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Probabilistic Semantics for Intuitionistic Logic

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Venturing into technically virgin territory, we isolate probability functions that: (i) stand to intuitionistic logic as Popper's functions do to classical logic,[1] and hence (ii) rate (we believe) the appellation "intuitionistic probability functions". We then define in terms of these functions a notion of *logical truth* and a notion of *entailment* (or, if preferred, *logical consequence*) such that, where IL is a first-order language, A is a statement of IL, and S is a set of statements of IL,

(a) A is logically true if provable by intuitionistic means (T2.8), and only if so (T3.6)

(b) A is entailed by S if provable from S by intuitionistic means (T2.10), and only if so (T3.8).

Our definitions and theorems constitute—in current parlance—a *probabilistic semantics* for intuitionistic logic, i.e., an adaptation to intuitionistic logic of the semantics that Popper and various writers in Popper's debt have devised for classical logic.[2] We concentrate in this article on the semantics that thus issues from our probability theory. In a sequel to it we shall discuss the interpretation of Pr which dictated the ten constraints placed in Section 1 on that function.

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Section 1 deals with the syntax and the (probabilistic) semantics of *IL*; Section 2 supplies proofs of our Soundness Theorems for *IL* (T2.8 and T2.10); and Section 3 supplies proofs of the Completeness ones (T3.6 and T3.8). All steps toward T2.8 and T2.10 are constructive. Some toward T3.6 and T3.8, on the other hand, are not. But that is the rule, *not* the exception, with completeness proofs for intuitionistic logic.

1 The syntax and semantics of IL The *signs* of *IL* will be:

(i) $\aleph_0$ (individual) variables
(ii) the $\aleph_0$ (individual) terms

$$t_1, t_2, t_3, \ldots,$$

listed here in their so-called *alphabetic order*
(iii) one or more predicates, each reported as being of a certain degree d
(iv) the statement f, which the reader may think of as some logical falsehood or other of *IL*
(v) $\supset$, &, $\vee$, $\forall$, and $\exists$
(vi) (,) and ,.[3]

Variables will be referred to by means of X; terms by means of T; individual signs (i.e., individual variables and individual terms) by means of I; predicates by means of P; finite strings of signs by means of A, B, C, D, E, and F; and, A being such a string, $I_1, I_2, \ldots, I_n (n \geqslant 0)$ being distinct individual signs, and $I'_1, I'_2, \ldots, I'_n$ being individual signs not necessarily distinct from one another nor from $I_1, I_2, \ldots, I_n$, the result of simultaneously substituting in A sign I'_1 for sign I_1, sign I'_2 for sign $I_2, \ldots$, sign I'_n for sign I_n will be referred to by means of $((A)(I'_1, I'_2, \ldots, I'_n/I_1, I_2, \ldots, I_n))$.

The *statements* of *IL* will be all strings of signs of the following sorts:

(i) f
(ii) $P(T_1, T_2, \ldots, T_d)$, where P is a predicate of degree d and $T_1, T_2, \ldots, T_d$ are not necessarily distinct terms
(iii) $(A \supset B)$, $(A \& B)$, and $(A \vee B)$, where A and B are not necessarily distinct statements
(iv) $(\forall X)A$ and $(\exists X)A$, where, for some term T, $(A(T/X))$ is a statement.[4]

We shall presume the statements of *IL* to be arranged in some specific order, called their *alphabetic order.*[5] For brevity's sake we shall drop various parentheses; and, T_i being the i^{th} term of *IL*, we shall write A'_i and B'_i $(i \geqslant 1)$ in place of $A(T_i/X)$ and $B(T_i/X)$, respectively; and at one juncture we shall write A''_i in place of $(A(X/T))(T_i/X)$.

Sets of statements will be referred to by means of S, and will be called *infinitely extendible* if there are $\aleph_0$ terms that are foreign to their members (i.e., that do not occur in any of their members). And, S being a finite set of statements, we shall understand by $B \& C(S)$ the statement B when S is ϕ, otherwise the conjunction $B \& ((\ldots (C_1 \& C_2) \& \ldots) \& C_n)$, where $C_1, C_2, \ldots, C_n$ are in alphabetic order the various members of S.

The *axioms* of *IL* will be all statements of the sixteen sorts:

A1	$A \supset (B \supset A)$
A2	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A3	$f \supset A$[6]
A4	$A \supset (B \supset (A \,\&\, B))$
A5	$(A \,\&\, B) \supset A$
A6	$(A \,\&\, B) \supset B$
A7	$A \supset (A \vee B)$
A8	$B \supset (A \vee B)$
A9	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
A10	$(\forall X)A \supset A(T/X)$
A11	$A \supset (\forall X)A$[7]
A12	$(\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B)$
A13	$A(T/X) \supset (\exists X)A$
A14	$(\exists X)A \supset A$
A15	$(\forall X)(A \supset B) \supset ((\exists X)A \supset (\exists X)B)$
A16	$(\forall X)(A(X/T))$ if A is an axiom.[8]

And the *ponential* of two statements A and $A \supset B$ of *IL* will be B. It follows from our account that any axiom of *IL* is of the sort

$$(\forall X_1)(\forall X_2) \ldots (\forall X_n)(A(X_1, X_2, \ldots, X_n/T_1, T_2, \ldots, T_n)),$$

where $n \geqslant 0$ and A is of one of the fifteen sorts A1-A15. It also follows, as the reader may wish to verify, that if A is an axiom, so is $A(T'/T)$ for any terms T and T'. We shall presume both facts below.

By a *proof in IL of a statement A of IL from a set S of statements of IL* we shall understand any finite column of statements of *IL* such that: (i) each entry in the column is a member of S, an axiom, or the ponential of two earlier entries in the column; and (ii) the last entry in the column is A. We shall say that A is *provable in IL from S* or, equivalently, that A is *provable in IL given the statements in S as assumptions* (for short, $S \vdash_I A$) if there is a proof of A from S in *IL*. We shall say that A is *provable in IL* (for short, $\vdash_I A$) if $\phi \vdash_I A$.[9] And we shall take S to be *inconsistent in IL* if $S \vdash_I f$, *consistent* otherwise.

By a *term extension* of *IL* we shall understand any language that is like *IL* except for *possibly* boasting countably many (individual) terms besides those of *IL*. (Note that because of the qualifier 'possibly' *IL* counts as one of its term extensions.) In general, we shall refer to term extensions of *IL* by means of IL^+, and write $S \vdash_I^+ A$ ($\vdash_I^+ A$, when $S = \phi$) for 'A is provable in IL^+ from S'. One term extension of *IL* plays a crucial role in Section 3. It boasts $\aleph_0$ extra terms over and above those of *IL*, and for that reason is known as IL^∞. The terms of IL^∞ are

$$t_1^\infty, t_2^\infty, t_3^\infty, \ldots,$$

where t_1^∞, t_3^∞, t_5^∞, . . ., are in alphabetic order the terms of *IL* (and hence t_2^∞, t_4^∞, t_6^∞, . . ., are the terms of IL^∞ peculiar to IL^∞). In keeping with the convention above we write $S \vdash_I^\infty A$ for A is provable in IL^∞ from S.

It is easily verified that, when S is a set of statements of IL and A a statement of IL, $S \vdash_I A$ if $S \vdash_I^{\infty} A$. For suppose the column made up of $B_1, B_2, \ldots, B_p$ constitutes a proof of A from S in IL^{∞}, and for each i from 1 through p let C_i be the result of replacing by t_1 (the first term of IL) every term in B_i that is not one of IL (and hence is peculiar to IL^{∞}). Then, as the reader will note, the column made up of $C_1, C_2, \ldots, C_p$ is sure to constitute a proof of A from S in IL. We shall bank on that fact below.

Now for matters of semantics, some concerning all term extensions of IL and the rest concerning just IL.

By an *intuitionistic probability function for* IL^+ we shall understand any function Pr^+ (Pr when IL^+ is IL, and Pr^{∞} when IL^+ is IL^{∞}) that takes (ordered) pairs of statements of IL^+ into real numbers and meets the following ten constraints:

C1 $\quad 0 \leqslant Pr^+(A, B) \leqslant 1$

C2 $\quad Pr^+(A, A) = 1$

C3 $\quad Pr^+(A, f) = 1$[10]

C4 $\quad Pr^+(A \supset B, C) = Pr^+(B, A \,\&\, C)$

C5 $\quad Pr^+(A \,\&\, B, C) = Pr^+(A, B \,\&\, C) \times Pr^+(B, C)$

C6 $\quad Pr^+(A \,\&\, B, C) = Pr^+(B \,\&\, A, C)$

C7 $\quad Pr^+(A, B \,\&\, C) = Pr^+(A, C \,\&\, B)$

C8 $\quad Pr^+(A, B \vee C) = Pr^+(A, B) \times Pr^+(A, C \,\&\, (B \supset A))$[11]

C9 $\quad Pr^+((\forall X)A, B) = \underset{i \to \infty}{Limit}\, Pr^+((\ldots(A'_1 \,\&\, A'_2) \,\&\, \ldots) \,\&\, A'_i, B)$

C10 $\quad Pr^+(A, (\exists X)B) = \underset{i \to \infty}{Limit}\, Pr^+(A, (\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_i)$.

This done, we shall say that a statement A of IL is *logically true in IL* if, for every intuitionistic probability function Pr for IL and every statement B of IL, $Pr(A, B) = 1$.[12] And we shall say that A is *entailed in IL by a set S of statements of IL* if, for every term extension IL^+ of IL, every intuitionistic probability function Pr^+ for IL^+, and every statement B of IL^+, $Pr^+(A, B \,\&\, C(S')) = 1$ for at least one finite subset S' of S.[13] To abridge matters we shall write $\models_I A$ in lieu of 'A is logically true in IL', and $S \models_I A$ in lieu of 'A is entailed in IL by S'.

2 *Soundness theorems* That under the account of '$\vdash$' and '$\models$' in Section 1 IL is both weakly and strongly sound hinges on three theorems:

(i) T2.2(a), the probability analogue of *Modus Ponens*
(ii) T2.2(b), according to which—no matter the finite set S or the member A of $C(S)$—$Pr(A, B \,\&\, C(S)) = 1$ for every B
(iii) T2.6, according to which—no matter the axiom A of IL—$Pr(A, B) = 1$ for every B.

T2.6 will follow from two ancillary theorems, one (T2.3) covering axiom schemata A1-A9 and the other (T2.5) covering the rest. Two multi-clause lemmas (L2.1 and L2.4) pave the way for T2.2, T2.4, and T2.5. L2.1-T2.6 hold for the remaining extensions of IL as well as for IL (hence with IL^+ and Pr^+ in place of IL and Pr, respectively); we shall take the fact for granted when proving our Strong Completeness Theorem. As announced earlier, all results in this section are obtained by strictly intuitionistic means.[14]

L2.1

(a) $Pr(A \mathbin{\&} B, C) \leqslant Pr(B, C)$.
(b) $Pr(A \mathbin{\&} B, C) \leqslant Pr(A, C)$.[15]
(c) $Pr(A_i, (\ldots(A_1 \mathbin{\&} A_2) \mathbin{\&} \ldots) \mathbin{\&} A_n) = 1$ *(for each i from 1 through n).*
(d) $Pr(A_i, A_i \mathbin{\&} (\ldots \mathbin{\&} (A_{n-1} \mathbin{\&} A_n) \ldots)) = 1$ *(for each i from 1 through n).*
(e) $Pr(A \supset B, C) = Pr(B, C \mathbin{\&} A)$.
(f) $Pr(A, (B \vee C) \mathbin{\&} D) = Pr(A, B \mathbin{\&} D) \times Pr(A, (C \mathbin{\&} (B \supset (D \supset A))) \mathbin{\&} D)$.
(g) $Pr(A, (B \vee C) \mathbin{\&} D) \leqslant Pr(A, B \mathbin{\&} D)$.[16]
(h) *Let 't' be short for 'f ⊃ f'. Then* $Pr(t, A) = 1$.
(i) *Let 't' be as in* (h). *Then* $Pr(A, B \mathbin{\&} t) = Pr(A, B)$.
(j) $Pr(A, B \vee C) = Pr(A, C \vee B)$.
(k) $Pr(A, (B \vee C) \mathbin{\&} D) \leqslant Pr(A, C \mathbin{\&} D)$.

Proof:
Ad(a): By C5

$$Pr(A \mathbin{\&} B, C) = Pr(A, B \mathbin{\&} C) \times Pr(B, C).$$

But by C1 the factors on the right each fall within the range [0,1]. Hence (a).
Ad (b): By (a) and C6.
Ad (c): By C2

$$Pr((\ldots(A_1 \mathbin{\&} A_2) \mathbin{\&} \ldots) \mathbin{\&} A_n, (\ldots(A_1 \mathbin{\&} A_2) \mathbin{\&} \ldots) \mathbin{\&} A_n) = 1,$$

hence by $n - i$ applications of (b) and C1

$$Pr((\ldots(A_1 \mathbin{\&} A_2) \mathbin{\&} \ldots) \mathbin{\&} A_i, (\ldots(A_1 \mathbin{\&} A_2) \mathbin{\&} \ldots) \mathbin{\&} A_n) = 1,$$

and hence (c) by (a) and C1.
Ad (d): By C2

$$Pr(A_1 \mathbin{\&} (\ldots \mathbin{\&} (A_{n-1} \mathbin{\&} A_n) \ldots), A_1 \mathbin{\&} (\ldots \mathbin{\&} (A_{n-1} \mathbin{\&} A_n) \ldots)) = 1,$$

hence by $i - 1$ applications of (a) and C1

$$Pr(A_i \mathbin{\&} (\ldots \mathbin{\&} (A_{n-1} \mathbin{\&} A_n) \ldots), A_1 \mathbin{\&} (\ldots \mathbin{\&} (A_{n-1} \mathbin{\&} A_n) \ldots)) = 1,$$

and hence (d) by (b) and C1.
Ad (e): By C4 and C7.
Ad (f): By C8

$$Pr(D \supset A, B \vee C) = Pr(D \supset A, B) \times Pr(D \supset A, C \mathbin{\&} (B \supset (D \supset A))).$$

Hence (f) by (e).
Ad (g): By (f)

$$Pr(A, (B \vee C) \mathbin{\&} D) = Pr(A, B \mathbin{\&} D) \times Pr(A, (C \mathbin{\&} (B \supset (D \supset A))) \mathbin{\&} D).$$

But by C1 the factors on the right each fall within the range [0, 1]. Hence (g).
Ad (h): By L2.1(c) $Pr(f, f \mathbin{\&} A) = 1$. Hence (h) by C4.
Ad (i): By C5 and (h)

$$Pr(A \mathbin{\&} t, B) = Pr(A, t \mathbin{\&} B)$$

and

$$Pr(t \,\&\, A, B) = Pr(A, B).$$

Hence by C6

$$Pr(A, t \,\&\, B) = Pr(A, B).$$

Hence (i) by C7.
Ad (j): By C8 and C4

$$Pr(A, B \vee C)$$

equals

$$Pr(A, B) \times Pr(C \supset A, B \supset A),$$

which by (i) and C4 equals

$$Pr(B \supset A, t) \times Pr(C \supset A, (B \supset A) \,\&\, t),$$

which by C5-C6 equals

$$Pr((B \supset A) \,\&\, (C \supset A), t),$$

which by C5 and (i) equals

$$Pr(B \supset A, C \supset A) \times Pr(C \supset A, t),$$

which by C4 and (i) equals

$$Pr(A, B \,\&\, (C \supset A)) \times Pr(A, C),$$

which by C8 equals

$$Pr(A, C \vee B).$$

Ad (k): By (j), C4, and C7

$$Pr(A, (B \vee C) \,\&\, D) = Pr(A, (C \vee B) \,\&\, D).$$

Hence (k) by (g).

T2.2

(a) *If* $Pr(A, C) = 1$ *and* $Pr(A \supset B, C) = 1$, *then* $Pr(B, C) = 1$.
(b) *Let* S *be a finite set of statements and* A *be an arbitrary member of* S. *Then* $Pr(A, B \,\&\, C(S)) = 1$ *for every* B.

Proof:
Ad (a): Suppose

$$Pr(A \supset B, C) = 1,$$

and hence by C4

$$Pr(B, A \,\&\, C) = 1;$$

and suppose

$$Pr(A, C) = 1.$$

Then by C5

$$Pr(B \,\&\, A, C) = 1,$$

and hence by L2.1(b) and C1

$$Pr(B, C) = 1.$$

Hence (a).
Ad (b): By L2.1(c) and C7.

T2.3

(a) $Pr(A \supset (B \supset A), C) = 1.$
(b) $Pr((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)), D) = 1.$
(c) $Pr(f \supset A, B) = 1.$
(d) $Pr(A \supset (B \supset (A \,\&\, B)), C) = 1.$
(e) $Pr((A \,\&\, B) \supset A, C) = 1.$
(f) $Pr((A \,\&\, B) \supset B, C) = 1.$
(g) $Pr(A \supset (A \vee B), C) = 1.$
(h) $Pr(B \supset (A \vee B), C) = 1.$
(i) $Pr((A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)), D) = 1.$

Proof:
Ad (a): By L2.1(d)

$$Pr(A, B \,\&\, (A \,\&\, C)) = 1.$$

Hence (a) by two applications of C4.
Ad (b): By L2.1(d)

$$Pr(A, A \,\&\, ((A \supset B) \,\&\, ((A \supset (B \supset C)) \,\&\, D))) = 1,$$
$$Pr(A \supset B, A \,\&\, ((A \supset B) \,\&\, ((A \supset (B \supset C)) \,\&\, D))) = 1,$$

and

$$Pr(A \supset (B \supset C), A \,\&\, ((A \supset B) \,\&\, ((A \supset (B \supset C)) \,\&\, D))) = 1.$$

Hence by three applications of T2.2(a)

$$Pr(C, A \,\&\, ((A \supset B) \,\&\, ((A \supset (B \supset C)) \,\&\, D))) = 1.$$

Hence (b) by three applications of C4.
Ad (c): By C3

$$Pr(A \,\&\, B, f) = 1,$$

hence by C5 and C3

$$Pr(A, B \,\&\, f) = 1,$$

hence by C7

$$Pr(A, f \,\&\, B) = 1,$$

and hence (c) by C4.
Ad (d): By L2.1(d)

$$Pr(A, B \,\&\, (B \,\&\, (A \,\&\, C))) = 1$$

and

$$Pr(B, B \,\&\, (A \,\&\, C)) = 1.$$

Hence by C5

$$Pr(A \,\&\, B, B \,\&\, (A \,\&\, C)) = 1,$$

and hence (d) by two applications of C4.
Ad (e)-(f): By L2.1(c) and C4.
Ad (g): By L2.1(c)

$$Pr(A \vee B, (A \vee B) \,\&\, C) = 1,$$

hence by L2.1(g) and C1

$$Pr(A \vee B, A \,\&\, C) = 1,$$

and hence (g) by C4.
Ad (h): Proof like that of (g), but using L2.1(k) in place of L2.1(g).
Ad (i): (i′) By L2.1(d)

$$Pr(A, A \,\&\, ((B \supset C) \,\&\, ((A \supset C) \,\&\, D))) = 1$$

and

$$Pr(A \supset C, A \,\&\, ((B \supset C) \,\&\, ((A \supset C) \,\&\, D))) = 1,$$

and hence by T2.2(a)

$$Pr(C, A \,\&\, ((B \supset C) \,\&\, ((A \supset C) \,\&\, D))) = 1.$$

(ii″) Similarly by L2.1(c)-(d) and T2.2(a)

$$Pr(C, (B \,\&\, (A \supset (((B \supset C) \,\&\, ((A \supset C) \,\&\, D)) \supset C)) \,\&\, ((B \supset C) \,\&\, ((A \supset C) \,\&\, D)))) = 1.$$

(i‴) By (i′), (i″), and L2.1(f)

$$Pr(C, (A \vee B) \,\&\, ((B \supset C) \,\&\, ((A \supset C) \,\&\, D))) = 1,$$

and hence (i) by C4.

L2.4

(a) $Pr(\underbrace{(\ldots(A \,\&\, A) \,\&\, \ldots) \,\&\, A}_{i\ A\text{'s}}, A \,\&\, B) = 1.$

(b) *Let* $Pr(A_h, B) = 1$ *for every* $h \leqslant i$ *and every statement* B. *Then* $Pr((\ldots(A_1 \,\&\, A_2) \,\&\, \ldots) \,\&\, A_i, B) = 1.$

(c) $Pr(A, (\exists X)B \,\&\, C) = \underset{i\to\infty}{\mathit{Limit}}\, Pr(A, ((\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_i) \,\&\, C).$[17]

(d) $Pr(A, (\underbrace{(\ldots(A \vee A) \vee \ldots) \vee A}_{i\ A\text{'s}}) \,\&\, B) = 1.$

(e) *Let* $Pr(A_h \supset B, C) = 1$ *for every* $h \leqslant i$ *and every statement* C. *Then* $Pr(((\ldots(A_1 \vee A_2) \vee \ldots) \vee A_i) \supset B, C) = 1.$

Proof:
Ad (a): The proof is by mathematical induction on i, with the Basis holding

true by L2.1(c). As for the Inductive Step, by C6

$$Pr(\underbrace{(\ldots(A \,\&\, A) \,\&\, \ldots) \,\&\, A}_{i\ A\text{'s}},\ A \,\&\, B)$$

equals

$$Pr(A \,\&\, (\underbrace{(\ldots(A \,\&\, A) \,\&\, \ldots) \,\&\, A)}_{i-1\ A\text{'s}},\ A \,\&\, B)$$

which by C5 equals the product of

$$Pr(A,\ (\underbrace{(\ldots(A \,\&\, A) \,\&\, \ldots) \,\&\, A)}_{i-1\ A\text{'s}} \,\&\, (A \,\&\, B))$$

and

$$Pr(\underbrace{(\ldots(A \,\&\, A) \,\&\, \ldots) \,\&\, A}_{i-1\ A\text{'s}},\ A \,\&\, B).$$

But by L2.1(d) the first of these factors equals 1, and by the hypothesis of the induction so does the second. Hence (a).

Ad (b): The proof is by mathematical induction on i. Note, as regards the Inductive Step, that by C5

$$Pr((\ldots(A_1 \,\&\, A_2) \,\&\, \ldots) \,\&\, A_i,\ B)$$

equals

$$Pr((\ldots(A_1 \,\&\, A_2) \,\&\, \ldots) \,\&\, A_{i-1},\ A_i \,\&\, B) \times Pr(A_i,\ B).$$

But by the hypothesis of the induction the first of these factors equals 1, and by the hypothesis on $Pr(A_i, B)$ so does the second. Hence (b).

Ad (c): By C10

$$Pr(C \supset A,\ (\exists X)B) = \underset{i\to\infty}{Limit}\ Pr(C \supset A,\ (\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_i).$$

Hence (c) by C4 and C7.

Ad (d): The proof is by mathematical induction on i, with the Basis holding true by L2.1(c). As for the Inductive Step, by L2.1(f)

$$Pr(A,\ (\underbrace{(\ldots(A \vee A) \vee \ldots) \vee A)}_{i\ A\text{'s}} \,\&\, B)$$

equals the product of

$$Pr(A,\ (\underbrace{(\ldots(A \vee A) \vee \ldots) \vee A)}_{i-1\ A\text{'s}} \,\&\, B)$$

and

$$Pr(A,\ (A \,\&\, ((\underbrace{(\ldots(A \vee A) \vee \ldots) \vee A)}_{i-1\ A\text{'s}} \supset A)) \,\&\, B).$$

But by the hypothesis of the induction the first of these factors equals 1, and by L2.1(c) so does the second. Hence (d).

Ad (e): The proof is by mathematical induction on i. Note, as regards the Inductive Step, that by C4

$$Pr(((\ldots(A_1 \vee A_2) \vee \ldots) \vee A_i) \supset B, C)$$

equals

$$Pr(B, ((\ldots(A_1 \vee A_2) \vee \ldots) \vee A_i) \,\&\, C),$$

which by L2.1(f) equals the product of

$$Pr(B, ((\ldots(A_1 \vee A_2) \vee \ldots) \vee A_{i-1}) \,\&\, C)$$

and

$$Pr(B, A_i \,\&\, (((\ldots(A_1 \vee A_2) \vee \ldots) \vee A_{i-1}) \supset (C \supset B)) \,\&\, C).$$

But by C4 the first of these factors equals

$$Pr(((\ldots(A_1 \vee A_2) \vee \ldots) \vee A_{i-1}) \supset B, C),$$

which by the hypothesis of the induction equals 1; while by the hypothesis on $Pr(A_i \supset B, C)$

$$Pr(A_i \supset B, (A_i \,\&\, (((\ldots(A_1 \vee A_2) \vee \ldots) \vee A_{i-1}) \supset (C \supset B))) \,\&\, C) = 1,$$

by L2.1(c)

$$Pr(A_i, (A_i \,\&\, (((\ldots(A_1 \vee A_2) \vee \ldots) \vee A_{i-1}) \supset (C \supset B))) \,\&\, C) = 1,$$

and hence by T2.2(a) the second factor also equals 1. Hence (e).

T2.5

(a) $Pr((\forall X)A \supset A(T/X), B) = 1.$
(b) $Pr(A \supset (\forall X)A, B) = 1.$
(c) $Pr((\forall X)(A \supset B) \supset ((\forall X)A \supset (\forall X)B), C) = 1.$
(d) $Pr(A(T/X) \supset (\exists X)A, B) = 1.$
(e) $Pr((\exists X)A \supset A, B) = 1.$
(f) $Pr((\forall X)(A \supset B) \supset ((\exists X)A \supset (\exists X)B), C) = 1.$
(g) *Let* $Pr(A_i'', B) = 1$ *for every* i *and every statement* B.[18] *Then* $Pr((\forall X)A(X/T), B)$.

Proof:
Ad (a): Let T be the g^{th} term of IL, and hence $A(T/X)$ be A_g'. By L2.1(c)

$$Pr((\forall X)A, (\forall X)A \,\&\, B) = 1,$$

hence by C9

$$\operatorname*{Limit}_{i\to\infty} Pr((\ldots(A_1' \,\&\, A_2') \,\&\, \ldots) \,\&\, A_i', (\forall X)A \,\&\, B) = 1,$$

hence by L2.1(b) and C1

$$Pr((\ldots(A_1' \,\&\, A_2') \,\&\, \ldots) \,\&\, A_g', (\forall X)A \,\&\, B) = 1,$$

hence (by L2.1(a) in the case that $g > 1$)

$$Pr(A'_g, (\forall X)A \,\&\, B) = 1,$$

and hence (a) by C4 and the remark on A'_g.
Ad (b): By L2.4(a)

$$Pr(\underbrace{(\ldots(A \,\&\, A) \,\&\, \ldots) \,\&\, A}_{i\ A\text{'s}}, A \,\&\, B) = 1.$$

But, as X here is foreign to A, A'_i and A are the same for every i. Hence

$$Pr((\ldots(A'_1 \,\&\, A'_2) \,\&\, \ldots) \,\&\, A'_i, A \,\&\, B) = 1,$$

hence

$$\underset{i\to\infty}{Limit}\, Pr((\ldots(A'_1 \,\&\, A'_2) \,\&\, \ldots) \,\&\, A'_i, A \,\&\, B) = 1,$$

hence by C9

$$Pr((\forall X)A, A \,\&\, B) = 1,$$

and hence (b) by C4.
Ad (c): Let $1 \leqslant h \leqslant i$. By (a)

$$Pr((\forall X)(A \supset B) \supset (A'_h \supset B'_h), (\forall X)A \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1.$$

But by L2.1(d)

$$Pr((\forall X)(A \supset B), (\forall X)A \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1.$$

Hence by T2.2(a)

$$Pr(A'_h \supset B'_h, (\forall X)A \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1.$$

But by (a) and C4

$$Pr(A'_h, (\forall X)A \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1.$$

Hence by T2.2(a)

$$Pr(B'_h, (\forall X)A \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1,$$

hence by L2.4(b)

$$Pr((\ldots(B'_1 \,\&\, B'_2) \,\&\, \ldots) \,\&\, B'_i, (\forall X)A \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1.$$

Hence

$$\underset{i\to\infty}{Limit}\, Pr((\ldots(B'_1 \,\&\, B'_2) \,\&\, \ldots) \,\&\, B'_i, (\forall X)A \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1,$$

hence by C9

$$Pr((\forall X)B, (\forall X)A \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1,$$

and hence (c) by C4.
Ad (d): Let T and A'_g be as in the proof of (a). By L2.1(c)

$$Pr((\exists X)A, (\exists X)A \,\&\, B) = 1,$$

hence by L2.4(c)

$$\underset{i\to\infty}{Limit}\ Pr((\exists X)A, ((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \,\&\, B) = 1,$$

hence by L2.1(g) and C1

$$Pr((\exists X)A, ((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_g) \,\&\, B) = 1,$$

hence (by L2.1(k) in the case that $g < 1$)

$$Pr((\exists X)A, A'_g \,\&\, B) = 1,$$

and hence (d) by C4 and the remark on A'_g.
Ad (e): By L2.4(d)

$$Pr(A, \underbrace{((\ldots(A \vee A) \vee \ldots) \vee A)}_{i\ A\text{'s}} \,\&\, B) = 1;$$

hence by the same reasoning as in the proof of (b), but using L2.4(c) in place of C9,

$$Pr(A, (\exists X)A \,\&\, B) = 1;$$

and hence (e) by C4.
Ad (f): Let $1 \leqslant h \leqslant i$. By (a)

$$Pr((\forall X)(A \supset B) \supset (A'_h \supset B'_h), ((\ldots((A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \,\&\, ((\forall X)(A \supset B) \,\&\, C))).$$

But by L2.1(d)

$$Pr((\forall X)(A \supset B), ((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1.$$

Hence by T2.2(a)

$$Pr(A'_h \supset B'_h, ((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1,$$

hence by C4

$$Pr(B'_h, A'_h \,\&\, (((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \,\&\, ((\forall X)(A \supset B) \,\&\, C))) = 1,$$

hence by (d) and T2.2(a)

$$Pr((\exists X)B, A'_h \,\&\, (((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \,\&\, ((\forall X)(A \supset B) \,\&\, C))) = 1,$$

hence by C4

$$Pr(A'_h \supset (\exists X)B, ((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1,$$

and hence by L2.4(e)

$$Pr(((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \supset (\exists X)B, ((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1.$$

But by L2.1(d)

$$Pr((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i, ((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1.$$

Hence by T2.2(a)

$$Pr((\exists X)B, ((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A'_i) \,\&\, ((\forall X)(A \supset B) \,\&\, C)) = 1,$$

hence

$$\underset{i\to\infty}{Limit}\ Pr((\exists X)B, ((\ldots(A'_1 \vee A'_2) \vee \ldots) \vee A_i) \& ((\forall X)(A \supset B) \& C)) = 1,$$

hence by L2.4(c)

$$Pr((\exists X)B, (\exists X)A \& ((\forall X)(A \supset B) \& C)) = 1,$$

and hence (f) by 2 applications of C4.
Ad (g): By L2.4(b) and the hypothesis on $\mathrm{Pr}(A''_i, B)$

$$Pr((\ldots(A''_1 \& A''_2) \& \ldots) \& A''_i, B) = 1,$$

hence

$$\underset{i\to\infty}{Limit}\ Pr((\ldots(A''_1 \& A''_2) \& \ldots) \& A''_i, B) = 1,$$

and hence by C9

$$Pr((\forall X)A(X/T), B) = 1.$$

T2.6 *Let A be an axiom of IL. Then $Pr(A, B) = 1$ for every statement B of IL.*

Proof: Since A is sure to be of the sort

$$(\forall X_1)(\forall X_2)\ldots(\forall X_n)(A'(X_1, X_2, \ldots, X_n/T_1, T_2, \ldots, T_n)),$$

where (i) $n \geqslant 0$ and (ii) in the case that $n > 0$

$$(\forall X_2)\ldots(\forall X_n)(A'(X_2, \ldots, X_n/T_2, \ldots, T_n))$$

is an axiom of IL, proof of T2.6 is by mathematical induction on n.
Basis: $n = 0$, in which case A' is of one of the fifteen sorts A1-A15. Then

$$Pr(A', B) = 1$$

by T2.3(a)-(i) and T2.5(a)-(f).
Inductive Step: $n > 0$. Since

$$(\forall X_2)\ldots(\forall X_n)(A'(X_2, \ldots, X_n/T_2, \ldots, T_n))$$

is an axiom of IL, so is

$$(((\forall X_2)\ldots(\forall X_n)(A'(X_2, \ldots, X_n/T_2, \ldots, T_n)))(X_1/T_1/t_i/X_1)$$

for every i from 1 on. Hence by the hypothesis of the induction

$$Pr((((\forall X_2)\ldots(\forall X_n)(A'(X_2, \ldots, X_n/T_2, \ldots, T_n)))(X_1/T_1))(t_i/X_1), B) = 1$$

for every i from 1 on and every statement B of IL. Hence by T2.5(g)

$$Pr((\forall X_1)(((\forall X_2)\ldots(\forall X_n)(A'(X_2, \ldots, X_n/T_2, \ldots, T_n)))(X_1/T_1)), B) = 1.$$

But $(\forall X_1)(\forall X_2)\ldots(\forall X_n)(A'(X_1, X_2, \ldots, X_n/T_1, T_2, \ldots, T_n))$ is the same as $(\forall X_1)(((\forall X_2)\ldots(\forall X_n)(A'(X_2, \ldots, X_n/T_2, \ldots, T_n)))(X_1/T_1))$. Hence

$$Pr((\forall X_1)(\forall X_2)\ldots(\forall X_n)(A'(X_1, X_2, \ldots, X_n/T_1, T_2, \ldots, T_n)), B) = 1.$$

Our Soundness Theorems are now at hand:

T2.7 *Let $S \vdash_I A$; let Pr be an arbitrary intuitionistic probability function for IL; and let B be an arbitrary statement of IL. Then $Pr(A, B \,\&\, C(S')) = 1$ for at least one finite subset S' of S.*

Proof: Let the column made up of $A_1, A_2, \ldots, A_p$ constitute a proof of A from S in IL, and let S' consist of those statements of IL among $A_1, A_2, \ldots, A_p$ that belong to S. It is easily shown by mathematical induction on i that, for each i from 1 through p,

$$Pr(A_i, B \,\&\, C(S')) = 1$$

and hence

$$Pr(A, B \,\&\, C(S')) = 1.$$

For suppose A_i belongs to S. Then $Pr(A_i, B \,\&\, C(S')) = 1$ by T2.2(b). Or suppose A_i is an axiom. Then $Pr(A_i, B \,\&\, C(S')) = 1$ by T2.6. Or suppose A_i is the ponential of A_g and A_h, where $g, h < i$. Then by the hypothesis of the induction $Pr(A_g, B \,\&\, C(S')) = Pr(A_h, B \,\&\, C(S')) = 1$, and hence $Pr(A_i, B \,\&\, C(S')) = 1$ by T2.2(a).

Hence, taking S to be ϕ, in which case $Pr(A, B \,\&\, C(S'))$ is just $Pr(A, B)$:

T2.8 (The Weak Soundness Theorem for *IL*) *If $\vdash_I A$, then $\models_I A$.*

Hence, as all the preceding lemmas and all theorems through T2.6 hold with IL^+, Pr^+, $\vdash_I^+$, and t_i^+ in place of IL, Pr, $\vdash_I$, and t_i:

T2.9 *Let $S \vdash_I A$. Then for every term extension IL^+ of IL, every intuitionistic probability function Pr^+ for IL^+, and every statement B of IL^+, $Pr^+(A, B \,\&\, C(S')) = 1$ for at least one finite subset S' of S.*

Hence:

T2.10 (The Strong Soundness Theorem for *IL*) *If $S \vdash_I A$, then $S \models_I A$.*

3 Completeness theorems The key theorem in this section, T3.5, will concern sets (of statements of *IL*) that are *infinitely extendible.* Since ϕ is infinitely extendible, T3.5 will ensure that $\vdash_I A$ if $\models_I A$ (T3.6). The argument used to prove T3.5 will then be extended to cover *all* sets. The resulting theorem, T3.7, will ensure that $S \vdash_I A$ if $S \models_I A$ (T3.8).

The proof of T3.5 hinges on two ancillary theorems: (i) T3.3, according to which a certain function *Pr* meets constraints C1-C11, and (ii) T3.4, a provability counterpart of T2.2. T3.3 itself will follow from two multiclause lemmas regarding provability in *IL*, L3.1-2. We sketch proofs of only two clauses of L3.1 and only three of L3.2; the rest are obvious. Lemmas L3.1-2 and Theorem T3.4 hold true for IL^∞ as well as for *IL* (hence with IL^∞, $\vdash^\infty$, and t_i^∞ in place of IL, $\vdash$, and t_i), a fact we shall take for granted when proving T3.7.

The definition of *Pr* in T3.3, the proof of T3.6, and that of T3.7 are *not* constructive. All else in the section, however, meets intuitionistic demands.

L3.1

(a) $S \vdash_I A \supset A$.
(b) $S \vdash_I f \supset A$.
(c) $S \vdash_I C \supset (A \supset B)$ *if and only if* $S \vdash_I (A \mathbin{\&} C) \supset B$.
(d) $S \vdash_I C \supset (A \mathbin{\&} B)$ *if and only if* $S \vdash_I (B \mathbin{\&} C) \supset A$ *and* $S \vdash_I C \supset B$.
(e) $S \vdash_I C \supset (A \mathbin{\&} B)$ *if and only if* $S \vdash_I C \supset (B \mathbin{\&} A)$.
(f) $S \vdash_I (B \mathbin{\&} C) \supset A$ *if and only if* $S \vdash_I (C \mathbin{\&} B) \supset A$.
(g) $S \vdash_I (B \vee C) \supset A$ *if and only if* $S \vdash_I B \supset A$ *and* $S \vdash_I (C \mathbin{\&} (B \supset A)) \supset A$.

Proof:
Ad (d): Suppose, on one hand, that $S \vdash_I C \supset (A \mathbin{\&} B)$. Then $S \vdash_I C \supset A$ and $S \vdash_I C \supset B$, and hence $S \vdash_I (B \mathbin{\&} C) \supset A$ and $S \vdash_I C \supset B$. Suppose, on the other hand, that $S \vdash_I (B \mathbin{\&} C) \supset A$ and $S \vdash_I C \supset B$. Then $S \vdash_I C \supset A$ and $S \vdash_I C \supset B$, and hence $S \vdash_I C \supset (A \mathbin{\&} B)$. Hence (d).
Ad (g): Suppose, on one hand, that $S \vdash_I (B \vee C) \supset A$. Then $S \vdash_I B \supset A$ and $S \vdash_I C \supset A$, and hence $S \vdash_I B \supset A$ and $S \vdash_I (C \mathbin{\&} (B \supset A)) \supset A$. Suppose, on the other hand, that $S \vdash_I B \supset A$ and $S \vdash_I (C \mathbin{\&} (B \supset A)) \supset A$. Then $S \vdash_I B \supset A$ and $S \vdash_I C \supset A$, and hence $S \vdash_I (B \vee C) \supset A$.

L3.2

(a) *If* $S \vdash_I (\forall X)A$, *then* $S \vdash_I B \supset ((\ldots(A'_1 \mathbin{\&} A'_2) \mathbin{\&} \ldots) \mathbin{\&} A'_i)$.
(b) *If* $S \vdash_I B \supset A(T/X)$ *for any term* T *foreign to* S *and* $B \supset (\forall X)A$, *then* $S \vdash_I B \supset (\forall X)A$.
(c) *If* $S \vdash_I B \supset ((\ldots(A'_1 \mathbin{\&} A'_2) \mathbin{\&} \ldots) \mathbin{\&} A'_h)$, *then* $S \vdash_I B \supset A'_g$ *for each* $g \leqslant h$.
(d) *If* $S \vdash_I (\exists X)B \supset A$, *then* $S \vdash_I ((\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_i) \supset A$.
(e) *If* $S \vdash_I B(T/X) \supset A$ *for any term* T *foreign to* S *and* $(\exists X)B \supset A$, *then* $S \vdash_I (\exists X)B \supset A$.
(f) *If* $S \vdash_I ((\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_h) \supset A$, *then* $S \vdash_I B'_g \supset A$ *for each* $g \leqslant h$.

Proof:
Ad (b): Suppose $S \vdash_I B \supset A(T/X)$, where T is foreign to S and $B \supset (\forall X)A$ (and hence to $S \cup \{B\}$ and $(\forall X)A$). Then $S \cup \{B\} \vdash_I A(T/X)$, hence $S \cup \{B\} \vdash_I (\forall X)A$ by Theorem 4 in [11] and the hypothesis on T, and hence $S \vdash_I B \supset (\forall X)A$. Hence (*b*).
Ad (d): Suppose $S \vdash_I (\exists X)B \supset A$. Then $S \vdash_I (\forall X)(B \supset A)$ by *96 in [7], hence $S \vdash_I B'_i \supset A$, and hence $S \vdash_I ((\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_i) \supset A$. Hence (d).
Ad (e): Suppose $S \vdash_I B(T/X) \supset A$, where T is foreign to S and $(\exists X)B \supset A$ (and hence to S and $(\forall X)(B \supset A)$). Then $S \vdash_I (\forall X)(B \supset A)$ by Theorem 4 in [11] and the hypothesis on T, and hence $S \vdash_I (\exists X)B \supset A$ by *96 in [7]. Hence (e).

T3.3 *Let* S *be an arbitrary set of statements of IL that is infinitely extendible, and for any statements* A *and* B *of IL let* $Pr(A, B)$ *equal 1 or 0 according as* $S \vdash_I B \supset A$ *or not. Then Pr constitutes an intuitionistic probability function for IL.*[19]

Proof:
(i) That Pr meets Constraint C1 follows from the definition of Pr, and that it meets Constraints C2-C8 follows from the definition and L3.1(a)-(h).
(ii) Suppose, on one hand, that $Pr((\forall X)A, B) = 1$, and hence by definition that

$S \vdash_I B \supset (\forall X)A$. Then by L3.2(a) $S \vdash_I B \supset ((\ldots(A'_1 \,\&\, A'_2) \,\&\, \ldots) \,\&\, A'_i)$ for each i from 1 on, hence by definition $Pr((\ldots(A'_1 \,\&\, A'_2) \,\&\, \ldots) \,\&\, A'_i, B) = 1$ for each i from 1 on and hence $\underset{i\to\infty}{Limit}\, Pr((\ldots(A'_1 \,\&\, A'_2) \,\&\, \ldots) \,\&\, A'_i, B) = 1$. Suppose, on the other hand, that $Pr((\forall X)A, B) = 0$, and hence by definition that $S \nvdash_I B \supset (\forall X)A$. Then by L3.2(b) $S \nvdash_I B \supset A(T/X)$ for any term T foreign to S and $B \supset (\forall X)A$. But, as S is infinitely extendible, there is sure to be such a term. So $S \nvdash_I B \supset A'_g$ for some g, hence by L3.2(c) $S \nvdash_I B \supset ((\ldots(A'_1 \,\&\, A'_2) \,\&\, \ldots) \,\&\, A'_h)$ for any $h \geqslant g$, hence by definition $Pr((\ldots(A'_1 \,\&\, A'_2) \,\&\, \ldots) \,\&\, A'_h, B) = 0$ for any $h \geqslant g$, and hence $\underset{i\to\infty}{Limit}\, Pr((\ldots(A'_1 \,\&\, A'_2) \,\&\, \ldots) \,\&\, A'_i, B) = 0$. So Pr meets Constraint C9.

(iii) Suppose, on one hand, that $Pr(A, (\exists X)B) = 1$, and hence by definition that $S \vdash_I (\exists X)B \supset A$. Then by L3.2(d) $S \vdash_I ((\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_i) \supset A$ for each i from 1 on, hence by definition $Pr(A, (\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_i) = 1$ for each i from 1 on, and hence $\underset{i\to\infty}{Limit}\, Pr(A, (\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_i) = 1$. Suppose, on the other hand, that $Pr(A, (\exists X)B) = 0$, and hence by definition that $S \nvdash_I (\exists X)B \supset A$. Then by L3.2(e) $S \nvdash_I B(T/X) \supset A$ for any term T foreign to S and $(\exists X)B \supset A$. But, as S is infinitely extendible, there is sure to be such a term. So $S \nvdash_I B'_g \supset A$ for some g, hence by L3.2(f) $S \nvdash_I ((\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_h) \supset A$ for any $h \geqslant g$, hence by definition $Pr(A, (\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_h) = 0$ for any $h \geqslant g$, and hence $\underset{i\to\infty}{Limit}\, Pr(A, (\ldots(B'_1 \vee B'_2) \vee \ldots) \vee B'_i) = 0$. So Pr meets Constraint C10.

T3.4

(a) *Let $S \vdash_I D$. If $S \nvdash_I A$, then $S \nvdash_I D \supset A$.*
(b) *Let B be an arbitrary axiom of IL, and S' be an arbitrary finite subset of S. Then $S \vdash_I B \,\&\, C(S')$.*

Our Completeness Theorems are now at hand:

T3.5 *Let S be an arbitrary set of statements of IL that is infinitely extendible. If $S \nvdash_I A$, then there exists an intuitionistic probability function Pr for IL and a statement B of IL such that $Pr(A, B \,\&\, C(S')) \neq 1$ for every finite subset S' of S.*

Proof: Suppose $S \nvdash_I A$; let B be some axiom or other of IL; let S' be an arbitrary finite subset of S; and let Pr be defined as in T3.3. $S \vdash_I B \,\&\, C(S')$ by L3.4(b), hence $S \nvdash_I (B \,\&\, C(S')) \supset A$ by T3.4(a), and hence $Pr(A, B \,\&\, C(S')) = 0$ by the definition of Pr. So, if $S \nvdash_I A$, then by T3.3 there is sure to be an intuitionistic probability function Pr for IL and a statement B of IL such that $Pr(A, B \,\&\, C(S')) \neq 1$ for every finite subset S' of S.

Hence, taking S to be ϕ, in which case $Pr(A, B \,\&\, C(S'))$ is just $Pr(A, B)$:

T3.6 (The Weak Completeness Theorem for IL) *If $\models_I A$, then $\vdash_I A$.*

Proof: Suppose $\nvdash_I A$. Then by T3.5 $\not\models_I A$. Hence by (the classical Law of) Contraposition $\vdash_I A$ if $\models_I A$.[20]

When as in T3.5 S is presumed to be *infinitely extendible*, there is sure to be, as we noted, a term of IL that is foreign to S and $B \supset (\forall X)A$ (Constraint

C9) or to S and $(\exists X)B \supset A$ (Constraint C10). Not so, however, when S is an *arbitrary* set of statements of IL, the case we must now tackle. One way out of this predicament is to call on the language IL^{∞} (Section 1) an extension of IL that boasts $\aleph_0$ terms over and above those of IL, and use in place of our original Pr the function Pr^{∞} such that, for any statements A and B of IL^{∞}, $Pr^{\infty}(A, B) = 1$ if $B \supset A$ is provable in IL^{∞} from the present S, $Pr^{\infty}(A, B) = 0$ otherwise. Since $\aleph_0$ terms of IL^{∞} are foreign to S (a set of statements of IL, recall), there is sure to be a term of IL^{∞} that is foreign to either: (i) S and any conditional $B \supset (\forall X)A$ of IL^{∞} you please, or (ii) S and any conditional $(\exists X)B \supset A$ you please. Further, the terms of IL^{∞} (like those of IL) were alphabetically ordered. So, we may now think of T_i in $A(T_i/X)$ and $B(T_i/X)$ as the i^{th} term of IL^{∞} (rather than IL); and, writing again A_i' for $A(T_i/X)$ and B_i' for $B(T_i/X)$, we can count on there being for any term T of IL^{∞} a g such that $A(T/X)$ is A_g' and $B(T/X)$ is B_g'. Finally, as demonstrated in Section 1, $S \vdash_I A$ if $S \vdash_I^{\infty} A$ (and hence, of course, $S \nvdash_I^{\infty} A$ if $S \nvdash_I A$) when, as is the case here, S is a set of statements of IL and A is a statement of IL.

Minimal editing of the proof of T3.3 will thus ensure that Pr^{∞} constitutes an intuitionistic probability function for IL^{∞}; and minimal editing of the proof of T3.5 will ensure that if $S \nvdash_I A$ and hence $S \nvdash_I^{\infty} A$, then $Pr^{\infty}(A, B \,\&\, C(S')) \neq 1$, where B is some axiom or other of IL^{∞} and S' is any finite subset of S.

Hence:

T3.7 *Let S be an arbitrary set of statements of IL (and A be an arbitrary statement of IL). If $S \nvdash_I A$, then there exists a term extension IL^+ of IL, an intuitionistic probability function Pr^+ for IL^+, and a statement B of IL^+ such that $Pr^+(A, B \,\&\, C(S')) \neq 1$ for every finite subset S' of S.*

Hence by Contraposition:

T3.8 (The Strong Completeness Theorem for IL) *If $S \models_I A$, then $S \vdash_I A$.*

Incidentally, since our account of $\vdash_I$ is such that $S \vdash_I A$ if and only if $S' \vdash_I A$ for some finite subset S' of S, we have by T2.10 and T3.8 that:

T3.9 (The Compactness Theorem for IL) *$S \models_I A$ if and only if $S' \models_I A$ for some finite subset S' of S.*

NOTES

1. To be more exact, "as Popper's and Gaifman's functions do to classical logic". Popper limited himself in [15] to the probability of negations and conjunctions, and Constraint C9 below, which is commonly used to handle quantifications, stems from a paper of Gaifman's [3].
2. Some of the pertinent references are [2], [3], [5], [6], [8]-[10], [12], [15], and [16].
3. Most formulations of intuitionistic logic have a negation operator, say $\sim$, in place of f. Here, $\sim A$ may be thought of as short for $A \supset f$. See Notes 6 and 10 for further details on $\sim$.

4. The statements of IL are what some would call *closed* statements. Indeed, we shall make no mention here of *open* statements. Note, however, that IL has terms ($\aleph_0$ of them, as a matter of fact). So the statements of IL, though closed, are *not* all closures of open ones. Because of (iv) identical quantifiers cannot overlap in a statement, a departure from common practice that is quite opportune (and conceptually immaterial).

5. There are various ways of alphabetically ordering the statements of a first-order language. The one in [13], which stems from [17], is particularly simple and easily adapted to suit IL.

6. We borrowed A3 from Exercise 26.19 in [1]. With $\sim$ substituting for f as a (primitive) sign of IL, the following two axiom schemata:

$$(A \supset B) \supset ((A \supset \sim B) \supset \sim A)$$

and

$$A \supset (\sim A \supset B),$$

or equivalents thereof, would be wanted in place of A3.

7. Note that with $A \supset (\forall X)A$—and hence A—presumed to be a statement of IL, X is sure not to occur in A (see Note 4). $(\forall X)A$ here is thus a *vacuous* quantification, as is the $(\exists X)A$ of A14.

8. A16 stems from an axiom schema of Fitch's. It permits one to dispense with Generalization as a primitive rule of inference and makes for a *strongly* complete axiomatization of intuitionistic logic. Many axiomatizations of first-order logic, be it classical or intuitionistic, that use Generalization as a primitive rule of inference are only *weakly* complete, as shown by Montague and Henkin. For further information on the whole matter, see [11].

9. Following recent practice we shall write '$S \nvdash_I A$' for 'It is not the case that $S \vdash_I A$'; $S \not\models_I A$ for 'It is not the case that $S \models_I A$', etc.

10. With $\sim$ substituting for f as a primitive sign of IL, the following two constraints:

$$Pr^+(\sim A, B) = Pr^+(\sim A, A \,\&\, B)$$

and

$$Pr^+(A, B \,\&\, \sim B) = 1,$$

or equivalents thereof, would be wanted in place of C3.

11. The following constraint, incidentally, would do the same work as C8:

C8′. $Pr^+((A \vee B) \supset C, t) = Pr^+((A \supset C) \,\&\, (B \supset C), t)$,

where 't' is short for '$f \supset f$'. C8′ holds in classical probability theory, whereas C8 does not. See the sequel to this paper for further information on the matter.

12. The account stems from [16], where (given a different choice of probability functions) it characterized logical truth in the conditional logic of Stalnaker and Thomason. Use of it was later made in [5], [2], [9], etc., where (given again different choices of probability functions) it characterized logical truth in classical logic (sometimes with, but most frequently without, identity).

13. The account is a simplification of one we used in [14] to characterize (given a different choice of probability functions) entailment in classical logic.

14. Readers in a hurry may take L2.1-T2.6 for granted and move on to T2.7.

15. Because of L2.1(b) the sequence consisting of $Pr(A'_1, B)$, $Pr(A'_1 \& A'_2, B)$, $Pr((A'_1 \& A'_2) \& A'_3, B)$, etc., is nonincreasing. So, if its limit equals 1, each term is sure by C1 to equal 1, too. The fact will prove crucial when it comes to proving T2.5(a).

16. Because of L2.1(g) the sequence consisting of $(Pr((\exists X)A, A'_1 \& B)$, $Pr((\exists X)A, (A'_1 \vee A'_2) \& B)$, $Pr((\exists X)A, ((A'_1 \vee A'_2) \vee A'_3) \& B)$, etc., is nonincreasing. So, if its limit equals 1, each term is sure by C1 to equal 1 too. The fact will prove crucial when it comes to proving T2.5(d).

17. Recall that B'_i and (later in the text) A'_i are short for $B(T_i/X)$ and $A(T_i/X)$, where T_i is the i^{th} term of IL.

18. Recall that A''_i is short for $(A(X/T))(T_i/X)$, where T_i is the i^{th} term of IL.

19. As there is (in point of fact, can be) *no* effective way of ascertaining for arbitrary S, A, and B whether or not $S \vdash_I A \supset B$, the definition of Pr is not constructive. It would be, though, if IL were a nonquantificational language and S were finite: under these circumstances there *is*, as Gentzen showed in [4], an effective way of ascertaining whether or not $S \vdash_I A \supset B$.

20. Contraposition, as the reader well knows, is *not* intuitionistically acceptable as a rule of inference.

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