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SUBSTITUTIONS FOR PREDICATE VARIABLES AND FUNCTIONAL VARIABLES

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The customary descriptions regarding substitution for predicate variables are quite involved and sometimes more restrictive than would be necessary. A precise statement of the conditions under which this type of substitution is permissible is very complex and often has been inadequate in rendering a validity-preserving substitution rule (for a detailed account with references see Church [1], pp. 289-290). Some of the difficulties arise from the restrictions placed on the formation rules for (well-formed) formulas; these restrictions range from the inadmissibility of vacuous quantifiers or quantifiers within the scope of quantifiers using the same variables, to the requirement that no variable may occur free and bound in the same formula. Although such restrictions placed on formulas appear to be impractical in many respects, even less restrictive formulation rules do not eliminate the complications inherent in the process involving substitution for predicate variables. Additional difficulties occur when an adequate formulation of substitution for functional variables is considered.

In this paper* we present recursive definitions of substitution for predicate variables as well as for functional variables which seem to be at least as general as the usual formulations and which at the same time avoid complex descriptions. The scope of our definition of substitution for predicate variables is essentially the same as the description given by Church [1], pp. 192-193. The adequacy of our formulations for both types of substitution will be established by showing that each type of substitution preserves validity.

1 *The formal language*

1.1 The list of *primitive symbols* of our formal language includes

*A description of both types of substitution was first presented at the 1974 meeting of the Association for Symbolic Logic in New York [3].

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(1) a denumerable set of individual variables
(2) a countable set of individual constants
(3) for each integer $n > 0$ a countable set of n-ary functional variables
(4) for each integer $n \geqslant 0$ a countable set of n-ary predicate variables
(5) the identity symbol $\equiv$
(6) the propositional connectives $\neg$ and $\wedge$
(7) the universal quantifier $\forall$
(8) the parentheses (,).

Other propositional connectives and the existential quantifier may be introduced by definition in the customary manner; our further formulations are easily modified to accommodate these additional symbols.

1.2 *Terms* are defined inductively as follows:

(1) Each individual variable and each individual constant is a term (of length 1).
(2) If f is an n-ary functional variable and $t_1, \ldots, t_n$ are terms (of lengths $m_1, \ldots, m_n$, respectively) then $ft_1 \ldots t_n$ is a term (of length $m_1 + \ldots + m_n + 1$).

Atomic formulas are defined as follows:

(1) Each 0-ary predicate variable is an atomic formula.
(2) If $t_1, \ldots, t_n$ are terms and P is an n-ary predicate variable then $Pt_1 \ldots t_n$ is an atomic formula.
(3) If t_1 and t_2 are terms then $t_1 \equiv t_2$ is an atomic formula.

Formulas are defined inductively by the following conditions:

(1) Each atomic formula is a formula (of rank 0).
(2) If B is a formula (of rank n) then $(\neg B)$ is a formula (of rank $n + 1$).
(3) If B and C are formulas (of ranks m and n, respectively) then $(B \wedge C)$ is a formula (of rank $m + n + 1$).
(4) If B is a formula (of rank n) and x is any individual variable then $(\forall xB)$ is a formula (of rank $n + 1$).

1.3 If t_0 is any term in which each occurrence of the individual variable x is replaced simultaneously by the term t, then the resulting term will be indicated by $t_0[x/t]$. Furthermore, we shall write $t_0[x^n/t^n]$ in place of $t_0[x_1/t_1][x_2/t_2] \ldots [x_n/t_n]$. Note that if $x_1, \ldots, x_n$ are distinct individual variables not occurring in any of the terms $t_1, \ldots, t_n$ then $t_0[x^n/t^n]$ can be obtained by simultaneous replacements of $t_1, \ldots, t_n$ for $x_1, \ldots, x_n$, respectively, in t_0.

The notion of *free* occurrence of a term t in a formula A can be described inductively according to the rank of A as follows:

(1) Any occurrence of a term t in an atomic formula is free.
(2) If t occurs free in the formula B then t occurs free in the formula $(\neg B)$.
(3) If t occurs free in the formula B or in the formula C then t occurs free in the formula $(B \wedge C)$.

(4) If t occurs free in the formula B and x is an individual variable not occurring in t, then t occurs free in the formula $(\forall xB)$.

An occurrence of an individual variable in a formula which is not a free occurrence is said to be a *bound* occurrence.

If A is any formula, x any individual variable, and t any term, and there exists a formula B which is the result of replacing in A each free occurrence of x by a free occurrence of t, then B is said to be obtained from A by a *free substitution* of t for x, abbreviated: $\mathbf{Sf}A(x/t)B$. An inductive definition of $\mathbf{Sf}A(x/t)B$ is:

(1) If A is an atomic formula and B results from A upon simultaneously replacing in A each occurrence of x by t, then $\mathbf{Sf}A(x/t)B$.

(2) If $A = (\neg A_1)$ and $\mathbf{Sf}A_1(x/t)B_1$ then $\mathbf{Sf}A(x/t)B$ with $B = (\neg B_1)$.

(3) If $A = (A_1 \wedge A_2)$, $\mathbf{Sf}A_1(x/t)B_1$ and $\mathbf{Sf}A_2(x/t)B_2$, then $\mathbf{Sf}A(x/t)B$ with $B = (B_1 \wedge B_2)$.

(4) (a) If $A = (\forall yA_1)$ and x does not occur free in A, then $\mathbf{Sf}A(x/t)A$.
(b) If $A = (\forall yA_1)$, x is free in A, y does not occur in t, and $\mathbf{Sf}A_1(x/t)B_1$, then $\mathbf{Sf}A(x/t)B$ with $B = (\forall yB_1)$.

We shall indicate by $\mathbf{Sf}A(x^n/t^n)B$ that there exist formulas $B_1, \ldots, B_n (= B)$ such that $\mathbf{Sf}A(x_1/t_1)B_1, \ldots, \mathbf{Sf}B_{n-1}(x_n/t_n)B_n$. Note that if $x_1, \ldots, x_n$ are distinct individual variables not occurring in any of the terms $t_1, \ldots, t_n$, then this consecutive free substitution leads to the same result as a simultaneous free substitution of x_i by t_i for $i = 1, \ldots, n$.

2 *Semantical concepts*

2.1 Let ω be any non-empty domain of individuals. I is said to be an *ω-interpretation* iff I is a function whose domain consists of all individual variables, all individual constants, all functional variables, and all predicate variables, and whose range is such that:

$I(x) \in \omega$ for each individual variable x
$I(c) \in \omega$ for each individual constant c
$I(f^n) \in \omega^{(\omega^n)}$ for each n-ary functional variable f^n
$I(P^0) \in \{\mathbf{T}, \mathbf{F}\}$ for each 0-ary predicate variable P^0
$I(P^n) \subseteq \omega^n$ for each n-ary predicate variable P^n where $n > 0$.

If I is any ω-interpretation, x any individual variable, and d any individual of ω, then I_x^d denotes the function which coincides with I for all arguments other than x and for which $I_x^d(x) = d$. Clearly, I_x^d is again an ω-interpretation with $I_x^d(f) = I(f)$ and $I_x^d(P) = I(P)$ for any functional variable f and any predicate variable P. Furthermore, we note that $I_x^{I(x)} = I$, and $I_{x\,y}^{dd'} = I_{y\,x}^{d'd}$ for $x \neq y$.

According to the above definition, $I(t)$ is already defined for any term t which is an individual variable or an individual constant; on this basis we extend the definition inductively to any term t by the stipulation: $I(ft_1 \ldots t_n) = I(f)(I(t_1), \ldots, I(t_n))$. Induction on the length of the term t_0

shows that for any ω-interpretation I: $I_x^{I(t)}(t_0) = I(t_0[x/t])$. This relationship can be generalized:

Lemma 1 *If $z_1, \ldots, z_n$ are distinct individual variables not occurring in any of the terms $t_1, \ldots, t_n$, then for any ω-interpretation I and for any term t_0:*

$$I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(t_0) = I(t_0[z^n/t^n])$$

Proof: (by induction on the length of t_0)

1. (a) If $t_0 = c$ then
$$\begin{aligned} I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(t_0) &= I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(c) \\ &= I(c) = I(c[z^n/t^n]) = I(t_0[z^n/t^n]) \end{aligned}$$
(b) If $t_0 = x$ with $x \neq z_i$ for $i = 1, \ldots, n$, then
$$\begin{aligned} I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(t_0) &= I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(x) \\ &= I(x) = I(x[z^n/t^n]) = I(t_0[z^n/t^n]) \end{aligned}$$
(c) If $t_0 = x$ with $x = z_i$ for some (and hence exactly one) $i = 1, \ldots, n$ then $I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(t_0) = I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(z_i)$
$= I(t_i)$ since z_i occurs exactly once among $z_1, \ldots, z_n$
$= I(z_i[z^n/t^n])$ since z_i does not occur in $t_1, \ldots, t_n$
$= I(x[z^n/t^n]) = I(t_0[z^n/t^n])$.

2. If $t_0 = ft'_1 \ldots t'_r$ and, by induction hypothesis, for $k = 1, \ldots, r$: $I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(t'_k) = I(t'_k[z^n/t^n])$, then:
$$\begin{aligned} I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(t_0) &= I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(ft'_1 \ldots t'_r) \\ &= I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(f)(I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(t'_1), \ldots, I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}(t'_r)) \\ &= I(f)(I(t'_1[z^n/t^n]), \ldots, I(t'_r[z^n/t^n])) \qquad \text{Induction Hypothesis} \\ &= I(ft'_1[z^n/t^n] \ldots t'_r[z^n/t^n]) = I(t_0[z^n/t^n]). \end{aligned}$$

2.2 Let I be any ω-interpretation; the notion that I is an ω-*model* of a formula A, abbreviated $\mathrm{Mod}_\omega I A$, is defined inductively according to the rank of A as follows:

(1) (a) $\mathrm{Mod}_\omega I\, P^0$ iff $I(P^0) = T$
(b) $\mathrm{Mod}_\omega I\, Pt_1 \ldots t_n$ iff $\langle I(t_1), \ldots, I(t_n)\rangle \in I(P)$
(c) $\mathrm{Mod}_\omega I\, t_1 \equiv t_2$ iff $I(t_1) = I(t_2)$
(2) $\mathrm{Mod}_\omega I\, (\neg B)$ iff not $\mathrm{Mod}_\omega I\, B$
(3) $\mathrm{Mod}_\omega I\, (B \wedge C)$ iff $\mathrm{Mod}_\omega I\, B$ and $\mathrm{Mod}_\omega I\, C$
(4) $\mathrm{Mod}_\omega I\, (\forall x B)$ iff $\mathrm{Mod}_\omega I_x^d B$ for each $d \in \omega$.

A formula A is said to be ω-*valid* iff $\mathrm{Mod}_\omega I A$ for each ω-interpretation I; A is called *valid* iff A is ω-valid for each (non-empty) individual domain ω.

The notions of ω-interpretation and ω-model as formulated here are adapted from Hermes [2], pp. 78-79.

2.3 Two ω-interpretations I and J are said to *coincide* with respect to a formula A iff $I(f) = J(f)$ for each functional variable f occurring in A, $I(P) = J(P)$ for each predicate variable P occurring in A, $I(c) = J(c)$ for each individual constant c occurring in A, and $I(x) = J(x)$ for each individual variable x occurring *free* in A.

Proofs for the following two theorems can be found in [2], pp. 83-85.

Coincidence Theorem: *If I and J are two ω-interpretations which coincide with respect to a formula A then: $Mod_\omega\, I\, A$ iff $Mod_\omega\, J\, A$.*

Substitution Theorem: *If $\mathbf{Sf}A(x/t)B$ and I is any ω-interpretation, then: $Mod_\omega\, I_x^{I(t)}\, A$ iff $Mod_\omega\, I\, B$.*

This substitution theorem can be generalized as follows:

Lemma 2 *If $z_1, \ldots, z_n$ are distinct individual variables not occurring in any of the terms $t_1, \ldots, t_n$ and $\mathbf{Sf}A(z^n/t^n)B$, then for any ω-interpretation I: $Mod_\omega\, I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}A$ iff $Mod_\omega\, I\, B$.*

Proof: Since $\mathbf{Sf}A(z^n/t^n)B$, there exist formulas $B_1, \ldots, B_n = B$ such that $\mathbf{Sf}A(z_1/t_1)B_1, \ldots, \mathbf{Sf}B_{n-1}(z_n/t_n)B_n$. Now:

$\mathrm{Mod}_\omega\, I_{z_1 \ldots z_n}^{I(t_1) \ldots I(t_n)}A$

iff $\mathrm{Mod}_\omega\, I_{z_n \ldots z_1}^{I(t_n) \ldots I(t_1)}A$ since $z_1, \ldots, z_n$ are all distinct

iff $\mathrm{Mod}_\omega\, I_{z_n \ldots z_2}^{I(t_n) \ldots I(t_2)\; I_{z_n \ldots z_2}^{I(t_n) \ldots I(t_2)}(t_1)}A$ since $z_2, \ldots, z_n$ do not occur in t_1 and hence $I_{z_n \ldots z_2}^{I(t_n) \ldots I(t_2)}(t_1) = I(t_1)$

iff $\mathrm{Mod}_\omega\, I_{z_n \ldots z_2}^{I(t_n) \ldots I(t_2)}B_1$ Substitution Theorem

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iff $\mathrm{Mod}_\omega\, I_{z_n z_{n-1}}^{I(t_n)\, I(t_{n-1})}B_{n-2}$

iff $\mathrm{Mod}_\omega\, I_{z_n z_{n-1}}^{I(t_n)\; I_{z_n}^{I(t_n)}(t_{n-1})}B_{n-2}$ since z_n does not occur in t_{n-1} and hence $I_{z_n}^{I(t_n)}(t_{n-1}) = I(t_{n-1})$

iff $\mathrm{Mod}_\omega\, I_{z_n}^{I(t_n)}B_{n-1}$ Substitution Theorem

iff $\mathrm{Mod}_\omega\, I\, B_n$ Substitution Theorem

It might be noticed that in this lemma the conditions placed on $z_1, \ldots, z_n$ could be relaxed; the proof of the lemma indicates that it would suffice to require that $z_1, \ldots, z_n$ are distinct individual variables with $z_{k+1}, \ldots, z_n$ not occurring in t_k for $k = 1, \ldots, n - 1$.

3 *Substitution for predicate variables*

3.1 Consider a formula A together with an n-ary predicate variable P, and let $z_1, \ldots, z_n$ be n distinct individual variables not occurring in A. The atomic formula $Pz_1 \ldots z_n$ will be called a *name form* of P with the *name variables* $z_1, \ldots, z_n$. Furthermore, let H^* be a formula, called a *substituend* for the name form $Pz_1 \ldots z_n$ of P, whose free individual variables other than $z_1, \ldots, z_n$ are referred to as the *parameters* of H^*. If now $t_1, \ldots, t_n$ are any terms, then $Pt_1 \ldots t_n$ is called a *derivative* of the name form $Pz_1 \ldots z_n$ of P, and $H^*(z_1/t_1, \ldots, z_n/t_n)$ is called the *corresponding derivative* of the substituend H^*; here $H^*(z_1/t_1, \ldots, z_n/t_n)$ indicates the formula which is obtained from H^* upon replacing simultaneously each free occurrence of z_k in H^* by t_k for $k = 1, \ldots, n$.

In terms of these notions, substitution for predicate variables can be described as follows: The formula A is said to be transformed into the formula B by a substitution of H^* for $Pz_1 \ldots z_n$, abbreviated: Sub $A(Pz^n/H^*)B$, iff B is obtained from A upon replacing in A each occurrence of a derivative of the name form $Pz_1 \ldots z_n$ by the corresponding derivative of the substituend H^*, provided that: (i) P does not occur in a component formula $(\forall x A_1)$ of A if x is a parameter of H^*, and (ii) the name variable z_k, $k = 1, \ldots, n$, is not free in a component formula $(\forall x H)$ of H^* if $Pt_1 \ldots t_n$ occurs in A with x occurring in t_k. If conditions (i) and (ii) are not satisfied, then the indicated substitution for predicate variables is left undefined.

It should be noted that the restriction (i) prevents the binding of parameters; and the conditions (ii) ensures that the corresponding derivatives of the substituend are obtained by *free* term substitutions.

3.2 The above description of substitution for predicate variables can be presented in form of a recursive definition in accordance with the recursive definition of formula.

Recursive definition of Sub $A(Pz^n/H^*)B$ It is assumed that $z_1, \ldots, z_n$ are distinct individual variables which do not occur in A.

(1) (a) If A is an atomic formula not containing P then Sub $A(Pz^n/H^*)A$.
(b) If $A = Pt_1 \ldots t_n$ and $\mathbf{Sf}\, H^*(z^n/t^n)B$, then Sub $A(Pz^n/H^*)B$.

(2) If $A = (\neg A_1)$ and Sub $A_1(Pz^n/H^*)B_1$, then Sub $A(Pz^n/H^*)(\neg B_1)$.

(3) If $A = (A_1 \wedge A_2)$, Sub $A_1(Pz^n/H^*)B_1$, and Sub $A_2(Pz^n/H^*)B_2$, then Sub $A(Pz^n/H^*)(B_1 \wedge B_2)$.

(4) (a) If $A = (\forall x A_1)$ and P does not occur in A then Sub $A(Pz^n/H^*)A$.
(b) If $A = (\forall x A_1)$, P occurs in A, x is not free in H^*, and Sub $A_1(Pz^n/H^*)B_1$, then Sub $A(Pz^n/H^*)(\forall x B_1)$.

This definition of substitution for predicate variables includes the degenerate case where $n = 0$; in this case the above definition reduces to a substitution for 0-ary predicate variables (i.e., propositional variables).

3.3 Substitution for predicate variables, as defined here, preserves validity; our proof is based on the following lemma:

Lemma 3 *Let Sub $A(Pz^n/H^*)B$; let I and J be any ω-interpretations which differ at most with respect to P and which are such that $J(P) = \{\langle d_1, \ldots, d_n\rangle \in \omega^n | Mod_\omega\, I^{d_1 \ldots d_n}_{z_1 \ldots z_n} H^*\}$; then $Mod_\omega\, I\, B$ iff $Mod_\omega\, J\, A$.*

Proof: Suppose: (i) Sub $A(Pz^n/H^*)B$, (ii) I and J are ω-interpretations which differ at most with respect to P, and which are such that (iii) $J(P) = \{\langle d_1, \ldots, d_n\rangle \in \omega^n | \mathrm{Mod}_\omega\, I^{d_1 \ldots d_n}_{z_1 \ldots z_n} H^*\}$. First, if P does not occur in A, then $B = A$ and I and J coincide with respect to A so that by the coincidence theorem we get trivially $\mathrm{Mod}_\omega I\, B$ iff $\mathrm{Mod}_\omega J\, A$. Hence it can be assumed that P *occurs in* A. The proof of the lemma is by induction on the rank of A.

(1) If A is an atomic formula then $A = Pt_1 \ldots t_n$ for some terms $t_1, \ldots, t_n$ since by assumption P occurs in A. By (i) it follows that $\mathbf{Sf}\, H^*(z^n/t^n)B$. Hence:

$\mathrm{Mod}_\omega\, I\, B$ iff $\mathrm{Mod}_\omega I^{I(t_1) \ldots I(t_n)}_{z_1 \ldots z_n} H^*$ Lemma 2

iff $\langle I(t_1), \ldots, I(t_n)\rangle \in J(P)$ by (iii)
iff $\langle J(t_1), \ldots, J(t_n)\rangle \in J(P)$ by (ii)
iff $\mathrm{Mod}_\omega J\ Pt_1 \ldots t_n$ iff $\mathrm{Mod}_\omega J\ A$.

(2) If $A = (\neg A_1)$ then by (i) there exists a formula B_1 such that $B = (\neg B_1)$ and $\mathrm{Sub}\, A_1(Pz^n/H^*)B_1$. By induction hypothesis we have: $\mathrm{Mod}_\omega I\ B_1$ iff $\mathrm{Mod}_\omega J\ A_1$; thus:

$\mathrm{Mod}_\omega I\ B$ iff $\mathrm{Mod}_\omega I(\neg B_1)$
iff not $\mathrm{Mod}_\omega I\ B_1$
iff not $\mathrm{Mod}_\omega J\ A_1$ Induction Hypothesis
iff $\mathrm{Mod}_\omega J(\neg A_1)$ iff $\mathrm{Mod}_\omega J\ A$.

(3) If $A = (A_1 \wedge A_2)$ then by (i) there exist formulas B_1 and B_2 such that $B = (B_1 \wedge B_2)$, $\mathrm{Sub}\, A_1(Pz^n/H^*)B_1$ and $\mathrm{Sub}\, A_2(Pz^n/H^*)B_2$. The induction hypothesis yields: $\mathrm{Mod}_\omega I\ B_1$ iff $\mathrm{Mod}_\omega J\ A_1$ and $\mathrm{Mod}_\omega I\ B_2$ iff $\mathrm{Mod}_\omega J\ A_2$. Hence:

$\mathrm{Mod}_\omega I\ B$ iff $\mathrm{Mod}_\omega I(B_1 \wedge B_2)$
iff $\mathrm{Mod}_\omega I\ B_1$ and $\mathrm{Mod}_\omega I\ B_2$
iff $\mathrm{Mod}_\omega J\ A_1$ and $\mathrm{Mod}_\omega J\ A_2$ Induction Hypothesis
iff $\mathrm{Mod}_\omega J(A_1 \wedge A_2)$ iff $\mathrm{Mod}_\omega J\ A$.

(4) Let $A = (\forall x A_1)$; since by assumption P occurs in A, it follows from (i) that x is not free in H^* and that there exists a formula B_1 with $B = (\forall x B_1)$ and $\mathrm{Sub}\, A_1(Pz^n/H^*)B_1$. The induction hypothesis states: If I' and J' are any ω-interpretations which differ at most with respect to P and which are such that $J'(P) = \{\langle d_1, \ldots, d_n\rangle \in \omega^n \mid \mathrm{Mod}_\omega I'^{d_1 \ldots d_n}_{z_1 \ldots z_n} H^*\}$, then $\mathrm{Mod}_\omega I'\ B_1$ iff $\mathrm{Mod}_\omega J'\ A_1$. Now I^d_x and J^d_x are ω-interpretations which differ at most with respect to P in view of (ii); moreover,

$\langle d_1, \ldots, d_n\rangle \in J^d_x(P)$
iff $\langle d_1, \ldots, d_n\rangle \in J(P)$ since J and J^d_x differ at most in x
iff $\mathrm{Mod}_\omega I^{d_1 \ldots d_n}_{z_1 \ldots z_n} H^*$ by (iii)
iff $\mathrm{Mod}_\omega I^{d_1 \ldots d_n d}_{z_1 \ldots z_n x} H^*$ Coincidence Theorem; x is not free in H^*
iff $\mathrm{Mod}_\omega I^{d\, d_1 \ldots d_n}_{x\, z_1 \ldots z_n} H^*$ since x is different from $z_1, \ldots, z_n$ which in turn follows from the fact that x occurs in $A = (\forall x A_1)$ while $z_1, \ldots, z_n$ do not occur in A.

From $\langle d_1, \ldots, d_n\rangle \in J^d_x(P)$ iff $\mathrm{Mod}_\omega I^{d\, d_1 \ldots d_n}_{x\, z_1 \ldots z_n} H^*$ it follows that $J^d_x(P) = \{\langle d_1, \ldots, d_n\rangle \in \omega^n \mid \mathrm{Mod}_\omega I^{d\, d_1 \ldots d_n}_{x\, z_1 \ldots z_n} H^*\}$ and hence (#) $\mathrm{Mod}_\omega I^d_x\ B_1$ iff $\mathrm{Mod}_\omega J^d_x\ A_1$, by the induction hypothesis. Thus:

$\mathrm{Mod}_\omega I\ B$ iff $\mathrm{Mod}_\omega I(\forall x B_1)$
iff $\mathrm{Mod}_\omega I^d_x\ B_1$ for each $d \in \omega$
iff $\mathrm{Mod}_\omega J^d_x\ A_1$ for each $d \in \omega$ by (#)
iff $\mathrm{Mod}_\omega J(\forall x A_1)$ iff $\mathrm{Mod}_\omega J\ A$.

This completes the proof of Lemma 3. The following substitution rule is an immediate consequence of this lemma.

Rule for substitution of predicate variables: *If Sub* $A(Pz^n/H^*)B$ *and* A *is valid, then* B *is valid.*

Proof: Suppose B is not valid; then there exists an individual domain ω and an ω-interpretation I such that not $\mathrm{Mod}_\omega I\, B$. Let J be the ω-interpretation which differs from I at most with respect to P and which is such that $J(P) = \{\langle d_1, \ldots, d_n\rangle \in \omega^n | \mathrm{Mod}_\omega I^{d_1 \ldots d_n}_{z_1 \ldots z_n} H^*\}$. It follows from Lemma 3 that $\mathrm{Mod}_\omega I\, B$ iff $\mathrm{Mod}_\omega J\, A$. Since not $\mathrm{Mod}_\omega I\, B$, we get thus not $\mathrm{Mod}_\omega J\, A$, and hence A is not valid. Thus the validity of A implies the validity of B.

4 *Substitution for functional variables*

4.1 A substitution of an n-ary functional variable f by a term t^* in a formula A is a replacement of each occurrence of a term starting with f by a "corresponding" replacement instance of t^*, subject to certain conditions so as to ensure that validity is preserved. A descriptive formulation analogous to that given for substitution of predicate variables using name forms, etc., is complicated by the fact that an occurrence of the functional variable f in a formula can be within the "scope" of another occurrence of f in the same term. Since substitution for a functional variable f requires replacement at *each* occurrence of f in a formula, such iterated occurrences of f in a formula must be replaced in a corresponding iterated manner. Because of this intricate situation we shall not attempt to give a descriptive formulation of substitution for functional variables, but rather proceed at once with a recursive definition.

4.2 In order to facilitate our recursive definition of substitution for functional variables, we introduce first as an auxiliary notion a term substitution operator $\Delta^{t^*}_{fz^n}$ which is applicable to any term t.

Definition of $\Delta^{t^*}_{fz^n}(t)$ By induction on the length of t:

(1) If t is an individual variable or constant then $\Delta^{t^*}_{fz^n}(t) = t$.

(2) (a) If $t = gt_1 \ldots t_r$ where g is any r-ary functional variable different from f then $\Delta^{t^*}_{fz^n}(t) = g\Delta^{t^*}_{fz^n}(t_1) \ldots \Delta^{t^*}_{fz^n}(t_r)$.

(b) If $t = ft_1 \ldots t_n$ then $\Delta^{t^*}_{fz^n}(t) = t^*[z^n/\Delta^{t^*}_{fz^n}(t^n)]$ where $t^*[z^n/\Delta^{t^*}_{fz^n}(t^n)]$ stands for $t^*[z_1/\Delta^{t^*}_{fz^n}(t_1)] \ldots [z_n/\Delta^{t^*}_{fz^n}(t_n)]$.

With the help of this notion of term substitution, the concept of substitution for functional variables in a formula can be defined inductively in a similar manner as was done in the case of substitution for predicate variables. We shall indicate by Sub $A(fz^n/t^*)B$ that B is the formula obtainable from the formula A upon substituting in A for each occurrence of a term involving the n-ary functional variable f a corresponding term of the substituend t^*.

Recursive definition of Sub $A(fz^n/t^*)B$ It is assumed that $z_1, \ldots, z_n$ are distinct individual variables which do not occur in A.

(1) (a) If $A = P^0$ then Sub $A(fz^n/t^*)A$.

(b) If $A = Pt_1 \ldots t_m$ then Sub $A(fz^n/t^*)P\Delta^{t^*}_{fz^n}(t_1) \ldots \Delta^{t^*}_{fz^n}(t_m)$.
(c) If $A = t_1 \equiv t_2$ then Sub $A(fz^n/t^*)\Delta^{t^*}_{fz^n}(t_1) \equiv \Delta^{t^*}_{fz^n}(t_2)$.

(2) If $A = (\neg A_1)$ and Sub $A_1(fz^n/t^*)B_1$, then Sub $A(fz^n/t^*)(\neg B_1)$.

(3) If $A = (A_1 \wedge A_2)$, Sub $A_1(fz^n/t^*)B_1$, and Sub $A_2(fz^n/t^*)B_2$, then Sub $A(fz^n/t^*)(B_1 \wedge B_2)$.

(4) (a) If $A = (\forall x A_1)$ and f does not occur in A, then Sub $A(fz^n/t^*)A$.
(b) If $A = (\forall x A_1)$, f occurs in A, x does not occur in t^*, and Sub $A_1(fz^n/t^*)B_1$, then Sub $A(fz^n/t^*)(\forall x B_1)$.

From the definition it follows at once that Sub $A(fz^n/t^*)A$ if A is a formula which does not contain f. Furthermore, if $(\forall x A_1)$ is a component formula of A which also contains the functional variable f and x occurs in t^*, then according to our definition there does not exist a formula B with Sub $A(fz^n/t^*)B$.

4.3 In order to show that this type of substitution preserves validity, we establish first the following lemma:

Lemma 4 *Let Sub $A(fz^n/t^*)B$; let I and J be ω-interpretations which differ at most with respect to f and which are such that for all $d_1, \ldots, d_n \in \omega$: $J(f)(d_1, \ldots, d_n) = I^{d_1 \ldots d_n}_{z_1 \ldots z_n}(t^*)$; then $Mod_\omega\, I\, B$ iff $Mod_\omega\, J\, A$.*

Proof: Suppose: (i) Sub $A(fz^n/t^*)B$, (ii) I and J are ω-interpretations which differ at most with respect to f, and which are such that (iii) $J(f)(d_1, \ldots, d_n) = I^{d_1 \ldots d_n}_{z_1 \ldots z_n}(t^*)$ for all $d_1, \ldots, d_n \in \omega$. By induction on the length of t we show first that for any term t:

$$(\#) \quad J(t) = I(\Delta^{t^*}_{fz^n}(t)) \quad .$$

Indeed, if t is an individual variable or an individual constant, then $\Delta^{t^*}_{fz^n}(t) = t$ and by (ii) it follows that $J(t) = I(\Delta^{t^*}_{fz^n}(t))$.

Next, if $t = gt_1 \ldots t_r$ with $g \neq f$ then by definition of $\Delta^{t^*}_{fz^n}$: $\Delta^{t^*}_{fz^n}(t) = g\Delta^{t^*}_{fz^n}(t_1) \ldots \Delta^{t^*}_{fz^n}(t_r)$ and hence

$$\begin{aligned}
J(t) &= J(gt_1 \ldots t_r) \\
&= J(g)(J(t_1), \ldots, J(t_r)) \\
&= I(g)(J(t_1), \ldots, J(t_r)) && \text{by (ii)} \\
&= I(g)(I(\Delta^{t^*}_{fz^n}(t_1)), \ldots, I(\Delta^{t^*}_{fz^n}(t_r))) && \text{Induction Hypothesis} \\
&= I(g\Delta^{t^*}_{fz^n}(t_1) \ldots \Delta^{t^*}_{fz^n}(t_r)) = I(\Delta^{t^*}_{fz^n}(t)).
\end{aligned}$$

Finally, if $t = ft_1 \ldots t_n$ then by definition of $\Delta^{t^*}_{fz^n}$ we have $\Delta^{t^*}_{fz^n}(t) = t^*[z^n/\Delta^{t^*}_{fz^n}(t^n)]$ and therefore

$$\begin{aligned}
J(t) &= J(ft_1 \ldots t_n) \\
&= J(f)(J(t_1), \ldots, J(t_n)) \\
&= I^{J(t_1) \ldots J(t_n)}_{z_1 \ldots z_n}(t^*) && \text{by (iii)} \\
&= I^{I(\Delta^{t^*}_{fz^n}(t_1)) \ldots I(\Delta^{t^*}_{fz^n}(t_n))}_{z_1 \ldots z_n}(t^*) && \text{Induction Hypothesis} \\
&= I(t^*[z^n/\Delta^{t^*}_{fz^n}(t^n)]) && \text{by Lemma 1} \\
&= I(\Delta^{t^*}_{fz^n}(t)).
\end{aligned}$$

This completes the inductive proof of property (#). In order to prove now the lemma, we note first that if f does not occur in A then $B = A$ so that in view of (ii) and the coincidence theorem we get at once: $\mathrm{Mod}_\omega I\, B$ iff $\mathrm{Mod}_\omega J\, A$. Hence we can assume that *f occurs in A*. The lemma is now proved by induction on the rank of A.

(1) If A is an atomic formula then A has one of the following two forms: (a) $A = Pt_1 \ldots t_m$ or (b) $A = t_1 \equiv t_2$. By (i) we get in case (a): $B = P\Delta^{t^*}_{fz^n}(t_1) \ldots \Delta^{t^*}_{fz^n}(t_m)$ and therefore:

$$\begin{aligned}
\mathrm{Mod}_\omega I\, B &\text{ iff } \mathrm{Mod}_\omega I\, P\Delta^{t^*}_{fz^n}(t_1) \ldots \Delta^{t^*}_{fz^n}(t_m) \\
&\text{ iff } \langle I(\Delta^{t^*}_{fz^n}(t_1)), \ldots, I(\Delta^{t^*}_{fz^n}(t_m))\rangle \in I(P) \\
&\text{ iff } \langle J(t_1), \ldots, J(t_m)\rangle \in J(P) && \text{by (\#) and (ii)} \\
&\text{ iff } \mathrm{Mod}_\omega J\, Pt_1 \ldots t_m \text{ iff } \mathrm{Mod}_\omega J\, A.
\end{aligned}$$

In case (b) we have by (i): $B = \Delta^{t^*}_{fz^n}(t_1) \equiv \Delta^{t^*}_{fz^n}(t_2)$ and therefore:

$$\begin{aligned}
\mathrm{Mod}_\omega I\, B &\text{ iff } \mathrm{Mod}_\omega I\, \Delta^{t^*}_{fz^n}(t_1) \equiv \Delta^{t^*}_{fz^n}(t_2) \\
&\text{ iff } I(\Delta^{t^*}_{fz^n}(t_1)) = I(\Delta^{t^*}_{fz^n}(t_2)) \\
&\text{ iff } J(t_1) = J(t_2) && \text{by (\#)} \\
&\text{ iff } \mathrm{Mod}_\omega J\, t_1 \equiv t_2 \text{ iff } \mathrm{Mod}_\omega J\, A.
\end{aligned}$$

(2) If $A = (\neg A_1)$ then by (i) there exists a formula B_1 such that $B = (\neg B_1)$ and $\mathrm{Sub}\, A_1(fz^n/t^*)B_1$. By induction hypothesis we have: $\mathrm{Mod}_\omega I\, B_1$ iff $\mathrm{Mod}_\omega J\, A_1$. Thus:

$$\begin{aligned}
\mathrm{Mod}_\omega I\, B &\text{ iff } \mathrm{Mod}_\omega I(\neg B_1) \\
&\text{ iff not } \mathrm{Mod}_\omega I\, B_1 \\
&\text{ iff not } \mathrm{Mod}_\omega J\, A_1 && \text{Induction Hypothesis} \\
&\text{ iff } \mathrm{Mod}_\omega J(\neg A_1) \text{ iff } \mathrm{Mod}_\omega J\, A.
\end{aligned}$$

(3) If $A = (A_1 \wedge A_2)$ then by (i) there exist formulas B_1 and B_2 such that $B = (B_1 \wedge B_2)$, $\mathrm{Sub}\, A_1(fz^n/t^*)B_1$, and $\mathrm{Sub}\, A_2(fz^n/t^*)B_2$. By induction hypothesis we have: $\mathrm{Mod}_\omega I\, B_1$ iff $\mathrm{Mod}_\omega J\, A_1$ and $\mathrm{Mod}_\omega I\, B_2$ iff $\mathrm{Mod}_\omega J\, A_2$. Hence

$$\begin{aligned}
\mathrm{Mod}_\omega I\, B &\text{ iff } \mathrm{Mod}_\omega I(B_1 \wedge B_2) \\
&\text{ iff } \mathrm{Mod}_\omega I\, B_1 \text{ and } \mathrm{Mod}_\omega I\, B_2 \\
&\text{ iff } \mathrm{Mod}_\omega J\, A_1 \text{ and } \mathrm{Mod}_\omega J\, A_2 && \text{Induction Hypothesis} \\
&\text{ iff } \mathrm{Mod}_\omega J(A_1 \wedge A_2) \text{ iff } \mathrm{Mod}_\omega J\, A.
\end{aligned}$$

(4) If $A = (\forall x A_1)$ then by (i) there exists a formula B_1 such that $B = (\forall x B_1)$, x does not occur in t^*, and $\mathrm{Sub}\, A_1(fz^n/t^*)B_1$. The induction hypothesis states: If I' and J' are any ω-interpretations which differ at most with respect to f and which are such that $J'(f)(d_1, \ldots, d_n) = I'^{d_1 \ldots d_n}_{z_1 \ldots z_n}(t^*)$ for all $d_1, \ldots, d_n \in \omega$, then: $\mathrm{Mod}_\omega I'\, B_1$ iff $\mathrm{Mod}_\omega J'\, A_1$. Consider now the ω-interpretations I^d_x and J^d_x which in view of (ii) differ at most with respect to f. Moreover, we have for all $d_1, \ldots, d_n \in \omega$:

$$\begin{aligned}
J^d_x(f)(d_1, \ldots, d_n) &= J(f)(d_1, \ldots, d_n) && \text{since } J \text{ and } J^d_x \text{ differ at most with respect to } x \\
&= I^{d_1 \ldots d_n}_{z_1 \ldots z_n}(t^*) && \text{by (iii)}
\end{aligned}$$

$= I^{d_1 \ldots d_n d}_{z_1 \ldots z_n x}(t^*)$ since x does not occur in t^*

$= I^{d\, d_1 \ldots d_n}_{x\, z_1 \ldots z_n}(t^*)$ since x is different from $z_1, \ldots, z_n$ which in turn follows from the fact that x occurs in $A = (\forall x A_1)$ whereas $z_1, \ldots, z_n$ do not occur in A.

Since $J^d_x(f)(d_1, \ldots, d_n) = I^{d\, d_1 \ldots d_n}_{x\, z_1 \ldots z_n}(t^*)$, it follows by induction hypothesis that $\text{Mod}_\omega\, I^d_x\, B_1$ iff $\text{Mod}_\omega\, J^d_x\, A_1$ for all $d \in \omega$, and hence:

$\text{Mod}_\omega\, I\, B$ iff $\text{Mod}_\omega\, I(\forall x B_1)$
iff $\text{Mod}_\omega\, I^d_x\, B_1$ for all $d \in \omega$
iff $\text{Mod}_\omega\, J^d_x\, A_1$ for all $d \in \omega$
iff $\text{Mod}_\omega\, J(\forall x A_1)$ iff $\text{Mod}_\omega\, J\, A$.

This completes the inductive proof of Lemma 4. On the basis of this lemma we get at once the following substitution rule.

Rule for substitution of functional variables: *If Sub $A(fz^n/t^*)B$ and A is valid, then B is valid.*

Proof: Suppose Sub $A(fz^n/t^*)B$ and assume that B is not valid. Then there is an individual domain ω and an ω-interpretation I such that not $\text{Mod}_\omega I\, B$. Let J be the ω-interpretation which differs from I at most with respect to f and which is such that for all $d_1, \ldots, d_n \in \omega$: $J(f)(d_1, \ldots, d_n) = I^{d_1 \ldots d_n}_{z_1 \ldots z_n}(t^*)$. By Lemma 4 it follows that $\text{Mod}_\omega\, I\, B$ iff $\text{Mod}_\omega\, J\, A$. Since not $\text{Mod}_\omega\, I\, B$, we have thus not $\text{Mod}_\omega\, J\, A$ so that A is not valid. Therefore, the validity of A implies the validity of B.

5 As an *illustration* for the two types of substitution discussed in this paper, consider the formula

$$A = (\forall x(Q^2 x g^1 y \rightarrow (\exists z Q^2 xz)))$$

where Q^2 is a binary predicate variable and g^1 is a unary functional variable (with $\rightarrow$ and $\exists$ taken as abbreviations in the familiar manner). Choosing $Q^2 z_1 z_2$ as the name form and $P^2 f^2 z_1 z_2 f^2 z_1 z_2$ as substituend with P^2 as a binary predicate variable and f^2 as a binary functional variable, predicate variable substitution in A yields Sub $A(Qz^2/P^2 f^2 z_1 z_2 f^2 z_1 z_2)B$ where $B = (\forall x(P^2 f^2 x g^1 y f^2 x g^1 y \rightarrow (\exists z P^2 f^2 xz f^2 xz)))$.

Functional variable substitution in B with $f^2 z_1 z_2$ as name form and $f^2 f^2 z_1 z_2 u$ as substituend yields Sub $B(fz^2/f^2 f^2 z_1 z_2 u)C$ where $C = (\forall x(P^2 f^2 f^2 x g^1 yu f^2 f^2 x g^1 yu \rightarrow (\exists z P^2 f^2 f^2 xzu f^2 f^2 xzu)))$.

Predicate variable substitution in B with $P^2 z_1 z_2$ as name form and $(\forall u P^2 z_1 u)$ as substituend leads to Sub $B(Pz^2/(\forall u P^2 z_1 u))D$ where $D = (\forall x((\forall u P^2 f^2 x g^1 yu) \rightarrow (\exists z(\forall u P^2 f^2 xzu))))$.

Observe that in this last substitution the substituend did not contain the name variable z_2. Now A is a valid formula and hence it follows from the rules of substitution for predicate variables and functional variables that B, C, and D are also valid formulas.

Note that the predicate variable substitution $Pz^2/(\forall u P^2 z_1 u)$ which was applied to formula B to obtain D, is not applicable to the formula C since such a substitution in C would bind the free occurrence of u in C. Again, the functional variable substitution $fz^2/f^2f^2z_1z_2u$ which was applied to B to obtain C, is not applicable to the formula D since such a substitution would bind the parameter u in the substituend $f^2f^2z_1z_2u$.

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