Notre Dame Journal of Formal Logic
Volume XV, Number 4, October 1974
NDJFAM

UNARY PREDICATES

JAMES ANDREW FULTON

The following concerns what seems to be a mistake in the construction of a formal semantics for the predicate calculus. I have found this mistake in three books: Mendelson's [3], Shoenfeld's [4], and Leblanc and Wisdom's [1]. Other books verge on the mistake; of course, I cannot claim to have searched all developments of formal semantics. These three books differ in metalogical terminology. I shall follow the usage of Mendelson; but for purposes of cross-reference when a term is introduced, I shall indicate in parentheses the terms used by the other authors. When definitions differ among the authors only in terminology, I shall quote only Mendelson; but I shall provide in the footnote page references for all three books.

Any formal semantics involves two steps: An *interpretation* ([4]: *structure*; [1]: *D - interpretation*) assigns elements from a particular non-empty set, the *domain* ([4]: *universe*), to certain elements of the syntax including *individual constants* ([4]: *constants* (i.e., *O - ary functions*); [1]: *terms*) and *predicate letters* ([4]: *predicate symbols*; [1]: *predicates*). Predicate letters have associated with them a certain positive integer [Shoenfeld also permits O] which is the *degree* of the predicate letter; a predicate letter of degree n is an *n-ary* or *n-place* predicate. The second step is a definition of *satisfaction* ([4]: *truth*; [1]: *truth on a D - interpretation*) in terms of an interpretation.

In their respective definitions of satisfaction all three systems treat atomic *wfs* ([4]: *closed formulas*; [1]: *statements*) constructed from unary predicates as a special case of atomic wfs constructed from n-ary predicates:

If $\mathcal{A}$ is an atomic wf $A_j^n(t_1, \ldots, t_n)$ and B_j^n is the corresponding relation ([4]: predicate; [1]: subset of n-tuples on the domain) of the interpretation, then the sequence s satisfies $\mathcal{A}$ if and only if $B_j^n(s^*(t_1), \ldots, s^*(t_n))$, i.e., if the n-tuple $(s^*(t_1), \ldots, s^*(t_n))$ is in the relation B_j^n.[1]

1. [3], p. 51; [4], p. 19; [1], p. 307.

Received July 12, 1972

In other words according to all three systems, a wf consisting of a predicate ϕ of degree n followed by the n individual constants $\alpha_1, \ldots, \alpha_n$ (where the subscripts indicate their order of appearance) is satisfied if and only if the ordered n-tuple, whose members are the elements of the domain assigned respectively to $\alpha_1, \ldots, \alpha_n$, is a member of the set assigned to ϕ. In order to know whether a given atomic wf is satisfied, we must know what the interpretation assigns to each individual constant and what it assigns to each predicate. The assignment to individual constants is of course a member of the domain. The assignment to predicates bears further attention.

In general the assignment to an n-ary predicate letter is going to be a set of n-tuples. Thus according to Mendelson an "*interpretation* consists [in part] of . . . an assignment to each predicate letter A_j^n of an n-place relation in D [the domain]"; to Shoenfeld, "For each n-ary predicate symbol P of L other than =, an n-ary predicate $\mathsf{P}_{\mathfrak{A}}$ in $|\mathfrak{A}|$ [the universe]"; and to Leblanc and Wisdom, "each one of a certain number of predicates [should be thought of] as applying to the members of a specific subset of D [the domain] when the predicate is 1-place, to those of a specific subset of $D \times D$ when the predicate is 2-place, . . . and, in general, to those of a specific subset of D when the predicate is m-place."[2] It is clear from the above that Leblanc and Wisdom would assign a subset of the domain to each 1-place or unary predicate. To determine this assignment from the systems of Mendelson and Shoenfeld, we must find what they mean by 'relation' and 'predicate' respectively. Thus Mendelson says, "An *n-place relation* . . . on a set X is a subset of X^n, i.e., a set of ordered n-tuples of elements of X. . . . A 1-place relation on X is a subset of X, and is called a property on X." Similarly Shoenfeld says, "A subset of the set of n-tuples in A is called an *n-ary predicate in A*. . . . Note that a unary predicate in A is a subset of A."[3] Thus it seems that the effect of each system is to assign subsets of the domain to unary predicates and subsets of the set of n-tuples of elements of the domain to predicates of degree n where $n > 1$.

Consider then the following example. We want to represent the English sentence 'John is a tall man' as '**Tj**' in the symbolic language. Therefore we take as our domain the class of men, we assign '**j**' to John, and we assign, as these systems instruct us, to '**T**' a subset of that domain, the set of tall men. Clearly the English sentence is true if and only if the individual assigned to '**j**' is a member of the class assigned to '**T**'. Does this interpretation satisfy '**Tj**'? No, it does not! According to our definition of satisfaction, an atomic wf consisting of an n-place predicate followed by n individual constants is satisfied if and only if the n-tuple, whose members are those elements of the domain assigned respectively to those individual constants, is a member of the set assigned to the predicate. Applying this

2. [3], p. 49; [4], p. 18; [1], p. 306.

3. [3], p. 6; [4], p. 10.

to the sentence '**Tj**' we find that '**Tj**' is satisfied under this interpretation if and only if the 1-tuple whose only member is the element of the domain assigned to '**j**' is a member of the set assigned to '**T**'. In other words, using '$\models$' to abbreviate 'is satisfied' and '$<$' and '$>$' to indicate an ordered set

$$\models \mathbf{Tj} \text{ if and only if } <\text{John}> \in \{x\colon\ x \text{ is a tall man}\}.$$

But our domain consists of *men* not of *1-tuples of men*; John may belong to our domain, but $<$John$>$ may not. Hence '**Tj**' is not satisfied.

There seem to be three solutions to this difficulty. One would be to say that for any x, $<x> = x$. Perhaps this is what some of these systems intend, although nowhere is any specific attention given to the nature of a 1-tuple. But surely we want to distinguish between an individual and the ordered 1-tuple of which he is the only member, just as we distinguish an individual from the set of which he is the only member. Indeed, it would seem quite reasonable to say that $<x> = \{x\}$.

Another solution would be to define the satisfaction of atomic wfs constructed from unary predicates as a separate case from, not a special case of, the definition of satisfaction for atomic wfs involving predicates of higher degree. Thus we might say

For an atomic wf $\phi^n \alpha_1 \ldots \alpha_n$, $\models \phi^n \alpha_1 \ldots \alpha_n$ if and only if either $n = 1$ and $\bar{\alpha}_1 \in \bar{\phi}$, or $n > 1$ and $<\bar{\alpha}_1, \ldots, \bar{\alpha}_n> \in \bar{\phi}$ (where $\bar{\phi}$ names the assignment to ϕ).

What seems to me to be the most reasonable approach is to avoid this complication by distinguishing D^1 from D. Since D^n is supposed to be the set of n-tuples on D, D^1 should be the set of 1-tuples on D. But if $x \neq <x>$ then $D^1 \neq D$. The effect of this distinction is that our assignment to '**T**' in our example should not be the set of tall men, a subset of the domain, but the set of all 1-tuples of tall men, a subset of the set of 1-tuples on the domain. Clearly, John is a member of the class of tall men if and only if $<$John$>$ is a member of the class of 1-tuples of tall men; thus the latter is necessary and sufficient to establish the conditions in which the English sentence is true. Moreover this latter is a special case of the general definition of satisfaction for atomic wfs. Certainly it is less convenient to speak of sets of 1-tuples on the domain than of subsets of the domain. We might therefore want to follow Massey in establishing an *informal* convention in which we associate subsets of the domain with unary predicates.[4] But we must remember that this *association* is not the relation of semantic assignment.

REFERENCES

[1] Leblanc, H., and W. A. Wisdom, *Deductive Logic*, Allyn and Bacon, Inc., Boston (1972).

4. [2], p. 229.

[2] Massey, G. J., *Understanding Symbolic Logic*, Harper & Row Publishers, New York (1970).

[3] Mendelson, E., *Introduction to Mathematical Logic*, D. Van Nostrand, Inc., Princeton, New Jersey (1964).

[4] Shoenfeld, J. R., *Mathematical Logic*, Addison-Wesley Publishing Co., Reading, Massachusetts (1967).

Wichita State University
Wichita, Kansas