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Some Elementary Closure Properties of n-Cylinders

M. B. THURAISINGHAM

1 Introduction The concept n-cylinder was originally defined [4] in order to construct noncylindrical decision problems for system functions, a kind of function defined by Cleave [1]. It is a generalization of Young's [9] concept of a semicylinder and it forms a link between a semicylinder and a cylinder. Its definition is as follows:

Definition A set P is an n-cylinder if and only if there is a recursive function g such that for all $x_1, x_2, \ldots, x_n$,

$$\{x_1, x_2, \ldots, x_n\} \subseteq P \Rightarrow g(x_1, x_2, \ldots, x_n) \in P$$
$$\{x_1, x_2, \ldots, x_n\} \subseteq \bar{P} \Rightarrow g(x_1, x_2, \ldots, x_n) \in \bar{P}\ .$$

This function g is called the n-cylinder function for P. It can be seen that a semicylinder is a 1-cylinder. Properties of n-cylinders and their relationship to cylinders were explored in [5]-[8], and it was subsequently shown that:

(i) A set is a cylinder if and only if it is an n-cylinder for each $n \geqslant 1$.
(ii) The class of all $(n+1)$-cylinders is a proper subset of the class of all n-cylinders for each $n \geqslant 1$.
(iii) For each $n \geqslant 1$, the class of all n-cylinders contains a simple set and hence a nonsplinter.
(iv) For each $n \geqslant 1$, the class of all n-cylinders contains a nonsimple nonsplinter.
(v) A set P is an n-cylinder ($n \geqslant 1$) if and only if there is a recursive function g such that for all x,

$$x \in P \Rightarrow D_{g(x)} \subseteq P$$
$$x \in \bar{P} \Rightarrow D_{g(x)} \subseteq \bar{P}$$

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and $D_{g(x)}$ has $(n + 1)$ members, where D_m is the m^{th} finite set in some standard enumeration.

From these results we can deduce that as n tends to infinity, the class of all n-cylinders coincides with the class of all cylinders.

In Section 3 of this paper, we study some elementary closure properties of n-cylinders and subsequently obtain the following results:

(i) The class of all n-cylinders $(n \geq 1)$ is closed under the operation of complementation.
(ii) If A and B are n-cylinders $(n \geq 1)$, then so is A Join B where $A \text{ Join } B = \{2x: x \in A\} \cup \{2x + 1: x \in B\}$.
(iii) If A or B is an n-cylinder $(n \geq 1)$ then so is $A \times B$. Where $A \times B$ is the Cartesian product of A and B.
(iv) It is not the case that for each $n \geq 1$, the class of all n-cylinders is closed under the operations of intersection and union.

The preliminary definitions needed for the proof of these results are given in Section 2. For the recursive function theory terminology used in this paper we refer to [3].

*2 **Definitional preliminaries*** The definition of system functions and the definitions in the theory of graphs given in this section have been obtained from [1] and [2]. In Section 3, these graph theoretic concepts are employed in formulating certain algorithms.

Let $f: N \to P_w(N)$ where N is the set of all natural numbers and $P_w(N)$ is the set of all finite subsets of N.

For $X \in P_w(N)$, define $f(X) = \cup \{f(x): x \in X\}$. For each $x \in N$, define $f^0(x) = x$, $f^{k+1}(x) = f(f^k(x))$. f^{-1}, the inverse of f, is defined by $f^{-1}(x) = \{y: x \in f(y)\}$.

By "y is directly derivable by f from x" we mean $y \in f(x)$. By "y is derivable by f from x" (denoted $y \in C_f x$ or $x \in C_{f^{-1}} y$) we mean either $y = x$ or $y \in f(x)$ or there exist $y_1, y_2, \ldots, y_n$ $(n \geq 1)$ such that $y_1 \in f(x)$, $y \in f(y_n)$ and for all $i (1 \leq i \leq n - 1)$, $y_{i+1} \in f(y_i)$.

A system function is a function f: $n \to P_w(N)$ such that there exist recursive functions a and b such that for all x, $f(x) = D_{a(x)}$ and $f^{-1}(x) = D_{b(x)}$ where D_n is the n^{th} finite set in some standard enumeration.

A system function f which has the property that for all x, $f(x)$ has at most one member is called a machine function. The class of all system functions is denoted by $\mathfrak{S}$ and the class of all machine functions is denoted by $\mathfrak{M}$.

Let D be a digraph whose points are in N. By $x \in D$ is meant: x is a point in D. If $x \in D$ and $y \in D$, then $x \overset{+}{\to} y$ is a directed line if there is a line from x to y in D.

We assume that there are no directed lines $x \overset{+}{\to} x$ and there is at most one line from x to y.

If $x \in D$, then $D(x)$ is the connected component (or component) of D which contains x as a point. Its graphical representation is shown in Figure 1. If $x \in D$ and $y \in D$, then by $x \to y(D)$ is meant: $x = y$ or $x \overset{+}{\to} y$ is a directed line or there exist distinct points $v_1, v_2, \ldots, v_n (n \geq 1)$ of D such that $x \overset{+}{\to} v_1$, $v_1 \overset{+}{\to} v_2$, $v_2 \overset{+}{\to} v_3, \ldots, v_{n-1} \overset{+}{\to} v_n$, $v_n \overset{+}{\to} y$ are all directed lines.

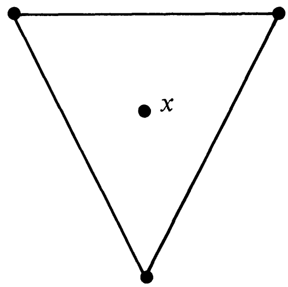

Figure 1

By $x \| y$ (D) is meant: x and y belong to different components of D. By $x | y$ (D) is meant: x and y belong to the same component of D, but it is not the case that $x \rightarrow y(D)$ or $y \rightarrow x(D)$.

The in-degree (out-degree respectively) of a point x of D is the number of points y of D such that $y \overset{\rightarrow}{\cdot} x$ ($x \overset{\rightarrow}{\cdot} y$ respectively) is a directed line. A point x is a root (leaf respectively) if its in-degree (out-degree respectively) is 0. $r(x)$ ($t(x)$ respectively) is the least number y such that y is a root (leaf respectively) of D and $y \rightarrow x(D)$ ($x \rightarrow y(D)$ respectively). A digraph is labeled if some of its points are distinguished from one another by names drawn from some given infinite list. By the expression "Introduce the labels $L_1, L_2, \ldots, L_n$ $(n \geqslant 1)$ to the digraph D" is meant the following: Find the least n numbers $x_1 < x_2 < \ldots < x_n$ which are not points of D. Adjoin these numbers as new points so that each point $x_i (1 \leqslant i \leqslant n)$ forms a new component. Name x_1 by L_1, x_2 by $L_2, \ldots, x_n$ by L_n.

A bigraph is a digraph whose points have in-degree of at most 2 and out-degree of at most 1. Let B be a bigraph and $x \in B$. Then a point z is denoted x^1 if there is a point y of B such that $x \overset{\rightarrow}{\cdot} y$ and $z \overset{\rightarrow}{\cdot} y$ are directed lines. By the expression "Extend the bigraph B to the bigraph B^1" is meant the following: Let $t_1, t_2, \ldots, t_k$ be the leaves of B and $r_1, r_2, \ldots, r_m$ be its roots. Find the least $2k + 2m$ numbers, say $x_1 < x_2 < x_3 < \ldots < x_{2k+2m}$, which are not points of B. Adjoin these numbers as new points and join the lines

$$t_1 \overset{\rightarrow}{\cdot} x_1,\ x_2 \overset{\rightarrow}{\cdot} x_1,\ t_2 \overset{\rightarrow}{\cdot} x_3,\ x_4 \overset{\rightarrow}{\cdot} x_3, \ldots, t_k \overset{\rightarrow}{\cdot} x_{2k-1},\ x_{2k} \overset{\rightarrow}{\cdot} x_{2k-1},\ x_{2k+1} \overset{\rightarrow}{\cdot} r_1,\ x_{2k+2} \overset{\rightarrow}{\cdot} r_1,$$
$$x_{2k+3} \overset{\rightarrow}{\cdot} r_2,\ x_{2k+4} \overset{\rightarrow}{\cdot} r_2, \ldots, x_{2k+2m-1} \overset{\rightarrow}{\cdot} r_m,\ x_{2k+2m} \overset{\rightarrow}{\cdot} r_m.$$

The resulting graph is B^1 (see Figure 2).

3 Properties of n-cylinders In this section we will prove the following result:

Result α For each $f \in S$, define A^f and K^f as follows:

$A^f = \{(x, y): y \in C_f x\}$ (general derivability problem for f)
$K^f = \{(x, y): x \in C_f y\}$ (inverse general derivability problem for f).

Then:

(i) If P is an n-cylinder $(n \geqslant 1)$, then so is $\overline{P}$.

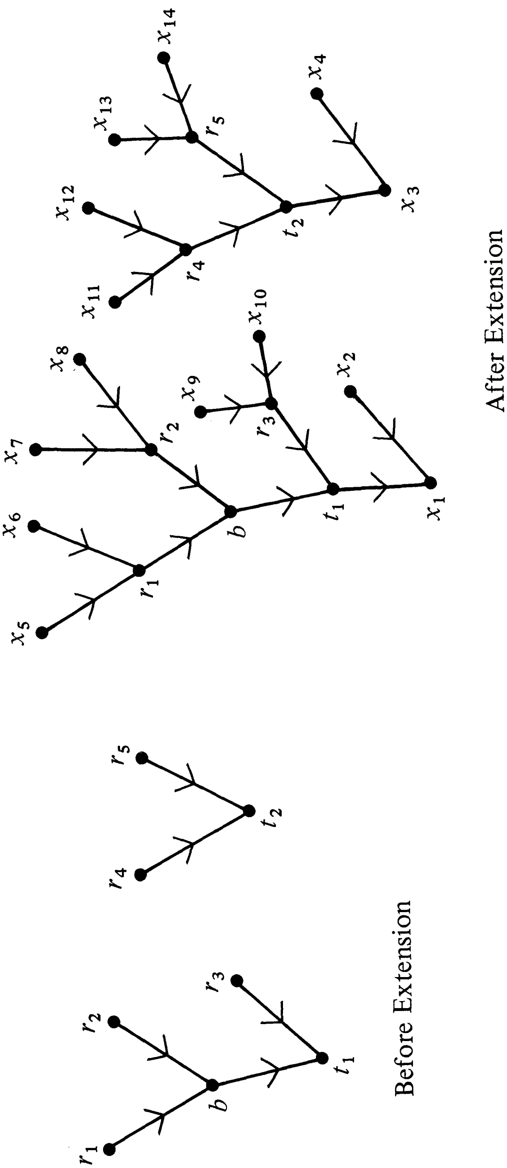
r_1
r_2
b
r_3
t_1
r_4
r_5
t_2
Before Extension
x_5
x_6
r_1
x_7
x_8
r_2
b
x_9
x_{10}
r_3
t_1
x_2
x_1
x_{11}
x_{12}
r_4
t_2
x_{13}
x_{14}
r_5
x_4
x_3
After Extension

Figure 2

(ii) If P and Q are n-cylinders ($n \geqslant 1$), then so is P Join Q.
(iii) If P or Q are n-cylinders ($n \geqslant 1$), then so is $P \times Q$.
(iv) (a) For all $f \in M$, A^f and K^f are either recursive or cylinders.
(b) There exist $g \in M$ and $h \in M$ such that both $A^g \cap B^g$ and $A^h \cup B^h$ are neither recursive nor 2-cylinders, where $\cap$ and $\cup$ stand for intersection and union, respectively.

This result shows that the class of all n-cylinders ($n \geqslant$) is closed under the operations of complementation, Cartesian product, and Join. Furthermore, since every cylinder is an n-cylinder for each $n \geqslant 1$, it is not the case that for each n, the class of all n-cylinders is closed under the operation $\cap$ and $\cup$. From this we can also deduce that the class of all cylinders is not closed under the operations of $\cap$ and $\cup$.

Proof of Result α: From the definition of n-cylinders, it can be easily seen that if P is an n-cylinder, then so is $\overline{P}$. The proofs of Result α(ii) and (iii) utilize a similar technique to the proof of Exercise 7-36(a) and (b) in [3]. The proof that A^f is either recursive or a cylinder for each $f \in M$ can be obtained in [1]. Since K^p is one-one equivalent to A^p for each $p \in S$, K^f is also either recursive or a cylinder for each $f \in M$. We will now prove Result α(iv,b) for the operation $\cap$. The proof of this result for the operation $\cup$ will utilize a similar technique.

Proof of Result α(iv,b) for the Operation $\cap$: The proof is divided into two parts. The first part consists of a programme in which labeled bigraphs $B^0, B^1, B^2, \ldots$ will be constructed with the following properties:

(i) There is a recursive function α such that $\alpha(m)$ is the Gödel number of B^m.
(ii) For each m, m is a point of B^m.
(iii) For each m, B^{m+1} is an extension of B^m; i.e., all points of B^m are points of B^{m+1}. If $m < p$, then for any point x of B^m, all lines incident with x in B^p are lines of B^{m+1}. Furthermore, B^{m+1} contains as a point the least number which is not a point of B^m.
(iv) For each m, a component of B^m has at most one label. These labels are taken from a given set $\bigcup_{i=1}^{2} \{P_e^i : e \geqslant 0\}$ of markers.

Also in this programme the dependence of a number on another number will be defined by induction.

The second part of the proof consists of three lemmas by means of which it will be proved that there is a $g \in M$ such that for no e is it true that ϕ_e^2 is a 2-cylinder function for $A^g \cap K^g$ where ϕ_e^2 is the e^{th} partial recursive function of two variables in some standard enumeration.

I Programme

Stage 0. B^0 consists of the points 0,1 labeled P_0^1, P_0^2, respectively.

Stage $m(m \geqslant 1)$, Step 1. Introduce the labels P_m^1, P_m^2 to B^{m-1} and extend the resulting graph to $\hat{B}$.

Step 2. Find the smallest number $e \leqslant m$ such that there exist numbers x, y, u, w, z all less than m satisfying the condition R where R is the conjunction of the conditions R_1, R_2, and R_3 where $R_1 \equiv T(e, \tau(x, y), \tau(y, x), z)$ and $U(z) = \tau(u, w)$. τ is a recursive function which maps N^2 one-one and onto N, T is the Kleene's T-predicate, and if z is the Gödel number of a Turing Machine computation, then $U(z)$ is the output of the Turing Machine for that computation.

$R_2 \equiv x$ and y are labeled P_e^1 and P_e^2, respectively, in $\hat{B}$. $R_3 \equiv \sim((x = u \wedge y = w) \vee (x = w \wedge y = u))$.

If there does not exist such an e, set $B^m = \hat{B}$. If there exists such an e, define

$$\begin{aligned}
e_m &= (\mu e)\,(\exists z, x, y, u, w \text{ all} \leqslant m)R(z, x, y, u, w, e, m)\\
x_m &= (\mu x)\,(\exists z, y, u, w \text{ all} \leqslant m)R(z, x, y, u, w, e_m, m)\\
y_m &= (\mu y)\,(\exists z, u, w \text{ all} \leqslant m)R(z, x_m, y, u, w, e_m, m)\\
u_m &= (\mu u)\,(\exists z, w \text{ both} \leqslant m)R(z, x_m, y_m, u, w, e_m, m)\\
w_m &= (\mu w)\,(\exists z \leqslant m)R(z, x_m, y_m, u_m, w, e_m, m).
\end{aligned}$$

For convenience let e_m, x_m, y_m, u_m, w_m be e, x, y, u, w, respectively. In Step 3 of the programme e will be attacked (in a sense to be made clear) and it will be ensured that: $(x \to y \wedge y \to x)(B^m) \equiv (u \to w \wedge w \to u)(B^m)$.

Step 3. (a) Delete P_e^1 and P_e^2. (b) Reintroduce all labels of the form P_j^1, P_j^2 such that j depends on e (i.e., if these labels are already assigned to points delete them and reintroduce them; if not, introduce them). Let the resulting graph be B^+.

Construct B^m from B^+ according to the various cases in Step 3(c). The graphical representation of these cases will be given at the end of the programme.

(c) *Case 1.* $B^+(u) = B^+(w)$.

1.1 If $(u \to w \wedge w \to u)(B^+)$, then set $B^m = B^+$ (Figure 3a).

1.2 If $u|w(B^+)$, then set $B^n = B^+ \cup \{t(x)\overset{\to}{\cdot}r(y), t(y)\overset{\to}{\cdot}r(x)\}$ (i.e., B^n results from B^+ by adjoining the lines $t(x)\overset{\to}{\cdot}r(y)$ and $t(y)\overset{\to}{\cdot}r(x)$) (Figure 3b).

1.3 If $[(u \to w \wedge \sim(w \to u))(B^+)] \vee [w \to u \wedge \sim(u \to w))(B^+)]$, then:

(i) If there is no i such that $B^+(u) = B^+(w)$ is labeled either P_i^1 or P_i^2, then:

(α) If $u, w \in B^+(x)$ or $u, w \in B^+(y)$, then set $B^n = B^+ \cup \{t(w)\overset{\to}{\cdot}r(u)\}$ if $(u \to w \wedge \sim(w \to u))(B^+)$ (Figure 3c). $B^n = B^+ \cup \{t(u)\overset{\to}{\cdot}r(w)\}$ if $(w \to u \wedge \sim(u \to w))(B^+)$ (Figure 3d).

(β) If $u, w \in B(x) \cup B^+(y)$, then set $B^n = B^+ \cup \{t(x)\overset{\to}{\cdot}r(y), t(y)\overset{\to}{\cdot}r(x)\}$ (Figure 3b).

(ii) If there is an i such that $B^+(u) = B^+(w)$ is labeled either P_i^1 or P_i^2, then:

(α) If $i > e$, delete P_i^1, P_i^2 and reintroduce them. Let the resulting graph be B^*. Set $B^n = B^* \cup \{t(x)\overset{\to}{\cdot}r(y), t(y)\overset{\to}{\cdot}r(x)\}$ (Figure 3b).

(β) If $i < e$, set $B^n = B^+ \cup \{t(x)\overset{\to}{\cdot}r(y), t(y)\overset{\to}{\cdot}r(x)\}$ and record that e depends on i (Figure 3b).

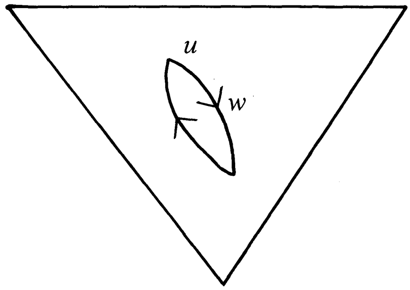

Figure 3a

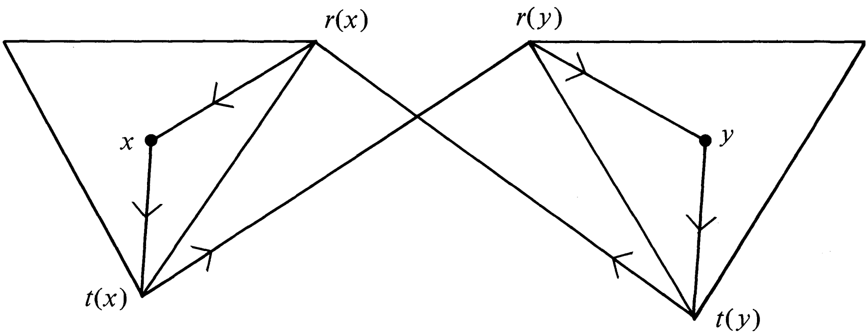

Figure 3b

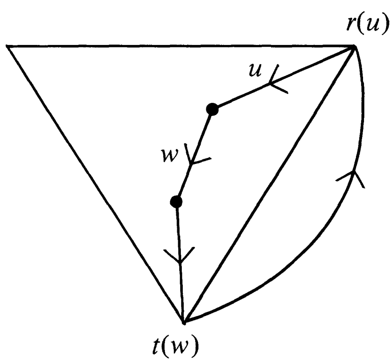

Figure 3c

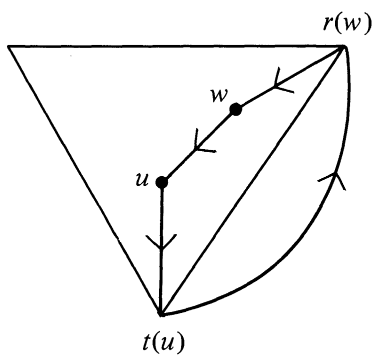

Figure 3d

Case 2. $B^+(u) \neq B^+(w)$.

2.1 $u \in B^+(x)$ and $w \in B^+(y)$.

(a) If $u \to x(B^+)$, then set $B^n = B^+ \cup \{t(x) \overset{+}{\to} r(y), t(y) \overset{+}{\to} r(u^1)\}$ (Figure 3e).
(b) If $x \to u(B^+)$, then set $B^n = B^+ \cup \{t(x) \overset{+}{\to} r(y), t(y) \overset{+}{\to} r(x^1)\}$ (Figure 3f).
(c) If $x | u(B^+)$, then set $B^n = B^+ \cup \{t(x) \overset{+}{\to} r(y), t(y) \overset{+}{\to} r(x)\}$ (Figure 3g).
(d) If $(x = u \wedge y \to w)(B^+)$, then set $B^n = B^+ \cup \{t(x) \overset{+}{\to} r(y^1), t(y) \overset{+}{\to} r(x)\}$ (Figure 3h).
(e) If $(x = u \wedge w \to y)(B^+)$, then set $B^n = B^+ \cup \{t(x) \overset{+}{\to} r(w^1), t(y) \overset{+}{\to} r(x)\}$ (Figure 3i).
(f) If $(x = u \wedge w | y)(B^+)$, then set $B^n = B^+ \cup \{t(x) \overset{+}{\to} r(y), t(y) \overset{+}{\to} r(x)\}$ (Figure 3j).

2.2 If $w \in B^+(x)$ and $u \in B^+(y)$, then Cases 2.2(a-f) are similar to the respective Cases 2.1(a-f) with u and w interchanged.

2.3 Cases 2.1 and 2.2 do not hold (Figure 3b).

(i) If there is an i such that $B^+(u)$ and $B^+(w)$ are labeled P_i^1, P_i^2, respectively, or vice versa, then:

(α) If $i > e$, delete P_i^1, P_i^2 and reintroduce them. Let the resulting graph be B^*. Set $B^n = B^* \cup \{t(x) \overset{+}{\to} r(y), t(y) \overset{+}{\to} r(x)\}$.
(β) If $i < e$, set $B^n = B^+ \cup \{t(x) \overset{+}{\to} r(y), t(y) \overset{+}{\to} r(x)\}$ and record that e depends on i.

(ii) If there is no i such that $B^+(u)$ and $B^+(w)$ are labeled P_i^1, P_i^2, respectively, or vice versa, then set $B^n = B^+ \cup \{t(x) \overset{+}{\to} r(y), t(y) \overset{+}{\to} r(x)\}$.

This ends the programme.

Set $B = \bigcup_{n=0}^{\infty} B^n$ where $B^i \cup B^j = B^j$ if $i \leqslant j$
$= B^i$ if $i > j$.

Clearly for any point x of B^m, all lines incident with x in B are lines of B^{n+1}.

$$\begin{aligned} \text{Define } g(x) &= \{y\colon (x, y) \text{ is a line of } B\} \\ &= \{y\colon (x, y) \text{ is a line of } B^{x+1}\} \\ g^{-1}(x) &= \{y\colon (y, x) \text{ is a line of } B\} \\ &= \{y\colon (y, x) \text{ is a line of } B^{x+1}\}. \end{aligned}$$

Then $g \in M$ and $A^g \cap K^g = \{(x, y)\colon (x \to y \wedge y \to x)(B)\}$

II Definition A label L is fixed at a stage numbered H if either: (i) L remains assigned to the same point at all stages numbered $n \geqslant H$, or (ii) L remains unassigned at all stages numbered $n \geqslant H$.

Lemma 1 *For each e, there is a stage $H(e)$ at which all labels P_i^1, P_i^2 where $i \leqslant e$ are fixed.*

Proof: There are four ways in which labels P_e^1, P_e^2 where $i \leqslant e$ can be moved. They are as follows:

G1. Introduction at Stage e via Step 1
G2. Deletion via an attack on e (Step 3(a))

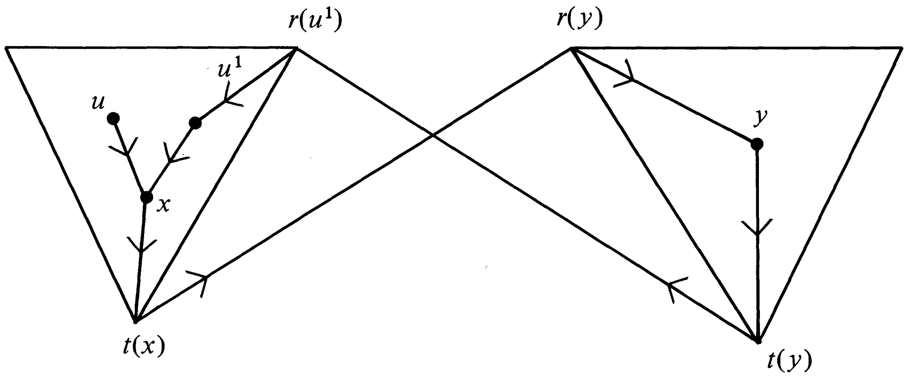

Figure 3e

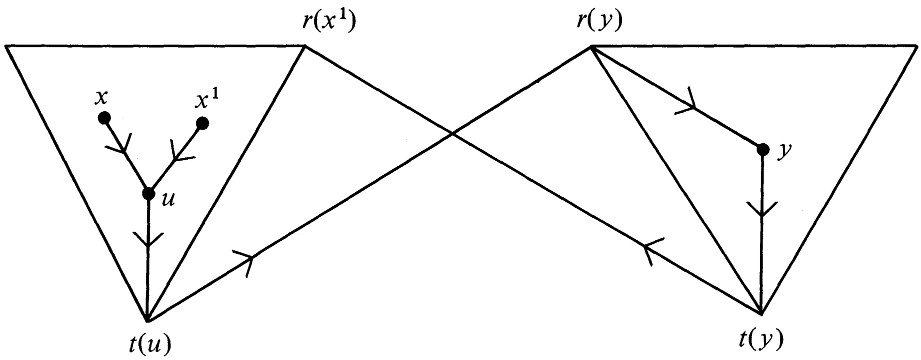

Figure 3f

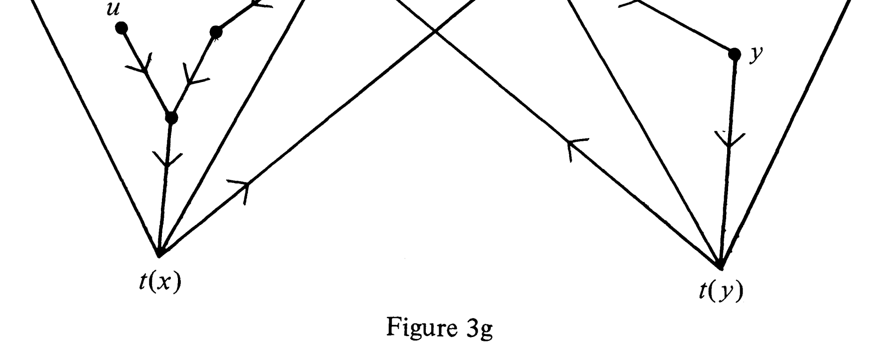

Figure 3g

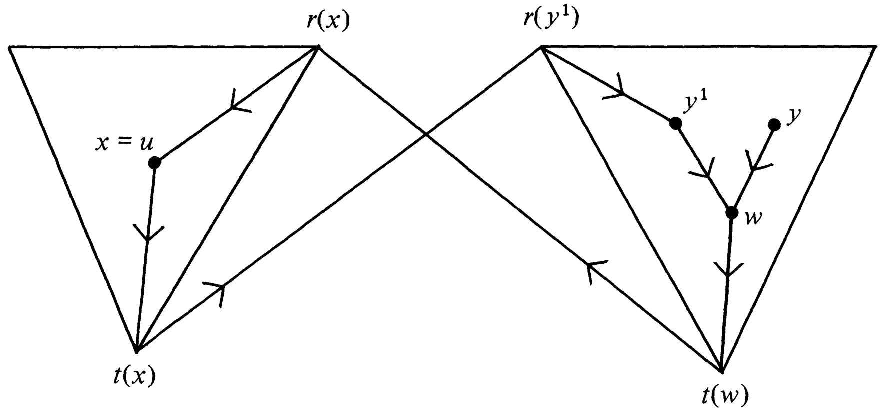

Figure 3h

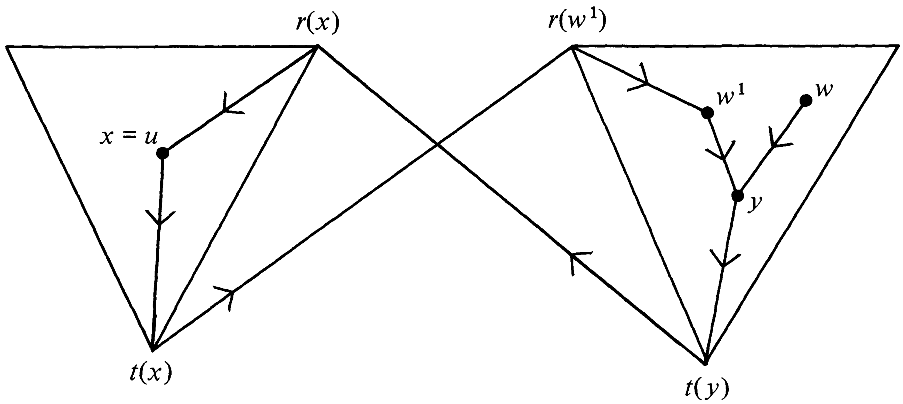

Figure 3i

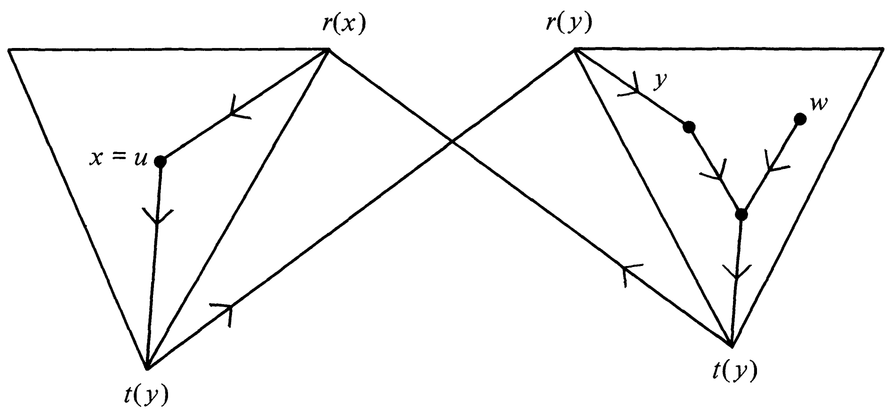

Figure 3j

G3. Reintroduction via an attack on i where e depends on i (Step 3(b))
G4. Reintroduction via an attack on i where $i < e$ under Cases 1.3(ii, α) and 2.3(i, α).

The proof of the lemma is by induction on e.

Basis e = 0 P_0^1, P_0^2 are introduced at Stage 0 (G1). As 0 does not depend on any number, and as there is no i such that $i < 0$, P_0^1, P_0^2 cannot be moved by G3 or G4. Therefore if 0 is never attacked, then P_0^1, P_0^2 are fixed and assigned at Stage 0; i.e., $H(0) = 0$. But if 0 is attacked at a Stage m, then P_0^1, P_0^2 are fixed and unassigned at Stage m; i.e., $H(0) = m$.

Inductive Step As inductive hypothesis, assume that all labels P_i^1, P_i^2 where $i < e$ are fixed at Stage $H(e - 1)$. P_e^1, P_e^2 are introduced at Stage e (G1). They cannot be moved by G3 or G4 at a stage $m \geqslant H = \text{Max.}\ (H(e - 1), e)$. For, if not, then a number $i < e$ will be attacked at Stage m. This is a contradiction as P_i^1, P_i^2 are fixed at Stage $H(e - 1)$. Suppose e is never attacked at a Stage $m \geqslant H$. Then P_e^1, P_e^2 are fixed and assigned at Stage H; i.e., $H(e) = H$. If e is attacked at a Stage $m \geqslant H$, then P_e^1, P_e^2 are fixed and unassigned at Stage m; i.e., $H(e) = m$.

Thus the inductive hypothesis implies that all labels P_i^1, P_i^2 where $i \leqslant e$ are fixed at some Stage $H(e)$. The statement of the lemma now follows by induction on e.

Lemma 2 *For each e, if P_e^1, P_e^2 are fixed and unassigned at Stage $H(e)$, then ϕ_e^2 is not a 2-cylinder function for $A^g \cap K^g$.*

Proof: P_e^1, P_e^2 are introduced at Stage e. If they are fixed and unassigned at Stage $H(e)$, there is a Stage $m(\leqslant m \leqslant H(e))$ at which they were last deleted. Thus e was attacked at Stage m. Suppose at Stage m P_e^1, P_e^2 were assigned to the respective points x, y. Then there exist $u, w \leqslant m$ where $\sim[(u = x \wedge w = y) \vee (w = x \wedge u = y)]$ such that $\phi_e^2(\tau(x, y), \tau(y, x)) = \tau(u, w)$. Furthermore, in Step 3c of Stage m, it would have been ensured that $(x \to y \wedge y \to x)(B^m) \not\equiv (u \to w \wedge w \to u)(B^m)$. It now suffices to prove the following statement (θ).

$$(\theta) \qquad (x \to y \wedge y \to x)(B) \not\equiv (u \to w \wedge w \to u)(B).$$

For, if (θ) holds, then $\{(x, y), (y, x)\} \subseteq A^g \cap K^g \not\equiv (u, w) \in A^g \cap K^g$. Now, $(x, y) \in A^g \cap K^g \longleftrightarrow (y,x) \in A^g \cap K^g$. Therefore, as $\phi_e^2(\tau(x, y), \tau(y, x)) = \tau(u, w)$, ϕ_e^2 is not a 2-cylinder function for $A^g \cap K^g$.

Proof of (θ): Suppose Case 1.1 occurred in Step 3(c) of Stage m. Then it would have been ensured that $(u \to w \wedge w \to u)(B^m)$ and $x \| y(B^m)$. As B is an extension of B^m, $(u \to w \wedge w \to u)(B^m) \Rightarrow (u \to w \wedge w \to u)(B)$.

It will now be proved by induction on s that for all $s \geqslant o$, $x \| y(B^{m+s})$. Basis $s = o$: Since $x \| y(B^m)$, we have that $x \| y(B^{m+o})$.

Inductive Hypothesis Assume that for all $s < s_0$, $x \| y(B^{m+s})$.

Consider the process of constructing B^{m+s_o}. As the labels P_e^1, P_e^2 which were assigned to the respective points x, y during Step 2 of Stage m were deleted during Step 3(a) of Stage m, $B^{m+s_o-1}(x)$ and $B^{m+s_o-1}(y)$ have no labels. If no number is attacked during Stage $m + s_o$, then $x \| y(B^{m+s_o-1}) \Rightarrow x \| y(B^{m+s_o})$.

Suppose a number is attacked during Stage $m + s_o$ and Case 1.1 occurs in Step 3(c). Then during Stage $m + s_o$, the component containing x as a point is not connected to that containing y as a point. Therefore $x \| y(B^{m+s_o})$. Similarly if one of the other cases occurs in Step 3c of Stage $m + s_o$, it can be shown that $x \| y(B^{m+s_o})$. Thus the inductive hypothesis implies that for all $s \geqslant o$, $x \| y(B^{m+s})$; i.e., $x \| y(B)$. Therefore $(u \rightarrow w \wedge w \rightarrow u)(B)$ and $x \| y(B)$.

Similarly if one of the other cases occurred in Step 3c of Stage m, it can be shown that $(x \rightarrow y \wedge y \rightarrow x)(B) \not\equiv (u \rightarrow w \wedge w \rightarrow u)(B)$.

This proves (θ) and hence the lemma.

Lemma 3 *For each e, ϕ_e^2 is not a 2-cylinder function for $A^g \cap K^g$.*

Proof: If ϕ_e^2 is not total, then it cannot be a 2-cylinder function for $A^g \cap K^g$. Assume that ϕ_e^2 is total. If P_e^1, P_e^2 are fixed and unassigned at Stage $H(e)$, then by Lemma 2, ϕ_e^2 cannot be a 2-cylinder function for $A^g \cap K^g$. Suppose P_e^1, P_e^2 are fixed and assigned to the respective points x, y at Stage $H(e)$. Then as ϕ_e^2 is total, $\phi_e^2(\tau(x, y), \tau(y, x))$ is defined. Let $\phi_e^2(\tau(x, y), \tau(y, x)) = \tau(u, w)$. If $[(x = u \wedge y = w) \vee (x = w \wedge y = u)]$ holds, then as $(x, y) \in A^g \cap K^g \longleftrightarrow (y, x) \in A^g \cap K^g$, ϕ_e^2 cannot be a 2-cylinder function for $A^g \cap K^g$. If $[(x = u \wedge y = w) \vee (x = w \wedge y = u)]$ does not hold, i.e., if x, y, u, w satisfy condition R_2 of Step 2 of the programme, then at some Stage $m \geqslant H(e)$, e will be attacked and P_e^1, P_e^2 will be deleted via Step 3(a) of Stage m. This is a contradiction as P_e^1, P_e^2 are fixed and assigned at Stage $H(e)$.

Hence the condition $[(x = u \wedge y = w) \vee (x = w \wedge y = u)]$ holds. Therefore ϕ_e^2 cannot be a 2-cylinder function for $A^g \cap K^g$. This proves the lemma and hence Result α(iv, b) for the case $\cap$.

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Department of Computer Science
New Mexico Tech
Socorro, New Mexico 87801