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A CONCISE METHOD FOR TRANSLATING PROPOSITIONAL FORMULAE CONTAINING THE STANDARD TRUTH-FUNCTIONAL CONNECTIVES INTO A SHEFFER STROKE EQUIVALENT; PLUS AN EXTENSION OF THE METHOD

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1.0 Using Smullyan's rules for the construction of an analytic tableau[1] or Jeffrey's rules for the construction of a truth tree,[2] construct a tree for the formula so that no branch contains a point which has not been used. For purposes of this note, call a point used if it is either an instance of an atomic formula, an instance of the negation of an atomic formula, or has had one of the rules applied to it.

1.1 Utilize the tableau rule for the Sheffer stroke, which is

$$\begin{array}{ccc} & p\,|\,q & \\ & \diagup \quad \diagdown & \\ -p & & -q \end{array} \qquad\qquad \begin{array}{c} -(p\,|\,q) \\ p \\ q \end{array}$$

and work up from the end point of each branch to the origin of the tree.

1.2 Helpful Hints:

(a) Erase from the constructed tree every formula except atomic formulae and their negations, while retaining the basic tree structure.
(b) It is useful to keep the negative sign attached to a formula so long as it negates an entire formula, except, of course, at the origin. However, whenever a negation sign is to be embedded within a formula, change it to its Sheffer stroke equivalent

$$-p = p\,|\,p.$$

1. R. M. Smullyan, *First-Order Logic*, New York (1968).

2. Richard C. Jeffrey, *Formal Logic: Its Scope and Limits*, New York (1967).

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For example, when moving up the tree from a pair of formulae such as

$$-p \qquad q$$

to $p|-q$, since the negative sign is now embedded within the formula, both it and the formula which it negates should in this case be replaced by $q|q$, yielding as the next line in the tree $p|(q|q)$.

1.3 Illustrations:

(a) Finding a Sheffer stroke equivalent for

$$p \supset (q \supset p)$$

Step 1: (working down)

$$p \supset (q \supset p)$$
$$-p \qquad q \supset p$$
$$-q \qquad p$$

Step 2: (working up)

$$p|\{[q|(p|p)]|[q|(p|p)]\}$$
$$-p \qquad q|(p|p)$$
$$-q \qquad p$$

Solution: $p|\{[q|(p|p)]|[q|(p|p)]\}$.

(b) Finding a Sheffer stroke equivalent for

$$(p \supset q) \cdot (q \supset p)$$

Step 1: (working down)

$$(p \supset q) \cdot (q \supset p)$$
$$p \supset q$$
$$q \supset p$$
$$-p \qquad q$$
$$-q \qquad p \qquad -q \qquad p$$

Step 2: (working up)

$$\{(p|p)|[q|(p|p)]\}|\{q|[q|(p|p)]\}$$
$$-\{(p|p)|[q|(p|p)]\} \qquad -\{q|[q|(p|p)]\}$$
$$-p \qquad q$$
$$q|(p|p) \qquad q|(p|p)$$
$$-q \qquad p \qquad -q \qquad p$$

Solution: $\{(p|p)|[q|(p|p)]\}|\{q|[q|(p|p)]\}$.

(c) Finding a Sheffer stroke equivalent for

$$[p \cdot (p \supset q)] \cdot (r \cdot s)$$

Step 1: (working down)[3]

3. The rules for tree construction tell us that when applying a rule to a given point in the tree, we must place the result of applying that rule at the end points of every open branch of which the point on which we are operating is a member. A branch is open just in case it does not contain contradictory points. Consequently, even though $r \cdot s$ is a member of the left branch in (c) here, the left branch was not open when we applied the rule to $r \cdot s$, a fact which prohibited us from extend-

$$\begin{array}{c} [p \cdot (p \supset q)] \cdot (r \cdot s) \\ p \cdot (p \supset q) \\ r \cdot s \\ p \\ p \supset q \end{array}$$

$$\begin{array}{cc} -p & q \\ & r \\ & s \end{array}$$

Step 2: (working up)

$$\begin{array}{c} \{p|\{p|\{q|[(r|s)|(r|s)]\}\}\}|\{p|\{p|\{q|[(r|s)|(r|s)]\}\}\} \\ p \\ p|\{q|[(r|s)|(r|s)]\} \end{array}$$

$$\begin{array}{cc} & -\{q|[(r|s)|(r|s)]\} \\ -p & q \\ & -(r|s) \\ & r \\ & s \end{array}$$

Solution: $\{p|\{p|\{q|[(r|s)|(r|s)]\}\}\}|\{p|\{p|\{q|[(r|s)|(r|s)]\}\}\}$.

2.0 It is interesting to note that this method can be used to translate propositional sentences using any truth-functional connectives into sentences using only a truth-functionally complete connective or binary set of connectives, e.g., $\{\downarrow\}$, $\{-, .\}$, $\{-, \vee\}$, $\{-, \supset\}$. One simply plugs into the method the tableau rule(s) for these connectives and follows the same procedure. The tableau rules for $\downarrow$, ., $\vee$, $\supset$ are given here for the convenience of the reader:

$$\begin{array}{cccc} \begin{array}{c} p \downarrow q \\ -p \\ -q \end{array} & \begin{array}{c} -(p \downarrow q) \\ p \quad q \end{array} & \begin{array}{c} p \cdot q \\ p \\ q \end{array} & \begin{array}{c} -(p \cdot q) \\ -p \quad -q \end{array} \\ \begin{array}{c} p \vee q \\ p \quad q \end{array} & \begin{array}{c} -(p \vee q) \\ -p \\ -q \end{array} & \begin{array}{c} p \supset q \\ -p \quad q \end{array} & \begin{array}{c} -(p \supset q) \\ p \\ -q \end{array} \end{array}$$

2.1 Example: Translating $(p \supset q) \cdot (q \supset p)$ into a truth-functionally equivalent formula containing only "-" and "$\supset$" as logical connectives:

ing the left branch in a manner similar to our extending the right branch. Nonetheless, we can say that every point in the left branch was used. We could have obliterated the distinction between closed branches and open branches for purposes of this method, thus obtaining logically equivalent, but more complicated results.

Step 1: (working down)

$$
\begin{array}{cccc}
\multicolumn{4}{c}{(p \supset q) \cdot (q \supset p)} \\
\multicolumn{4}{c}{p \supset q} \\
\multicolumn{4}{c}{q \supset p} \\
\multicolumn{2}{c}{-p} & \multicolumn{2}{c}{q} \\
-q & p & -q & p
\end{array}
$$

Step 2: (working up)

$$
\begin{array}{cccc}
\multicolumn{4}{c}{[(q \supset p) \supset p] \supset -[(q \supset p) \supset -q]} \\
\multicolumn{2}{c}{-[(q \supset p) \supset p]} & \multicolumn{2}{c}{-[(q \supset p) \supset -q]} \\
\multicolumn{2}{c}{-p} & \multicolumn{2}{c}{q} \\
\multicolumn{2}{c}{q \supset p} & \multicolumn{2}{c}{q \supset p} \\
-q & p & -q & p
\end{array}
$$

Solution: $[(q \supset p) \supset p] \supset -[(q \supset p) \supset -q]$.

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