as in $< \mathbf{L}, \varepsilon >$. We call a model $< M, \varepsilon >$ *transitive* if $x \ \varepsilon\ M, y \ \varepsilon\ x \Rightarrow y \ \varepsilon\ M$. It is easily proved (by induction on the order of the elements of $M$) that if $< M, \varepsilon >$ is a submodel of $< \mathbf{L}, \varepsilon >$ then $< M, \varepsilon >$ is isomorphic to a transitive model. So we may assume $< M_1, \varepsilon >$ is transitive.

Now certain formulas $F(x)$ are *invariant* relative to the class of transitive models. By an $F(x)$ being *invariant* we mean that if $\alpha$ satisfies $F(x)$ in a transitive model $< M, \varepsilon >$ then $\alpha$ satisfies $F(x)$ in *every* transitive model in which the set $\alpha$ occurs. For example, "x is an ordinal" (the formula is $F(x) = \mathrm{trans}(x) \,\&\, (y)(z)(y \ \varepsilon\ x \,\&\, z \ \varepsilon\ x \,\&\, y \neq z \Rightarrow y \ \varepsilon\ z \vee z \ \varepsilon\ y)$ where trans(x) is short for $(y)(z)(z \ \varepsilon\ x \,\&\, y \ \varepsilon\ z \Rightarrow y \ \varepsilon\ x))$ is easily seen to be invariant relative to the class of transitive models, and so are "$x$ is the ordinal $\omega$", "x

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is a finite ordinal", "x is the set $M_\alpha$ for some ordinal $\alpha$" and many other notions. Since the proof that Gödel gave in [G] of the "absoluteness" of these notions in fact shows invariance, I omit details.

The sentence "there is an ordinal $\alpha$ such that there is no set of integers in $M_{\alpha+1} - M_\alpha$" is true in $<\mathsf{L}, \varepsilon>$ since there is no set of integers in $M_\alpha$ when $\alpha \geqslant$ classical $\omega_1$ which is not already in $M_{\omega_1}$, by theorem 2 of [G*]. So there is no set of integers in $M_{\omega_1+1} - M_{\omega_1}$, for example. But the same sentences T are true in $<M_1, \varepsilon>$ as in $<\mathsf{L}, \varepsilon>$. Hence, "there is an ordinal $\alpha$ such that $M_{\alpha+1} - M_\alpha$ contains no set of integers (finite ordinals)" is true in $<M_1, \varepsilon>$. Since this sentence contains only invariant notions, there is indeed an ordinal $\alpha \varepsilon M_1$ such that $M_{\alpha+1} - M_\alpha$ contains no set of integers. Since $M_1$ is countable, every ordinal in $M_1$ is countable, and we have thus proved that there is an $\alpha <$classical $\omega_1$ such that $M_{\alpha+1} - M_\alpha$ contains no set of integers.

But in fact we have proved more. For essentially the preceding argument can be formalized in VB. Of course, we cannot construct a model for *all* of VB in VB and also prove that it is a model; but we can construct the structure[5] $<M_{\omega_1+2}, \varepsilon>$ in VB and define the set $T$ of sentences true in this structure. Since we never *employed* the fact that $<\mathsf{L}, \varepsilon>$ is a model for VB in the argument, but only the fact that the sentence "there is an ordinal $\alpha$ such that there is no set of integers in $M_{\alpha+2} - M_\alpha$" is true in $<\mathsf{L}, \varepsilon>$, this structure suffices.

Since all theorems of VB are true in $<\mathsf{L}, \varepsilon>$, the theorem "there is a *countable* $\alpha$ such that $M_{\alpha+1} - M_\alpha$ contains no set of integers", which we have just proved, is also true in $<\mathsf{L}, \varepsilon>$. Hence there is such an $\alpha$ which is "countable" even on the interpretation of "countable" afforded by $<\mathsf{L}, \varepsilon>$; i.e., $\alpha <$ *constructible* $\omega_1$. q.e.d.

Let us call an ordinal $\alpha$ invariant whenever there exists an invariant formula $F(x)$ which is satisfied only by $\alpha$. It is easily proved (by considering countable models) that no ordinal $\geq \omega_1$ can be invariant. In fact, the invariant ordinals are just the $\Delta^1_2$ ordinals[6] by an argument of Mostowski.[7] These are smaller[8] than constructible $\omega_1$, but still quite "large"; for example, all the ordinals for which there are names in the Church-Kleene system of notations (cf. [C], [CK]), are only a segment of the $\Delta^1_2$ ordinals. Thus there is some possible interest in the following theorem, which says that there are stretches of length $\beta$, for every invariant $\beta$, in which we get no "new" sets of integers in the hierarchy of constructible sets of order $< \omega_1$.

THEOREM 2. *If $\beta$ is an invariant ordinal, then there is an $\alpha <$ constructible $\omega_1$ such that there is no set of integers in $M_{\alpha+\gamma} - M_\alpha$ for any $\gamma \leq \beta$.*

Proof: Exactly like the proof of theorem 1, using $\alpha + \beta$ in place of $\alpha + 1$ throughout. Since the $M_\alpha$ are a chain $M_0 \subset M_1 \subset M_2 \subset \ldots$, if there is no set of integers in $M_{\alpha+\beta} - M_\alpha$, then neither is there a set of integers in $M_{\alpha+\gamma} - M_\alpha$ for any $\gamma < \beta$.

## NOTES

1. The term is used throughout in the sense of [G*]. See also [G] for a more detailed exposition.

2. Here $\omega_1$ means the least ordinal in the constructible third number class (i.e., the $\omega_1$ of Gödel's model V=L) unless the expression "classical $\omega_1$" is used. When I wish to remind the reader that this is how "$\omega_1$" is used, I also write "constructible $\omega_1$".

3. In [PC], which is unfortunately not published.

4. A model $<M,R>$ for set theory is an ordered pair consisting of a set of entities $M$ and a two place predicate $R$ defined (at least) on $M$ such that the axioms of set theory are satisfied by the interpretation: "set" means *member of M*, and "$\varepsilon$" means $R$. If there are no infinite-descending $\varepsilon$-chains, i.e., no chains $a_1, a_2, a_3, \ldots$ such that $a_2Ra_1, a_3Ra_2, \ldots$ the model is said to be *well-founded*.

5. By a *structure* I mean simply an ordered pair $< C, R>$ such that $C$ is a set and $R$ is a diadic relation on $C$. The invariance of all the notions needed holds in the structure $<M_{\omega_1+2}, \varepsilon>$ as well as in $<\mathbf{L}, \varepsilon>$.

6. A predicate $R(x,y)$ of integers is called "expressible in both two function-quantifier forms" if the predicate $R(x,y)$ is definable in second order number theory using just two second order quantifiers (and arbitrary first order quantifiers, i.e., quantifiers over numbers, since these can always be reduced to one when function quantifiers are present, by known tricks), which can be brought out in the order EA, and also definable (not necessarily by the same formula, but also using just two second order quantifiers) in such a way that the second order quantifiers can be brought out in the order AE. These predicates form a family referred to in the literature as $\Sigma_2^1 \cap \pi_2^1$, or simply as $\Delta_2^1$. This family is closely connected with well-founded models; in fact, a predicate of integers has the same extension in all well-founded models of some system of set theory if and only if it is in $\Delta_2^1$. I use the term "$\Delta_2^1$ ordinal" to mean *ordinal of some well ordering* R(x,y) *of integers such that the predicate* $R \,\varepsilon\, \Delta_2^1$.

7. Cf. [M]. Mostowski's argument was given for $\beta$-models of analysis but is easily extended to well-founded models of set theory.

8. Shoenfield has shown in [S] that all two function-quantifier notions are "absolute" in Gödel's sense. Since it is a theorem that there are only countably many $\Delta_2^1$ ordinals, they are all < classical $\omega_1$. The theorem must hold in all models, in particular in $<\mathbf{L}, \varepsilon>$, so all the $\Delta_2^1$ ordinals are less than *constructible* $\omega_1$. Since by the result of Shoenfield just mentioned, the $\Delta_2^1$ predicates of the model $<\mathbf{L}, \varepsilon>$ are provably the same as those of $<\mathbf{V}, \varepsilon>$, it follows that all $\Delta_2^1$ ordinals (and even all EA ordinals and all AE ordinals) are less than *constructible* $\omega_1$.

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