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Stability for Pairs of Equivalence Relations

CARLO TOFFALORI

Abstract We consider pairs of equivalence relations E_0, E_1 such that, for some nonnegative integer h, every class of the join of E_0 and E_1 contains at most h classes of either E_0 or E_1. We classify these structures under categoricity (in some infinite power), nonmultidimensionality and finite cover property.

1 Let T be a countable complete first-order theory with no finite models. As usual, we assume that all models of T are elementary substructures of some big model U (the universe of T). Our aim is to study stability for theories T of two equivalence relations E_0, E_1, with particular attention to the problem of classifying among them the ones that are categorical in $\aleph_0$ or in $\aleph_1$.

Notice that in the simple case $E_0 = E_1$, hence when there is a unique equivalence relation, the situation is quite clear. In fact T is ω-stable and one can easily prove:

Theorem 1 *Let T be the theory of an equivalence relation E. Then the following propositions are equivalent*:
1. *T is $\aleph_0$-categorical*
2. *T does not satisfy the finite cover property* (f.c.p.)
3. *there is $k \in \omega$ such that, for all $a \in U$, $E(U,a)$ has either $\leq k$ or infinitely many elements.*

Since, for every theory T, T's being $\aleph_1$-categorical implies T's being nmd and ω-stable (where 'nmd' signifies nonmultidimensionality), and this implies T's being ω-stable and without the f.c.p., it follows that, in the case of a unique equivalence relation,

$$T \text{ is } \aleph_1\text{-categorical} \Rightarrow T \text{ is } \aleph_0\text{-categorical}.$$

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Then one can easily see:

Theorem 2 *Let T be the theory of an equivalence relation E. Then the following propositions are equivalent*:

1. *T is $\aleph_1$-categorical*
2. *T is $\aleph_0$-categorical, and either there are at most finitely many finite E-classes and exactly one infinite E-class, or every E-class is finite and there is a unique $h \leq k$ such that there are infinitely many E-classes of power h.*

More generally, one can prove:

Theorem 2′ *Let T be as above. Then the following propositions are equivalent*:

1. *T is nmd*
2. *T is $\aleph_0$-categorical, and there are at most finitely many infinite classes.*

Finally, one can see that, if T is $\aleph_0$-categorical but not nmd, then $Dp\, T = 2$.

But, when we proceed to pairs of equivalence relations, the situation gets much more complicated. In fact, even if usually only refining and crosscutting equivalence relations are treated, there are many more intricate ways in which two equivalence relations can interact, and it is worth recalling that the theory of two equivalence relations is undecidable [8]. (See [6] for the troubles arising when one studies categoricity in this context.)

We shall prove here that, assuming the more restrictive condition (+) (see Section 2; essentially, (+) says that, if E is the equivalence relation generated by E_0 and E_1, then every E-class in U contains at most h classes of either E_0 or E_1 for a suitable $h \in \omega$), then some results similar to Theorems 1, 2, and 2′ above can be shown for theories of two equivalence relations. This will be obtained first by reducing the study of arbitrary pairs of equivalence relations (satisfying (+)) to the study of pairs of permuting equivalence relations, and then by giving a complete answer in this particular case.

We shall give a more general analysis of classification theory for theories of pairs of equivalence relations in [10]. (Main references are: [2] for basic model theory, [5] and [9] for stability theory.)

2 Let T be a theory of two equivalence relations E_0, E_1. For all $x,y \in U$ and $\epsilon \in \{0,1\}$, we define

$$(x,y) \in R_1^\epsilon \text{ if and only if } \models E_\epsilon(x,y)$$

and, for $h \in \omega$, $h > 0$,

$$(x,y) \in R_{h+1}^\epsilon \text{ if and only if } \models \exists z\ (E_\epsilon(x,z) \wedge R_h^{1-\epsilon}(z,y)).$$

Therefore, for all $x,y \in U$, $(x,y) \in R_{h+1}^\epsilon$ if and only if either h is even and $\models \exists z_1 \ldots \exists z_h (E_\epsilon(x,z_1) \wedge E_{1-\epsilon}(z_1,z_2) \wedge \ldots \wedge E_\epsilon(z_h,y))$ or h is odd and $\models \exists z_1 \ldots \exists z_h (E_\epsilon(x,z_1) \wedge E_{1-\epsilon}(z_1,z_2) \wedge \ldots \wedge E_{1-\epsilon}(z_h,y))$. Furthermore, for every $h \in \omega$ and $\epsilon \in \{0,1\}$, $R_h^\epsilon \subseteq R_{h+1}^{1-\epsilon}$.

Set now

$$(x,y) \in R_0 \text{ if and only if } \models E_0(x,y) \wedge E_1(x,y)$$

and, for every $h \in \omega$, $h > 0$,

$$(x,y) \in R_h \text{ if and only if } \models \bigvee_{\epsilon \in \{0,1\}} \text{“}(x,y) \in R_h^\epsilon\text{”}.$$

Notice that, for all $h \in \omega$, $R_h \subseteq R_{h+1}$. Finally, put

$$(x,y) \in E \text{ if and only if } \bigvee_{h \in \omega} \text{“}(x,y) \in R_h\text{”}.$$

Then E is an equivalence relation, and equals the join of E_0 and E_1. In the following we will call E the *connection relation*, and any E-class a *connected component*.

In general, E is not 0-definable (see example 2 below). In fact, E is 0-definable if and only if there is $h \in \omega$ such that $E = R_h$. However, if E_0 and E_1 are permuting (that is, if $R_2^0 = R_2^1$), then $E = R_2 = R_2^0 = R_2$, and hence E is 0-definable.

We can now state our assumption:

(+) There is $h \in \omega$ such that, for all $a \in U$, there exists $\epsilon \in \{0,1\}$ such that the connected component of a in U contains at most h elements pairwise inequivalent in E_ϵ.

Notice that, if $E_\epsilon \subseteq E_{1-\epsilon}$ for some $\epsilon \in \{0,1\}$, then (+) holds. It is easy to see that, if (+) holds, then E is 0-definable (in fact, $E = R_{2h}$). The following example shows that the converse is false.

Example 1. Let $T_1 = Th(Z \times Z, E_0, E_1)$ where, for all $\epsilon \in \{0,1\}$, $x = (x_0, x_1)$, $y = (y_0, y_1) \in Z \times Z$,

$$\models E_\epsilon(x,y) \quad \text{if and only if } x_\epsilon = y_\epsilon.$$

Then E is 0-definable in T_1 (as $E = R_2$), but (+) does not hold.

Example 2. Let $T_2 = Th(\{x \in Z \times Z\colon \text{either } x_0 = x_1 \text{ or } x_0 + 1 = x_1\}, E_0, E_1)$ where E_0 and E_1 are defined as above. Then E is not 0-definable in T_2 as, for all $h \in \omega$, there are x, y satisfying

$$(x,y) \in R_{h+1} - R_h.$$

In particular T_2 does not satisfy (+).

Owing to the results we want to show below, it is worth pointing out that: (i) T_2 is $\aleph_1$-categorical (hence T_2 is nmd and does not satisfy the f.c.p.), but T_2 is not $\aleph_0$-categorical; and (ii) T_1 is $\aleph_0$-categorical but does not satisfy (+). Thus it is not true for an arbitrary theory T of two equivalence relations that

$$T \text{ is } \aleph_1\text{-categorical} \Rightarrow T \text{ is } \aleph_0\text{-categorical} \Rightarrow T \text{ satisfies (+)}.$$

For completeness' sake, we notice that a simple Ryll-Nardzewski argument assures us that

$$T \text{ is } \aleph_0\text{-categorical} \Rightarrow E \text{ is 0-definable in } T$$

(see [6]).

Theorem 3 *Let T be a theory of two equivalence relations satisfying* (+). *Then T is superstable.*

Lemma 1 *If M is a model of a theory of two equivalence relations, there is a structure M^* in a language $L^* = \{E_0, E_1, P\}$ (with P a 1-ary relation symbol) such that $P(M^*) \simeq M$, E_0 and E_1 are permuting equivalence relations in M^*, if M satisfies $(+)$ then M^* satisfies $(+)$, and $\neg P(M^*)$ intersects each $E_0 \cap E_1$-class in one element.*

Proof: Build M^* in the following way:

(i) The domain of M^* contains M and, moreover, an element $x(X)$ for any class X of $E_0 \cap E_1$ in M, and an element $x(X_0, X_1)$ for any pair of classes X_0, X_1 of E_0, E_1 respectively in M such that $X_0 \cap X_1 = \emptyset$ but, if $a_0 \in X_0$ and $a_1 \in X_1$, then $M \models E(a_0, a_1)$.

(ii) $P(M^*) = M$.

(iii) For every $\epsilon \in \{0,1\}$, $E_\epsilon(M^{*2})$ is an equivalence relation extending in the natural way $E_\epsilon(M^2)$ (for instance, we put, for every class X of $E_0 \cap E_1$ in M, and for every $a \in X$, $M^* \models E_0(x(X), a) \wedge E_1(x(X), a)$, and, for every pair of classes X_0, X_1 of E_0, E_1 in M such that $x(X_0, X_1)$ is defined, and for all $a_0 \in X_0$ and $a_1 \in X_1$, $M^* \models E_0(x(X_0, X_1), a_0) \wedge E_1(x(X_0, X_1), a_1))$.

Claim 1 $E(M^{*2}) \cap M^2 = E(M^2)$.

$\supseteq$ is trivial.

$\subseteq$: Let $x, y \in M$ be such that $(x,y) \in E(M^{*2})$. Then there is $k \in \omega$ such that $M^* \models$ "$(x,y) \in R_k$". We can assume $k > 1$, k odd. Hence there are $z_1, \ldots, z_{k-1} \in M^*$ such that, for some $\epsilon \in \{0,1\}$,

$$M^* \models E_\epsilon(x, z_1) \wedge E_{1-\epsilon}(z_1, z_2) \wedge \ldots \wedge E_\epsilon(z_{k-1}, y).$$

Put for simplicity $x = z_0$ and $y = z_k$. If $z_1, \ldots, z_{k-1} \in M$, then we are done. Otherwise let i be the minimal index $\leq k$ such that $z_i \notin M$. Clearly $0 < i < k$. If $x_i = x(X)$ for some class X of $E_0 \cap E_1$ in M, then replace x_i with an arbitrary $a \in X$. If $x_i = x(X_0, X_1)$ for some pair X_0, X_1 of classes of E_0, E_1 in M such that $X_0 \cap X_1 = \emptyset$ but, for all $a_0 \in X_0$ and $a_1 \in X_1$, $M \models E(a_0, a_1)$, then fix $a_0 \in X_0$ and $a_1 \in X_1$. Suppose for simplicity $M^* \models E_\epsilon(x_{i-1}, x_i)$, then in M and consequently in M^* it is true that $\models E_\epsilon(x_{i-1}, a_\epsilon)$ and $\models E(a_\epsilon, a_{1-\epsilon})$; furthermore $M^* \models E_{1-\epsilon}(a_{1-\epsilon}, x_{i+1})$. Then we can replace x_i with a finite sequence of elements of M. By repeating this procedure, we get $M \models E(x,y)$.

Claim 2 *In M^*, E_0 and E_1 permute; in particular, E is 0-definable in M^*.*

It suffices to show that $E \subseteq R_2^0, R_2^1$. Let $x, y \in M^*$ be such that $(x,y) \in E(M^{*2})$. Fix $x', y' \in M$ satisfying

$$M^* \models E_0(x, x') \wedge E_1(y, y').$$

Then $(x', y') \in E(M^{*2})$, and so $M \models E(x,y)$. It follows that $E_0(M^*, x') \cap E_1(M^*, y') \neq \emptyset$, hence $E_0(M^*, x) \cap E_1(M^*, y) \neq \emptyset$. Then $M^* \models$ "$(x,y) \in R_2^0$". Similarly for R_2^1.

Notice that, for all $x \in M^*$, there exists $x' \in M$ such that $E(M^*, x) = E(M^*, x')$. Moreover, for all $x' \in M$, $E(M^*, x') = (E(M, x'))^*$.

Claim 3 *If M satisfies $(+)$, then M^* satisfies $(+)$.*

In fact, for all $x \in M$, $i \in \omega$, $\epsilon \in \{0,1\}$, $E(M,x)$ contains i elements pairwise inequivalent in E_ϵ if and only if $E(M^*, x)$ does.

Lemma 2 *If T is a theory in L^* such that the two equivalence relations permute, $(+)$ is satisfied and $\neg P$ intersects each $E_0 \cap E_1$-class uniquely, then T is superstable.*

Proof: Let $M \models T$; we have to calculate card $(S_1(M))$. Then let p be a non-algebraic type in $S_1(M)$.

Case 1: For all $m \in M$, $v \neq m \in p$, and there is $a \in M$ such that $E_0(v,a) \wedge E_1(v,a) \in p$. Clearly $P(v) \in p$. If x,y satisfy the previous conditions, then there exists an automorphism of U mapping x and y into each other, and fixing any further element of U. In particular $tp(x/M) = tp(y/M)$, so that p is uniquely determined by these conditions. Then there are at most card M 1-types over M satisfying Case 1.

Case 2: For all $m \in M$, $\neg E_1(v,m) \in p$, and there is $a \in M$ such that $E_0(v,a) \in p$. Notice that (+) implies that $E(M,a)$ contains only finitely many E_0-classes (say t E_0-classes, where $t \leq h$). Fix a_0 $(=a)$, $a_1, \ldots, a_{t-1} \in M$ pairwise equivalent in E_1 and inequivalent in E_0. For all $j < t$ and $x \models p$, let

$$\alpha_j(p) = \mathrm{card}(E_0(U,a_j) \cap E_1(U,x));$$

then $\alpha_j(p)$ is finite or equals card U. Let x,y in U satisfy the incomplete type in v given by the following formulas:

$\neg E_1(v,m)$ for all $m \in M$,
$E_0(v,a_0)$,
$P(v)$ or $\neg P(v)$ provided that $P(v) \in p$ or $\neg P(v) \in p$,
$\exists!\alpha_j(p)w(E_0(w,a_j) \wedge E_1(w,v))$ for all $j < t$ with $\alpha_j(p)$ finite,
$\exists > nw(E_0(w,a_j) \wedge E_1(w,v))(n \in \omega)$ for all $j < t$ with $\alpha_j(p) = \mathrm{card}\ U$.

Then one can build an automorphism of U interchanging x and y, and mapping, more generally, for all $j < t$, $E_0(U,a_j) \cap E_1(U,x)$ and $E_0(U,a_j) \cap E_1(U,y)$ into each other, and fixing any further element of U. Hence $tp(x/M) = tp(y/M)$, and p is uniquely determined by the previous list of formulas. Then there are at most $2 \cdot \aleph_0^h \cdot \mathrm{card}\ M = \mathrm{card}\ M$ 1-types over M corresponding to this case.

Case 2': For all $m \in M$, $\neg E_0(v,m) \in p$, and there is $a \in M$ such that $E_1(v,a) \in p$. This case can be handled exactly as Case 2.

Case 3: For all $m \in M$, $\neg E_0(v,m) \in p$ and $\neg E_1(v,m) \in p$. Then permutability and (+) yield that $\neg E(v,m) \in p$ for all $m \in M$. Let x,y satisfy the previous assumptions, and suppose $tp(x/\emptyset) = tp(y/\emptyset)$ (in particular, $\models P(x)$ if and only if $\models P(y)$); then there is an automorphism f of U mapping x into y (hence $E(U,x)$ into $E(U,y)$), and we can assume that f induces an isomorphism of $E(U,y)$ onto $E(U,x)$ and fixes any element of M. Then $tp(x/M) = tp(y/M)$. It follows that at most $2^{\aleph_0}$ 1-types over M satisfy this case.

In general, $\mathrm{card}(S_1(M)) \leq 2^{\aleph_0} + \mathrm{card}\ M$ for each $M \models T$. Hence T is superstable.

Theorem 3 follows immediately from the previous lemmas.

Corollary *Let T be a theory of two equivalence relations satisfying $(+)$. If T is $\aleph_0$-categorical, then T is ω-stable.*

Proof: This follows from the theorem of Lachlan [4] stating that a superstable $\aleph_0$-categorical theory is ω-stable.

Example 3. There exists a theory T of two equivalence relations E_0, E_1 satisfying (+) which is not ω-stable. In fact, consider the theory T_0 such that:

(i) $E_0 \subsetneqq E_1$ (in particular T_0 satisfies (+))
(ii) for all k, $n(1), \ldots, n(k) \in \omega - \{0\}$, there is $a \in U$ such that, for every $i = 1, \ldots, k$, $E_1(U, a)$ contains exactly $n(i)$ E_0-classes of power i.

Let T be any completion of T_0. Then T has at least $2^{\aleph_0}$ 1-types over $\varnothing$. Hence T is not ω-stable.

Example 4. If we drop (+), then we can meet even unstable examples of theories of pairs of permuting equivalence relations. For instance, let $T = Th(M, E_0, E_1)$ where

$$M = \{(a,b,c) \in \omega^3 : c = 0 \text{ when } a \geq b,\ c \in \{0,1\} \text{ otherwise}\}$$

and, for all (a,b,c), $(a',b',c') \in M$,

$$E_0((a,b,c),\ (a',b',c')) \text{ if and only if } a = a',$$

$$E_1((a,b,c),\ (a',b',c')) \text{ if and only if } b = b'.$$

Then E_0, E_1 are permuting (in fact crosscutting) equivalence relations, but (+) does not hold. T is unstable, since the subset of $M\{\bar{a} = (a,a,0) : a \in \omega\}$ is linearly ordered by

$$\bar{a} < \bar{b} \text{ if and only if } \mathrm{card}(E_0(M,\bar{a}) \cap E_1(M,\bar{b})) = 2.$$

3 In this section, T will always denote a theory of two equivalence relations satisfying (+). Our problem here is to find under which assumptions T in $\aleph_0$-categorical. Thus the following proposition arises in a natural way.

(P1) There exists $N \in \omega$ such that, for all $a \in U$, $(E_0 \cap E_1)(U, a)$ has either $\leq N$ or infinitely many elements.

In fact, we have

Lemma 3
(1) *If T is $\aleph_0$-categorical, then T satisfies* (P1).
(2) *If T does not admit the f.c.p., then T satisfies* (P1).

Proof: (1) is a consequence of (2) and the fact that any stable $\aleph_0$-categorical theory does not admit the f.c.p. (see [1]).

For (2), suppose that T does not satisfy (P1); then the formula $v \neq w \wedge E_0(v,w) \wedge E_1(v,w)$ admits the f.c.p., so that also T satisfies the f.c.p.

Thus we can restrict our attention to theories T satisfying (P1). For every structure $M = (M, E_0, E_1)$ such that $Th(M)$ satisfies (+) and (P1), we build a new structure M^* of the same language in the following way:

(i) The domain of M^* is composed by M and $N + 1$ elements $x_i(X_0, X_1)$ $(i \leq N)$ for any pair X_0, X_1 of classes of E_0, E_1 in M such that $X_0 \cap X_1 = \varnothing$ but, for any $a_0 \in X_0$ and $a_1 \in X_1$, $M \models E(a_0, a_1)$.

(ii) E_0, E_1 are equivalence relations in M^* defined in the obvious way in order to extend the relations of M.

One can easily see that:

(iii) $E(M^{*2}) \cap M^2 = E(M^2)$;
(iv) E_0 and E_1 permute in M^* (in particular, E is 0-definable in M^*);
(v) M^* satisfies (+) and (P1) (for $N+1$).

As we shall see in Section 5, elementary equivalence, categoricity in any infinite power, ¬f.c.p. etc. are preserved under passing from M to M^* and vice versa. Thus there is no loss of generality for our purposes in replacing a theory T satisfying (+) and (P1) with $T^* = Th(M^*)$ where M is any model of T. Hence we can assume that E_0 and E_1 permute in T.

Furthermore, if M is any structure with two permuting equivalence relations satisfying (+), then we can decompose M in the following way:

$$M = \bigcup_{i<h} (M_i^0 \cup M_i^1)$$

where, for all $i < h$,

$$M_i^0 = \{a \in M : E(U,a) \text{ contains exactly } i+1 \text{ classes of } E_0\},$$

$$M_i^1 = \left\{a \in M - \bigcup_{j<h} M_j^0 : E(U,a) \text{ contains exactly } i+1 \text{ classes of } E_1\right\}.$$

Clearly M satisfies (P1) if and only if, for all $i < h$, M_i^0 and M_i^1 satisfy (P1). Furthermore this decomposition preserves elementary equivalence, categoricity, ¬f.c.p., etc. in both senses (see Section 6 for the details). Hence there is no loss of generality for our purposes in assuming that in M (and consequently in all models of its theory) every E-class contains exactly h E_0-classes.

Thus we will suppose from now on in this section that T is a theory of two permuting equivalence relations such that (+) holds and each E-class contains exactly h E_0-classes.

Let T be such a theory. Another necessary condition for the $\aleph_0$-categoricity of T is the following.

(P2) For all $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \text{card } U\}$, there is $N(\alpha_0, \ldots, \alpha_{h-1}) \in \omega$ such that, for all $a_0, \ldots, a_{h-1} \in U$ pairwise equivalent in E_1 and inequivalent in E_0, the power of the set of E_1-classes X in U satisfying card $(X \cap E_0(U, a_j)) = \alpha_j$ for all $j < h$ is either $\leq N(\alpha_0, \ldots, \alpha_{h-1})$ or infinite (and hence = card U).

Theorem 4 *Let T satisfy the previous assumptions. Then the following propositions are equivalent*:
1. *T is $\aleph_0$-categorical*
2. *T does not satisfy the f.c.p.*
3. *T satisfies* (P1) *and* (P2).

Proof: 1. ⇒ 2. It suffices to recall that no stable $\aleph_0$-categorical theory satisfies the f.c.p. (see [1]).

2. ⇒ 3. We already saw that, if T does not have the f.c.p., then T satisfies

(P1). It remains to show that the same holds for (P2). Recall from Shelah [9] that the $\neg$f.c.p. fails exactly when there is a formula $\varphi(v, v', \bar{w})$ such that

(i) for all $\bar{a} \in U$, $\varphi(v, v', \bar{a})$ defines an equivalence relation,
(ii) for all $n \in \omega$, there is $\bar{a}(n) \in U$ such that $\varphi(v, v', \bar{a}(n))$ has finitely many but $>n$ equivalence classes.

Therefore it suffices to show that, if T does not satisfy (P2), then such a formula $\varphi(v, v', \bar{w})$ can be found. Let $\alpha_0, \ldots, \alpha_{h-1}$ be a counterexample to (P2), and $\bar{w}$ denote $(w_0, \ldots, w_{h-1})$; put

$$\vartheta(v, \bar{w}): \bigwedge_{i<j<h} (E_1(w_i, w_j) \wedge \neg E_0(w_i, w_j)) \wedge \bigwedge_{j<h} \exists! \alpha_j z (E_0(w_j, z) \wedge E_1(v, z))$$

(where $\exists!$ card U means $\exists > N$),

$$\varphi(v, v', \bar{w}): (\vartheta(v, \bar{w}) \wedge \vartheta(v', \bar{w}) \wedge E_1(v, v')) \vee (\neg\vartheta(v, \bar{w}) \wedge \neg\vartheta(v', \bar{w})).$$

3. $\Rightarrow$ 1. Assume that T satisfies (P1) and (P2). Let M be a denumerable model of T, X be a class of E in M; we wish to characterize the isomorphism type of X (considered as a structure with two equivalence relations E_0, E_1). Notice that a similar analysis can be done even when $M = U$, provided we replace $\aleph_0$ with card U. First, let us sketch informally our argument. X can be viewed as a matrix with entries from $\{1,2, \ldots, N, \aleph_0\}$, whose rows correspond to the E_0-classes, and whose columns correspond to the E_1-classes; hence there are h rows and countably many columns. Any entry gives the power of the intersection of the corresponding E_0-class and E_1-class; so any column is described by a sequence $(\alpha_0, \ldots, \alpha_{h-1})$ from $\{1,2, \ldots, N, \aleph_0\}$, and X is given by a function telling how many columns of each type there are. Now let us make our argument more precise. Let F be a bijection of h onto the set of E_0-classes of X, define a function $f = f(X,F)$ of $\{1, \ldots, N, \aleph_0\}^h$ into $\omega \cup \{\aleph_0\}$ by setting, for all $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$,

$f(\alpha_0, \ldots, \alpha_{h-1})$ = power of the set of E_1-classes $Y \subseteq X$ such that, for all $j < h$, card $(Y \cap F(j))$ is α_j.

Notice that

(a) $f \neq 0$;
(b) for all $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$, either $f(\alpha_0, \ldots, \alpha_{h-1}) \leq N(\alpha_0, \ldots, \alpha_{h-1})$ or $f(\alpha_0, \ldots, \alpha_{h-1}) = \aleph_0$

(in particular, there exist at most finitely many functions $f(X,F)$). Clearly, $f(X,F)$ depends not only on X but also on F. Hence let $f = f(X,F)$, $f' = f(X,F')$ where F, F' are bijections of h onto the set of E_0-classes of X. Then, for all $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$, we have

$$f(\alpha_0, \ldots, \alpha_{h-1}) = f'(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(h-1)})$$

where σ is the permutation on h defined by $\sigma = F'^{-1}F$. In the set Φ of all functions of $\{1, \ldots, N, \aleph_0\}^h$ in $\omega \cup \{\aleph_0\}$ satisfying (a) and (b), consider the following binary relation $\sim$: if $f, g \in \Phi$, then $f \sim g$ if and only if there is $\sigma \in S_h$ such that $\forall \alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$, $f(\alpha_0, \ldots, \alpha_{h-1}) = g(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(h-1)})$.

Clearly, $\sim$ is an equivalence relation on Φ; moreover, for every E-class X of a denumerable model of T, and for every pair F, F' of bijections of h onto the set of E_0-classes of $X, f(X,F) \sim f(X,F')$.

We claim that, if M, M' are denumerable models of T, and X, X' are E-classes of M, M' respectively, then

$$X \simeq X' \text{ if and only if } f \sim f'$$

where $f = f(X,F)$, $f' = f(X',F')$ and F, F' are bijections of h onto the set of E_0-classes of X, X' respectively (owing to what we noticed above, the choice of F, F' is inessential).

First, assume $f \sim f'$; then there is $\sigma \in S_h$ such that, for all $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$,

$$f(\alpha_0, \ldots, \alpha_{h-1}) = f'(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(h-1)}).$$

Put $F'' = F'\sigma^{-1}$, hence F'' is a bijection of h onto the set of E_0-classes of X'; let $f'' = f(X', F'')$. Then, for all $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}, f''(\alpha_0, \ldots, \alpha_{h-1})$ is the power of the set of E_1-classes $Y' \subseteq X'$ such that, for all $j < h$, card $(Y' \cap F''(j)) = \alpha_j$, and hence is the power of the set of E_1-classes $Y' \subseteq X'$ such that, for all $j < h$, card $(Y' \cap F'(j)) = \alpha_{\sigma(j)}$. Then

$$f''(\alpha_0, \ldots, \alpha_{h-1}) = f'(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(h-1)}) = f(\alpha_0, \ldots, \alpha_{h-1}).$$

Consequently there is $f'' = f(X', F'')$ such that $f = f''$. Without loss of generality, $f = f'$. For all $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$, there is a bijection of the set of E_1-classes Y of X such that, for all $j < h$, card $(Y \cap F(j)) = \alpha_j$ onto the set of E_1-classes Y' of X' such that, for all $j < h$, card $(Y' \cap F'(j)) = \alpha_j$. For every E_1-class Y of X (with card $(Y \cap F(j)) = \alpha_j$ for all $j < h$), let Y' be the E_1-class of X' corresponding to Y. Then there is a bijection of $Y \cap F(j)$ onto $Y' \cap F'(j)$ for all $j < h$; by combining these bijections, it is possible to build an isomorphism of Y onto Y'. A similar union provides an isomorphism of X onto X'.

Conversely, let i be an isomorphism of X onto X'. If F is a bijection of h onto the set of E_0-classes of X, then iF is a bijection of h onto the set of E_0-classes of X'. Put $f = f(X,F)$, $f' = f(X', iF)$. Then, for all $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}, f(\alpha_0, \ldots, \alpha_{h-1}) = f'(\alpha_0, \ldots, \alpha_{h-1})$. It follows that $f = f'$. (Notice that we have also proved that, if $f' = f(X', F')$, $f \in \Phi$ and $f \sim f'$, then there exists $f'' = f(X', F'')$ such that $f = f''$.)

Finally, let us show that, if M, M' are denumerable models of T, then $M \simeq M'$. For all $n \in \omega \cup \{\aleph_0\}$, put

$$\bar{n} = \begin{cases} n & \text{if } n \in \omega, \\ \text{card } U & \text{if } n = \aleph_0. \end{cases}$$

Let $f \in \Phi$, $\overline{w} = (w_0, \ldots, w_{h-1})$, and consider the following formulas:

(i) $\varphi_f'(\overline{w})$ is the conjunction of

$$\bigwedge_{i<j<h} (E_1(w_i, w_j) \wedge \neg E_0(w_i, w_j))$$

and the formula assuring that, for all $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \text{card } U\}$, there exist exactly $\overline{f(\alpha_0, \ldots, \alpha_{h-1})}$ elements $x \in U$ pairwise inequivalent in E_1 such that, for any $j < h$, card $(E_1(U,x) \cap E_0(U,w_j)) = \alpha_j$ (it is easy to see that this proposition really can be written by means of a first-order formula).

(ii) $\varphi_f(v)$ is the formula $\exists \overline{w}\ (\varphi'_f(\overline{w}) \wedge E(w_0,v))$.

Claim 1 *If $f \in \Phi$, M is a denumerable model of T and $a \in M$, then $M \models \varphi_f(a)$ if and only if there is a bijection F of h onto the set of E_0-classes of $E(M,a)$ such that $f = f(E(M,a),F)$.*

The proof of this claim is straightforward.

Claim 2 *If $f, f' \in \Phi$, and $f \sim f'$, then $T \models \forall v\ (\varphi_f(v) \leftrightarrow \varphi_{f'}(v))$.*

Let $a \in U$ be such that $\models \varphi_f(a)$, and let M be a denumerable model of T containing a. Then there is F such that $f = f(E(M,a),F)$ (Claim 1). As $f' \sim f$, there is F' such that $f' = f(E(M,a),F')$. Then $M \models \varphi_{f'}(a)$.

In particular, if $f, f' \in \Phi, f \sim f'$ and $T \models \neg(\exists v \varphi_f(v))$, then $T \models \neg(\exists v \varphi_{f'}(v))$.

Claim 3 *Let $f, f' \in \Phi$ be such that $T \models \exists v \varphi_f(v)$ and $T \models \forall v\ (\varphi_f(v) \leftrightarrow \varphi_{f'}(v))$; then $f \sim f'$.*

Let M be a denumerable model of T, $a \in M$, $M \models \varphi_f(a)$. Then $M \models \varphi_{f'}(a)$, so that there are F, F' satisfying $f = f(E(M,a),F)$, $f' = f(E(M,a),F')$. It follows that $f \sim f'$.

Therefore T is able to recognize

(i) which isomorphism types of E-classes occur in its denumerable models (in fact, these isomorphism types correspond to the functions $f \in \Phi$ satisfying $T \models \exists v\ \varphi_f(v)$; we point out that there are finitely many);

(ii) when two functions $f, f' \in \Phi$ correspond to the same isomorphism type (in fact this happens exactly when $T \models \exists v\ \varphi_f(v) \wedge \forall v\ (\varphi_f(v) \leftrightarrow \varphi_{f'}(v)))$;

(iii) how many times an isomorphism type is represented (this is given by the maximal number of elements pairwise inequivalent in E satisfying $\varphi_f(v)$ where f is any function of Φ corresponding to the isomorphism type).

Then, if M, M' are denumerable models of T, it is easy to build an isomorphism between M and M'.

4 The aim of this section is to characterize $\aleph_1$-categorical theories and, more generally, nmd theories of two equivalence relations satisfying (+). The next lemma shows that all these theories are $\aleph_0$-categorical.

Lemma 4 *Let T be a theory of two equivalence relations satisfying* (+). *If T is nmd, then T is $\aleph_0$-categorical. In particular, if T is $\aleph_1$-categorical, then T is $\aleph_0$-categorical.*

Proof: It is well known that, for every theory T,

$$T \text{ is } \aleph_1\text{-categorical} \Rightarrow T \text{ is nmd and superstable} \Rightarrow T \text{ satisfies } \neg\text{f.c.p.}$$

But in our case T is superstable (Theorem 3), and T does not admit the f.c.p. if and only if T is $\aleph_0$-categorical (Theorem 4). Then Lemma 4 follows immediately.

Thus we can assume that T satisfies the characterization of $\aleph_0$-categorical theories we gave in the last section. Moreover, as we already did in Section 3, we can assume that E_0, E_1 permute and every E-class contains exactly h E_0-classes (the justification of these further assumptions will be given in the next sections).

Notation: For every $f \in \Phi$, let $\nu(T,f)$ denote the power of the set of E-classes of U whose isomorphism type correspond to f.

Notice that either $\nu(T,f) < \aleph_0$ or $\nu(T,f) = \text{card } U$.

Let M be a model of T; it is easy to see that any nonalgebraic 1-type p over M is fully determined by one of the following list of conditions:

(a) $v \neq m \in p$ for all $m \in M$; $E_0(v,a) \wedge E_1(v,a) \in p$ for some $a \in M$ with $(E_0 \cap E_1)(M,a)$ infinite.

(b) $\neg E_1(v,m) \in p$ for all $m \in M$; for some $f \in \Phi$, $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$ with $f(\alpha_0, \ldots, \alpha_{h-1}) = \aleph_0$, and $a_0, \ldots, a_{h-1} \in M$ such that $\models \varphi_f'(\bar{a})$,

$$E_0(v,a_0) \in p,$$

$$\exists!\alpha_j w(E_0(w,a_j) \wedge E_1(v,w)) \in p \text{ for all } j < h.$$

(c) $\neg E_0(v,m) \in p$ for all $m \in M$; for some $f \in \Phi$ with $\nu(T,f) = \text{card } U$, $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$ such that $f(\alpha_0, \ldots, \alpha_{h-1}) \neq 0$, p contains

$$\exists\overline{w}(v = w_0 \wedge \varphi_f'(\overline{w}) \wedge \bigwedge_{j<h} \exists!\alpha_j z(E_0(w_j,z) \wedge E_1(w_j,z)).$$

Lemma 5 *Let $M \models T$, $A \supseteq M$, $p' \in S_1(A)$, $p = p' \restriction M$.*

1. *If p satisfies* (a), *then p' forks over M if and only if there is $x \in A$ such that $v = x \in p'$;*
2. *if p satisfies* (b), *then p' forks over M if and only if there is $x \in A$ such that $E_1(v,x) \in p'$;*
3. *if p satisfies* (c), *then p' forks over M if and only if there is $x \in A$ such that $E(v,x) \in p'$.*

Proof: 1. ($\Leftarrow$) is obvious, since p' represents $v = w$ which is not represented by p.

($\Rightarrow$) Let y, y' realize p with $y, y' \notin A$. The previous analysis of (a) provides an automorphism of U mapping y into y', and fixing in particular any element of A. Then $tp(y/A) = tp(y'/A)$. Hence there is a unique extension of p in $S_1(A)$ containing $v \neq x$ for all $x \in A$, and this extension must equal the heir $p|A$ of p. Consequently, when $p' \neq p|A$, there exists $x \in A$ such that $v = x \in p'$.

2 and 3 can be shown in a similar way.

Lemma 6 *Let $M \models T, p, q$ be nonalgebraic* 1*-types over M, $p \neq q$. Then $p \not\perp q$ if and only if one of the following conditions holds*:

1. *$\neg E_1(v,m) \in p \cap q$ for all $m \in M$; there are $a_0, \ldots, a_{h-1} \in M$ pairwise equivalent in E_1 but not in E_0, and $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N,$ card $U\}$ such that $E_0(v,a_0) \in p$, $E_0(v,a_1) \in q$ and, for all $j < h$,* "card $(E_0(U,a_j) \cap E_1(U,v) = \alpha_j$" *$\in p \cap q$;*
2. *$\neg E(v,m) \in p \cap q$ for all $m \in M$, and there is $f \in \Phi$ such that $\varphi_f(v) \in p \cap q$.*

Proof: Let $x \models q$, $A = M \cup \{x\}$, then $p \not\perp q$ if and only if there is $p' \in S_1(A)$ such that $p' \supseteq p$ but p' forks over M. First let us show that, if $p \not\perp q$, then 1 or 2 holds. We shall use Lemma 5, distinguishing three cases.

1st case: p satisfies (a). If $p' \in S_1(A)$, $p' \supseteq p$ and p' forks over M, then there is $y \in A$ such that $v = y \in p'$ (hence $y \models p$); as p is nonalgebraic, $y = x$; then $p = q$.

2nd case: p satisfies (b), in particular p is defined by the formulas $E_0(v, a_0)$, $\neg E_1(v, m)$ for all $m \in M$ and

$$\exists!\alpha_j w(E_0(w, a_j) \wedge E_1(v, w)) \qquad (j < h)$$

for a suitable choice of $a_j, \alpha_j (j < h)$. Let $p' \in S_1(A)$ extend p and fork over M; then there is $y \in M \cup \{x\}$ such that $E_1(v, y) \in p'$; furthermore, $y = x$ as $\neg E_1(v, m) \in p$ for all $m \in M$; hence $\neg E_1(v, m) \in q$ for all $m \in M$, there is $i < h$ such that $E_0(v, a_i) \in q$, and for all $j < h$ "card $(E_0(U, a_j) \cap E_1(U, v)) = \alpha_j$" $\in q$; as $p \neq q$, we have $i \neq 0$; with no loss of generality, $i = 1$.

3rd case: p satisfies (c). Let $p' \in S_1(A)$ extend p and fork over M; then there is $y \in M \cup \{x\}$ such that $E(v, y) \in p'$; as above, $y = x$, hence $\neg E(v, m) \in q$ for all $m \in M$ and $\varphi_f(v) \in q$ where f is the function associated to p in (c).

Conversely, assume that 1 or 2 holds; we claim that there is $p' \in S_1(A)$ extending p and forking over M.

If 1 holds, then take $y \in E_0(U, a_0) \cap E_1(U, x)$ and set $p' = tp(y/A)$. Then $p' \supseteq p$, but p' forks over M as p satisfies (b) and $E_1(v, x) \in p'$.

If 2 holds, then let $\bar{x}, \bar{y}, t \in U$ be such that $\models \varphi'_f(\bar{x}) \wedge E(x, x_0)$, $t \models p$ and $\models \varphi'_f(\bar{y}) \wedge E(t, y_0)$. With no loss of generality, we can assume $x = x_0$, $t = y_i$ for a suitable $i < h$. For all $j < h$, let $\beta_j \in \{1, \ldots, N, \aleph_0\}$ be such that $\bar{\beta}_j = \mathrm{card}((E_0 \cap E_1)(U, y_j))$; then $f(\beta_0, \ldots, \beta_{h-1}) > 0$ and there exists $y \in E(U, x)$ such that, for all $j < h$,

$$\mathrm{card}(E_0(U, x_j) \cap E_1(U, y)) = \bar{\beta}_j.$$

We can assume $y \in E_0(U, x_i)$; let $p' = tp(y/A)$. Then $p' \supseteq p$, in fact, if σ denotes the permutation $(i0) \in S_h$, then p is fully determined by the formulas $\neg E(v, m)$ for all $m \in M$ and, furthermore, by the sequence

$$(\beta_{\sigma(0)}, \ldots, \beta_{\sigma(h-1)}) \in \{1, \ldots, N, \aleph_0\}^h$$

together with the function $g \in \Phi$ such that, for all $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$,

$$g(\alpha_0, \ldots, \alpha_{h-1}) = f(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(h-1)}).$$

Moreover p' forks over M since $E(v, x) \in p'$.

Theorem 5 *Let T be a theory of two permuting equivalence relations satisfying $(+)$ and such that every E-class contains exactly h E_0-classes. Then the following propositions are equivalent:*

(i) *T is nmd*

(ii) *T is $\aleph_0$-categorical; for all $f \in \Phi$ with $\nu(T, f) > 0$ and $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$,*

- *if there is $j < h$ such that $\alpha_j = \aleph_0$ or $\nu(T, f) = \mathrm{card}\ U$, then $f(\alpha_0, \ldots, \alpha_{h-1}) \leq N(\alpha_0, \ldots, \alpha_{h-1})$;*

if there is $j < h$ *such that* $\alpha_j = \aleph_0$ *and* $\nu(T,f) = \text{card } U$, *then* $f(\alpha_0, \ldots, \alpha_{h-1}) = 0$.

Proof: (i) ⇒ (ii). We already saw that, if T is nmd, then T is $\aleph_0$-categorical. Let M_0 denote the (unique) denumerable model of T. Thus, for all $M \models T$ and for all nonalgebraic $p \in S_1(M)$, $p \not\perp M_0$. Let $f \in \Phi$, $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$, $\nu(T,f) > 0$.

Assume $f(\alpha_0, \ldots, \alpha_{h-1}) = \aleph_0$, $\alpha_j = \aleph_0$ for some $j < h$. Let $a_0, \ldots, a_{h-1} \in M_0$ satisfy $\models \varphi_f'(\bar{a})$, and $x \in U$ satisfy the type q given by

$$v \neq m \in q \text{ for all } m \in M_0,$$

$$E_0(v, a_j) \in q,$$

$$\exists! \alpha_i z (E_0(z, a_i) \wedge E_1(z, v)) \in q \text{ for all } i < h.$$

Let M be the model of T prime over $M \cup \{x\}$ (T is $\aleph_0$-categorical, hence ω-stable), and consider the 1-type p over M defined by

$$v \neq m \in p \text{ for all } m \in M,$$

$$E_0(v, x) \wedge E_1(v, x) \in p.$$

Then Lemma 6 implies $p \perp M_0$—a contradiction. Hence, if $\alpha_j = \aleph_0$ for some $j < h$, then $f(\alpha_0, \ldots, \alpha_{h-1}) < \aleph_0$.

Similar arguments show that, if $\nu(T,f) = \text{card } U$, then $f(\alpha_0, \ldots, \alpha_{h-1}) < \aleph_0$ and, if both $\alpha_j = \aleph_0$ for some $j < h$ and $\nu(T,f) = \text{card } U$, then $f(\alpha_0, \ldots, \alpha_{h-1}) = 0$.

(ii) ⇒ (i). Let M_0 denote the denumerable model of T. We have to show that, if (ii) holds, then, for every $M \models T$ and for every nonalgebraic $p \in S_1(M)$, $p \not\perp M_0$. This follows easily from Lemma 6.

Theorem 5′ *Let T be as in Theorem 5. Then the following propositions are equivalent*:

(i) *T is* $\aleph_1$*-categorical*;

(ii) *T is* $\aleph_0$*-categorical and satisfies one of the following conditions*:

1. *There exists exactly one* $\sim$*-class of functions* $f \in \Phi$ *satisfying* $\nu(T,f) = \text{card } U$, *and, for all* $f \in \Phi$ *with* $\nu(T,f) > 0$ *and* $\alpha_0, \ldots, \alpha_{h-1} \in \{1, \ldots, N, \aleph_0\}$, $f(\alpha_0, \ldots, \alpha_{h-1}) \leq N(\alpha_0, \ldots, \alpha_{h-1})$ (*and* $= 0$ *when there is* $j < h$ *such that* $\alpha_j = \aleph_0$);
2. *for all* $f \in \Phi$, $\nu(T,f) < \aleph_0$; *there exist* $g \in \Phi$ *with* $\nu(T,g) > 0$ *and* $\bar{\alpha} = (\alpha_0, \ldots, \alpha_{h-1}) \in \{1, \ldots, N, \aleph_0\}^h$ *such that either*

$$\textit{for all } j < h \ \alpha_j < \aleph_0 \textit{ and } g(\bar{\alpha}) = \aleph_0$$

or

$$\textit{there is } j < h \textit{ such that } \alpha_j = \aleph_0 \textit{ and } g(\bar{\alpha}) = 1$$

while, for all $f \in \Phi$ *and* $\bar{\beta} = (\beta_0, \ldots, \beta_{h-1}) \in \{1, \ldots, N, \aleph_0\}^h$ *with* $\nu(T,f) > 0$ *and* $f \nsim g$ *or* $\bar{\beta} \neq \bar{\alpha}$, $f(\bar{\beta}) < \aleph_0$ *and* $f(\bar{\beta}) = 0$ *when there is* $j < h$ *such that* $\beta_j = \aleph_0$.

Proof: Notice that Theorem 5 implies that, if T is unidimensional, then T is ω-stable. Hence T is $\aleph_1$-categorical if and only if T is unidimensional, namely if and only if, for all $M \models T$ and for all nonalgebraic $p, q \in S_1(M)$, $p \not\perp q$; 1 and 2 just provide a complete list of all cases in which this property holds.

In [10] it will be shown that, if T is any theory of two equivalence relations satisfying (+), then T is classifiable according to Shelah, and $Dp\ T \leq 3$. Theorem 5 provides a complete characterization of all theories T satisfying $Dp\ T = 1$. It may be interesting to point out that also the remaining cases ($Dp\ T = 2$, $Dp\ T = 3$) occur for $\aleph_0$-categorical T.

Suppose that $Dp\ T = 2$; then it suffices to assume that $E_1 \subseteq E_0$, and there is a unique class of E_0, containing infinitely many E_1-classes, all infinite. Suppose instead that $Dp\ T = 3$; then it suffices to assume that $E_1 \subseteq E_0$, moreover E_0 admits infinitely many classes, each class of E_0 contains infinitely many classes of E_1, and each class of E_1 is infinite.

5 In this section we will pay the first debt we contracted in Section 3, when we associated to any structure M with two equivalence relations E_0, E_1 satisfying (+) and (P1) a new structure M^* of the same language.

We said that there is no loss of generality for our purposes in replacing M with M^*. The aim of this section is to explain why. We tacitly assume from now on that M, M' denote structures with two equivalence relations satisfying (+) and (P1). Notice that, if M is such a structure, then M is 0-definable in M^*, for instance by the formula

$$\neg(\exists! N+1\ z(E_0(v,z) \wedge E_1(v,z))).$$

Lemma 7
(1) *For all M, M', $M \equiv M'$ if and only if $M^* \equiv M'^*$.*
(2) *For all M, if $\overline{M} \equiv M^*$, then there is $M' \equiv M$ such that $M'^* \simeq \overline{M}$.*

Proof: (Sketch) (1) ($\Leftarrow$) follows from the fact that M is 0-definable in M^*.

($\Rightarrow$) can be shown, for instance, by recalling that the first-order language for our structures contains only finitely many extralogical symbols and hence $\equiv\ = \ \simeq_\omega$ (see [3]). We leave to the reader the straightforward proof of the fact that, if $M \simeq_\omega M'$, then $M^* \simeq_\omega M'^*$.

(2) Just set $M' = \{x \in \overline{M} : \text{card}\ ((E_0 \cap E_1)(\overline{M}, x)) \neq N+1\}$ with the structure induced by $\overline{M}$.

In particular, if $T = Th(M)$ and $T^* = Th(M^*)$, then the models of T^* are just the structures M'^* where $M' \models T$.

One can easily see also that, for all M, M',

(i) any elementary embedding of M^* into M'^* contains an elementary embedding of M into M';
(ii) conversely, any elementary embedding of M into M' can be extended to an embedding of M^* into M'^*, and this embedding is elementary, too.

Lemma 8 *For all M,*
(1) *M is $\aleph_0$-categorical ($\aleph_1$-categorical) if and only if M^* is*
(2) *M is ω-stable if and only if M^* is*
(3) *M is nmd if and only if M^* is*
(4) *M does not have the f.c.p. if and only if M^* does not have the f.c.p.*

Proof: First notice that, if U is the universe of $T = Th(M)$, then U^* is the universe of $T^* = Th(M^*)$.

(1) follows from the remark that M is infinite if and only if M^* is and, in this case, M and M^* have the same power.

(2) ($\Leftarrow$) is a consequence of the fact that M is 0-definable in M^*; ($\Rightarrow$) can be shown by using a trivial counting types argument.

(3) It is simple to see that, for all $x, y \in U$,

$$tp(x/M) = tp(y/M) \text{ (in } T\text{) iff } tp(x/M^*) = tp(y/M^*) \text{ (in } T^*\text{)},$$

$$tp(x/M) \perp tp(y/M) \text{ (in } T\text{) iff } tp(x/M^*) \perp tp(y/M^*) \text{ (in } T^*\text{)}.$$

On the other hand, any nonalgebraic 1-type over M^* is $\not\perp$ to some type over M^* containing "$v \in U$". As this holds for all $M' \models T$, it follows that T is nmd if and only if T^* is.

(4) ($\Leftarrow$) follows again from the fact that M is 0-definable in M^*. ($\Rightarrow$) Assume that M does not satisfy the f.c.p.; we have to show that neither does M^*. We will use the Poizat criterion [7] saying that, if T is a stable theory, and T_1 is the theory of nice pairs of models of T, then T_1 is complete, and T does not have the f.c.p. if and only if any ω_1-saturated model of T_1 is a nice pair. As above, put $T = Th(M)$, $T^* = Th(M^*)$, and consider T_1 and $(T^*)_1$: both of these theories are complete, as T and T^* are superstable. We know that any ω_1-saturated model of T_1 is a nice pair of models of T, and we claim that the same holds for T^*.

First notice that, if (M_0, M_1) is a pair of models of T such that $M_1 < M_0$, then (M_0^*, M_1^*) is a pair of models of T^* again satisfying $M_1^* < M_0^*$; moreover (M_0, M_1) is a substructure of (M_0^*, M_1^*). Furthermore, if (M_0, M_1), (M_0', M_1') are such pairs of models of T, then $(M_0, M_1) \equiv (M_0', M_1')$ if and only if $(M_0^*, M_1^*) \equiv (M_0'^*, M_1'^*)$, and any pair $(\bar{M}_0, \bar{M}_1)$ of models of T^* with $\bar{M}_1 < \bar{M}_0$ is isomorphic to (M_0^*, M_1^*) for some pair (M_0, M_1) of models of T such that $M_1 < M_0$.

In particular, all pairs (M_0^*, M_1^*) with $(M_0, M_1) \models T_1$ are elementarily equivalent; let $(T_1)^*$ denote their theory.

Claim $\quad (T_1)^* = (T^*)_1$.

As $(T^*)_1$ is complete, it suffices to show that, if (M_0, M_1) is a nice pair of models of T, then (M_0^*, M_1^*) is a nice pair of models of T^* (for, in this case, every model of $(T_1)^*$ is elementarily equivalent to (M_0^*, M_1^*) for some nice pair (M_0, M_1) of models of T, and consequently is a model of $(T^*)_1$; hence $(T^*)_1 \subseteq (T_1)^*$ and, finally, $(T^*)_1 = (T_1)^*$). We already saw that $M_0^* > M_1^*$. Then we have to show that:

(a) M_1^* is ω_1-saturated
(b) for all $\bar{a} \in M_0^*$, every type over $M_1^* \cup \bar{a}$ (in T^*) is realized in M_0^*.

Proof of (a). Let X^* be a countable subset of M_1^*, $p \in S_1(X^*)$, $y \in U$ realize p. Let X be the union between $X^* \cap M_1$ and the set one gets by choosing, for any element $x \in X^* - M_1$, two elements $a, b \in M_1$ such that $x \in E_0(M_1^*, a) \cap E_1(M_1^*, b)$ (and taking a, b in X^* if possible). Then X is countable, and $X \subseteq M_1$.

If $y \in U$, then there exists $y_0 \in M_1$ satisfying $tp(y/X)$ in T, hence there exists an automorphism of U mapping y into y_0 and fixing any element of X. One can easily extend this automorphism to an automorphism of U^* fixing any element of X^*. Then $y_0 \models p$.

If $y \notin U$, let $y_0, y_1 \in U$ be such that $y \in E_0(U^*, y_0) \cap E_1(U^*, y_1)$; $tp(y_0, y_1/X)$ in T is realized in M_1, say by y_0', y_1', hence there is an automorphism of U mapping y_0, y_1 into y_0', y_1' and fixing any element of X; extend it to an automorphism of U^* fixing every element of X^*; the image y' of y in this automorphism belongs to M_1^* and realizes p.

(b) can be shown in a similar way, and this concludes the proof of the claim.

Consider now any ω_1-saturated model of $(T^*)_1 = (T_1)^*$; without loss of generality, this model is of the form (M_0^*, M_1^*) where $(M_0, M_1) \models T_1$. Moreover (M_0, M_1) is ω_1-saturated, as (M_0, M_1) is 0-definable in (M_0^*, M_1^*). Then (M_0, M_1) is a nice pair of models of T, and hence (M_0^*, M_1^*) is a nice pair of models of T^*.

Therefore T^* does not have the f.c.p.

6 We have to pay the second debt we ran into during the previous sections, when we decomposed any structure M with two equivalence relations E_0, E_1 satisfying (+) and permuting in the following way:

$$M = \bigcup_{i<h} (M_i^0 \cup M_i^1)$$

where, for all $i < h$, $M_i^0 = \{a \in M : E(U,a) \text{ contains exactly } i+1 \text{ classes of } E_0\}$ and $M_i^1 = \{a \in M - \bigcup_{j<h} M_j^0 : E(U,a) \text{ contains exactly } i+1 \text{ classes of } E_1\}$, and we agreed to replace M with M_h^0 in our analysis.

In fact, as M_i^0, M_i^1 are 0-definable in M for all $i < h$, it is straightforward to show:

Lemma 9 *Let M be as above, then*

(1) *if $M' \equiv M$, then $M_i'^{\epsilon} \equiv M_i^{\epsilon}$ for all $i < h$ and $\epsilon \in \{0,1\}$;*

(2) *if $M_i'^{\epsilon} \equiv M_i$ for all $i < h$ and $\epsilon \in \{0,1\}$ and $M' = \bigcup_{i,\epsilon} M_i'^{\epsilon}$, then $M' \equiv M$;*

(3) *M is $\aleph_0$-categorical if and only if, for all $i < h$ and $\epsilon \in \{0,1\}$, M_i^{ϵ} is $\aleph_0$-categorical;*

(4) *M is $\aleph_1$-categorical if and only if, for all $i < h$ and $\epsilon \in \{0,1\}$, M_i^{ϵ} is finite or $\aleph_1$-categorical, and M_i^{ϵ} is $\aleph_1$-categorical for exactly one choice of i and ϵ;*

(5) *M is ω-stable if and only if, for all $i < h$ and $\epsilon \in \{0,1\}$, M_i^{ϵ} is ω-stable;*

(6) *M is nmd if and only if, for all $i < h$ and $\epsilon \in \{0,1\}$, M_i^{ϵ} is nmd;*

(7) *M does not satisfy the f.c.p. if and only if, for all $i < h$ and $\epsilon \in \{0,1\}$, M_i^{ϵ} does not satisfy the f.c.p.*

(As regards ($\Leftarrow$) of (7), use Theorem 4, which is satisfied by M_i^{ϵ} for all $i < h$ and $\epsilon \in \{0,1\}$, and the fact that any $\aleph_0$-categorical stable theory does not have the f.c.p.)

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Dipartimento di Matematica
Università dell'Aquila
Via Roma 33
67100 L'Aquila, Italy