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Indiscernibility and Identity in Probability Theory

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TABLE OF CONSTRAINTS

ON STANDARD BOOLEAN ALGEBRAS

A1′ *For any* A *and* B *in* S, $A \cap B = B \cap A$

A2′ *For any* A, B, *and* C *in* S, $A \cap (B \cap C) = (A \cap B) \cap C$

A3′ *For any* A, B, *and* C *in* S, $A \cap -B = C \cap -C$ *if, and only if,* $A \cap B = A$

A4′ *For any* A *and* B *in* S, *if* $A = B$, *then* $-A = -B$

A5′ *For any* A, B, *and* C *in* S, *if* $A = B$, *then* $A \cap C = B \cap C$

ON POPPER'S RELATIVE PROBABILITY FUNCTIONS

B1 *For some* A *and* B *in* S, $P(A,B) \neq 1$ (*Existence*)

B2 *For any* A *and* B *in* S, $0 \leq P(A,B)$ (*Nonnegativity*)

B3 *For any* A *in* S, $P(A,A) = 1$ (*Normality*)

B4 *For any* A *and* B *in* S, *if* $P(C,B) \neq 1$ *for some* C *in* S, *then* $P(A,B) + P(-A,B) = 1$ (*Addition*)

B5 *For any* A, B, *and* C *in* S, $P(A \cap B,C) = P(A,B \cap C) \times P(B,C)$ (*Multiplication*)

B6 *For any* A, B, *and* C *in* S, $P(A \cap B,C) \leq P(B \cap A,C)$ (*Commutation*)

B7 *For any* A, B, *and* C *in* S, $P(A,B \cap C) \leq P(A,C \cap B)$ (*Commutation*)

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ON RÉNYI'S RELATIVE PROBABILITY FUNCTIONS

RB8′ *For any B in S, if* $P(A, B) = 1$ *for every A in S, then* $B = \Lambda$

$\Vert$

RB8 *For any A and B in S, if* $P(A, C) = P(B, C)$ *for every C in S, then* $A = B$

RB8** *For any A and B in S, if* $P(A, C) = P(B, C)$ *for every C in S, then* $P(A, C) = P(B, C)$ *for every C in S and every relative probability function P of Popper's of Type II defined on S*

ON KOLMOGOROV'S RELATIVE PROBABILITY FUNCTIONS

KB8′ *For any B in S, if* $P(B, \mathrm{V}) = 0$, *then* $P(A, B) = 1$ *for every A in S*

$\Vert$

KB8 *For any A and B in S, if* $P(C, A) = P(C, B)$ *for every C in S, then* $P(A, C) = P(B, C)$ *for every such C*

ON CARNAP'S RELATIVE PROBABILITY FUNCTIONS

CB8′ *For any A in S, if* $P(A, \mathrm{V}) = 0$, *then* $A = \Lambda$

$\Vert$

CB8 *For any A and B in S, if* $P(C, A) = P(C, B)$ *for every C in S, then* $A = B$

CB8** *For any A and B in S, if* $P(C, A) = P(C, B)$ *for every C in S, then* $P(C, A) = P(C, B)$ *for every such C and every relative probability function P of Popper's of Type II defined on S*

ON POPPER'S ABSOLUTE PROBABILITY FUNCTIONS

C1 *For any A in S,* $0 \leq P(A)$ (*Nonnegativity*)

C2 *For any A in S,* $P(A \cup -A) = 1$ (*Normality*)

C3 *For any A and B in S,* $P(A) = P(A \cap B) + P(A \cap -B)$ (*Special Addition*)

C4 *For any A and B in S,* $P(A \cap B) \leq P(B \cap A)$ (*Commutation*)

C5 *For any A, B, and C in S,* $P(A \cap (B \cap C)) \leq P((A \cap B) \cap C)$ (*Association*)

C6 *For any A in S,* $P(A) \leq P(A \cap A)$ (*Idempotence*)

ON CARNAP'S ABSOLUTE PROBABILITY FUNCTIONS

CC7′ *For any A in S, if* $P(A) = 0$, *then* $A = \Lambda$

$\Vert$

CC7 *For any A and B in S, if* $P(A) = P(B) = P(A \cap B)$, *then* $A = B$

CC7** *For any A and B in S, if* $P(A) = P(B) = P(A \cap B)$, *then* $P(A) = P(B)$ *for every absolute probability function P of Popper's of Type II defined on S*

1 Overview † It is usually in connection with properties that one talks of indiscernibility, saying that A and B are *indiscernible* if they have the same properties. Here it is in connection with functions, more specifically, probability functions, that we talk of indiscernibility. For example, given a relative (i.e. binary) probability function P defined on a set S,[1] we say that members A and B of S are

(i) *left-indiscernible under P* if

$$P(A,C) = P(B,C) \textit{ for every } C \textit{ in } S,$$

(ii) *right-indiscernible under P* if

$$P(C,A) = P(C,B) \textit{ for every } C \textit{ in } S,^2$$

(iii) *indiscernible tout court under P* if

A and B are both left- and right-indiscernible under P.

We further say that A and B are *indiscernible tout court* if A and B are indiscernible under *every* relative probability function defined on S.

The relative probability functions considered in this paper are those of Popper in [15], Appendices *iv–*v, plus three special kinds of Popper functions respectively associated with Rényi, Kolmogorov, and Carnap. We show that under all of these functions left-indiscernibility entails right-indiscernibility (Popper's Constraint), and investigate under which of them right-indiscernibility entails left-indiscernibility, under which indiscernibility (right- or left-) entails identity, etc. That these questions arise at all may come as a surprise, to readers of Leibniz especially. And, banking on these results, we obtain for Rényi's, Kolmogorov's, and Carnap's functions novel *characteristic formulas* or *constraints* that are phrased entirely in terms of indiscernibility under P and identity: constraints RB8, KB8, and CB8, respectively.

Rényi's functions in [16] and Carnap's in [4] are defined on sets known as *Boolean algebras*, while Popper's in [15] are defined on *arbitrary sets.* Proceeding as in [12] but more boldly, we initially take all of Popper's, Rényi's, Kolmogorov's, and Carnap's functions to be defined on Boolean algebras, and then take all of them to be defined on arbitrary sets. In the first case constraints are placed upon the sets on which the functions are defined and upon the functions themselves: constraints A1′–A5′ and B1–B5, respectively. Constraints A1′–A5′ are of course familiar postulates for Boolean algebras. In the second case constraints are placed upon the functions only, but there are more of them: constraints B1–B7.

Popper showed in [15] that constraints B1–B7 compel the arbitrary sets on which his functions are defined to be Boolean algebras in a special and weaker sense, i.e. *Boolean algebras with respect to indiscernibility under P* (Popper's Theorem). The same holds true by implication of Rényi's, Kolmogorov's, and Carnap's functions when they too are defined on arbitrary sets. However, we show that in the case of Rényi's functions constraint RB8, and in that of Car-

†The constraints mentioned in this overview are tabulated for convenience's sake on the preceding two pages. They will also be found in the relevant sections of the paper.

nap's functions constraint CB8, compel those supposedly arbitrary sets to be standard Boolean algebras after all. As a result, we end up at this point with (i) six (rather than the expected eight) distinct families of relative probability functions and (ii) two distinct characterizations each of Rényi's functions and Carnap's, one characterization explicitly requiring of the sets on which these functions are defined that they be standard Boolean algebras, the other *not* (Sections 3–5).

We turn next to Popper's absolute (i.e., unary) probability functions in [14]. We take these too to be defined on standard Boolean algebras first, this by means of constraints A1′–A5′ and constraints C1–C3, and on *arbitrary sets* second, this by means of constraints C1–C6. The functions in question, when defined on the Boolean algebras known as *fields of sets*, coincide with Kolmogorov's absolute probability functions in [6]. Only one special kind of Popper function merits attention, Carnap's in [4], for which we provide a characteristic constraint phrased entirely in terms of indiscernibility under P and identity: constraint CC7. Exploiting a result in [10], we first show that when Popper's functions and Carnap's are defined on arbitrary sets, Popper's constraints C1–C6 compel the sets in question to be Boolean algebras with respect to indiscernibility under P. We further show that in the case of Carnap's functions constraint CC7 compels those supposedly arbitrary sets to be standard Boolean algebras after all. So, we end up at this point with (iii) three (rather than the expected four) distinct families of absolute probability functions and (iv) two distinct characterizations of Carnap's functions, one explicitly requiring of the sets on which the functions are defined that they be standard Boolean algebras, the other *not* (Section 6).

Indiscernibility under a probability function, it turns out, is a matter of symmetric difference. Readers will recall the *symmetric difference* $A \dot{-} B$ of members A and B of a set S as

$$(A \cap -B) \cup (B \cap -A),$$

hence, when A and B are themselves sets, as the set consisting of every A not a B and every B not an A. We establish that *left*- and *right*-indiscernibility under a relative probability function, and indiscernibility *tout court* under a relative or an absolute one, can be rendered in terms of $\dot{-}$. The renditions, together with this theorem of Boolean algebra:

$$A \dot{-} B = \Lambda \textit{ if, and only if, } A = B,$$

will deliver constraints RB8, KB8, CB8, and CC7, i.e. *our* characterizations of Rényi's, Kolmogorov's, and Carnap's probability functions. They will also show Popper's Constraint (left-indiscernibility under P entails right-indiscernibility under P) to be a truth of quantifier logic (Sections 4 and 6).

Finally we obtain results which in effect round off Popper's probability theory. Strengthening what we call Popper's Theorem and its counterpart in Section 6 for absolute probability functions, we show that relative probability functions meeting constraints B1–B7 or absolute ones meeting constraints C1–C6 are definable on a set S if, and only if, there exists an equivalence relation on S with respect to which S constitutes a nondegenerate Boolean algebra. So probability functions, definable on *just* the sets that are nondegenerate Boolean algebras, are yet definable on *all* on them. Using the notion of *the least equiva-*

lence relation with respect to which a set constitutes a Boolean algebra, we also isolate—and briefly consider—three new families of probability functions: two of them "the lost families" of relative functions in Section 5, if you will, and the third "the lost family" of absolute ones in Section 6. Their characteristic constraints—RB8**, CB8**, and CC7**—are indeed akin to RB8, CB8, and CC7, but weaker; and they talk of indiscernibility *tout court* where RB8, CB8, and CC7 talked of identity. The functions are of particular interest when defined on sets of statements (Section 7).

2 Boolean algebras Let S be a nonempty set closed under a unary function $-$ and a binary one $\cap$,[3] and let $\approx$ be an equivalence relation (i.e. a reflexive, symmetrical, and transitive relation) on S. We say that

(i) S constitutes a *Boolean algebra with respect to* $\approx$ if these five constraints are met:

A1 *For any A and B in S, $A \cap B \approx B \cap A$* (*Commutation*)
A2 *For any A, B, and C in S, $A \cap (B \cap C) \approx (A \cap B) \cap C$* (*Association*)
A3 *For any A, B, and C in S, $A \cap -B \approx C \cap -C$ if, and only if, $A \cap B \approx A$* (*Special Complementation*)
A4 *For any A and B in S, if $A \approx B$, then $-A \approx -B$*
A5 *For any A, B, and C in S, if $A \approx B$, then $A \cap C \approx B \cap C$.*[4]

We also say that

(ii) S constitutes in particular a *nondegenerate Boolean algebra with respect to* $\approx$ if $A \not\approx B$ for some A and B in S.

And, with S as before, we say that S constitutes a (*nondegenerate*) *Boolean algebra tout court* if there is an equivalence relation with respect to which S constitutes a (nondegenerate) Boolean algebra.[5] The best known, and in all too many quarters the *only* known, Boolean algebras are those *with respect to identity*. As in Section 1, we shall often refer to them as *standard* Boolean algebras. But there are other Boolean algebras: in particular, the Boolean algebras *with respect to indiscernibility under a probability function P* which were mentioned in Section 1 and which are to play a major role in this paper. They first appeared in [15], where Popper talked of *substitutional equivalence* rather than indiscernibility. So there will be *identity* versions of A1–A5, and there will be *indiscernibility* ones. The former, the results of replacing '$\approx$' everywhere in A1–A5 by '$=$', are the constraints A1′–A5′ mentioned in Section I.[6]

A Boolean algebra S often consists of sets, but it may also consist of relations, propositions, individuals, etc. When infinite, S may be of *any* infinite cardinality. But, when S is finite and a standard algebra, S may be of *only* these cardinalities: 1, 2, 4, 16, . . . , i.e. 2^n for some n equal to or larger than 0.[7] As for the two functions $-$ and $\cap$, they are usually referred to as *complementation* and *intersection*, and they will be here. When S consists of sets or of relations, they may be interpreted set-theoretically; but they are open to other interpretations as well. For example, when S consists of propositions, $-$ is normally interpreted as the negation function and $\cap$ as the conjunction one. Importantly, when S consists of sets, $-$ and $\cap$ are interpreted set-theoretically, and $\approx$ is taken to be $=$, S is called a *field of sets.* Rényi's relative probability functions in [16] and Carnap's in [4] were defined on fields of sets; and so were Kolmogorov's absolute probability functions, as noted in Section 1.

Two additional functions on S turned up in Section 1: the union function $\cup$ and the symmetric difference function $\dot{-}$. We define the first thus:

$$A \cup B =_{def} -(-A \cap -B).$$

This done, we may *now* formally define the second thus:

$$A \dot{-} B =_{def} (A \cap -B) \cup (B \cap -A),$$

the *definiens* being short of course for $-(-(A \cap -B) \cap -(B \cap -A))$. Two particular members of any Boolean algebra also turned up in Section 1: the *null* or *zero element* Λ and its complement, the *universal* or *one element* V. We define Λ thus:

$$\Lambda =_{def} A \cap -A,$$

A here some arbitrary but fixed member of S; and we then define V thus:

$$\mathrm{V} =_{def} -\Lambda.$$

That A is a particular member of S is no limitation: by Lemma 1(b) in the Appendix we nonetheless have $\Lambda \approx A \cap -A$ and $\mathrm{V} \approx A \cup -A$ for *any* A in S, so that Λ is simply an arbitrary one of the $\approx$-equivalent elements $A \cap -A$, $B \cap -B$, etc. Hence, when $\approx$ is $=$, $\Lambda = A \cap -A$ and $\mathrm{V} = A \cup -A$ for *any* such A.

On several occasions we shall *reduce* certain Boolean algebras to others, more specifically, to *standard* ones. On each occasion a certain equivalence relation (an indiscernibility one, it so happens) will have been defined on the original algebras. Let S be one of the algebras; let $\approx$ be the equivalence relation defined on it; for each A in S, let $[A]$ — the so-called *equivalence set of A* — be the set consisting of every B in S such that $B \approx A$; let $[S]$ be the set consisting of the equivalence sets that result; for each $[A]$ in $[S]$, let $-[A]$ be the set consisting of every B in S such that $B \approx -A$; and for each $[A]$ and $[B]$ in S, let $[A] \cap [B]$ be the set consisting of every C in S such that $C \approx A \cap B$. It is obvious that

$$[A] = [B]$$

is tantamount to

$$A \approx B.$$

And, in consequence, it is easily verified that $[S]$ constitutes a standard Boolean algebra, i.e. a Boolean algebra with respect to identity. We shall call $[S]$ *the reduction of S with respect to $\approx$*.[8] $[S]$ is sure to be a nondegenerate algebra if S is.

The various equivalence relations on a set S with respect to which it constitutes a Boolean algebra are of course so many sets of pairs of members of S, and so is their intersection, the set consisting of the pairs of members of S that belong to *all* the equivalence relations with respect to which S constitutes a Boolean algebra. Significantly for our purposes in Section 7, that intersection proves to be one of the equivalence relations on S with respect to which S satisfies A1–A5; and, being by definition a subset of each of them, it is known as *the least* *equivalence relation on S with respect to which S constitutes a Boolean algebra.* For convenience's sake we shall refer to it by means of '$\approx^*$'. When S constitutes a *nondegenerate* Boolean algebra, $\approx^*$ proves to be one equivalence relation on S with respect to which S constitutes such an algebra. And, when S constitutes a *standard* one, i.e. a Boolean algebra *with respect to* $=$, $\approx^*$ proves to be $=$. Note indeed that: (i) by definition every pair of members of S of the sort $\langle A, A\rangle$ belongs to every equivalence relation $\approx$ on S, and (ii) no other pair belongs to $=$.

3 The relative probability functions of Type I For convenience's sake we call the probability functions that are defined on standard Boolean algebras *functions of Type I*, those that are defined on arbitrary sets *functions of Type II.* We study the relative ones of Type I in this section, the next, and Section 7, those of Type II in Sections 5 and 7. And, as noted in Section 1, we consider in each case four families of relative functions: Popper's, Rényi's, Kolmogorov's, and Carnap's.

Specifically, by a *relative probability function of Popper's of Type I* we understand any binary real-valued function P defined on a standard Boolean algebra S and meeting these five constraints, adaptations of constraints of Popper's in [15]:

B1 *For some A and B in S, $P(A,B) \neq 1$* (*Existence*)
B2 *For any A and B in S, $0 \leq P(A,B)$* (*Nonnegativity*)
B3 *For any A in S, $P(A,A) = 1$* (*Normality*)
B4 *For any A and B in S, if $P(C,B) \neq 1$ for some C in S, then $P(A,B) + P(-A,B) = 1$* (*Addition*)
B5 *For any A, B, and C in S, $P(A \cap B, C) = P(A, B \cap C) \times P(B,C)$* (*Multiplication*)[9]

And by a *relative probability function of Rényi's, Kolmogorov's*, or *Carnap's of Type I* we understand any relative probability function of Popper's of that type that meets this extra constraint in Rényi's case:

RB8′ *For any B in S, if $P(A,B) = 1$ for every A in S, then $B = \Lambda$,*

this one in Kolmogorov's case:

KB8′ *For any B in S, if $P(B,\mathrm{V}) = 0$, then $P(A,B) = 1$ for every A in S,*

and this one in Carnap's case:

CB8′ *For any A in S, if $P(A,\mathrm{V}) = 0$, then $A = \Lambda$.*

Notes: (a) Unlike most writers we left room in Section 2 for degenerate Boolean algebras. However, constraints B1 and B3 compel the set S on which a relative probability function P is defined to have at least two members *left*-discernible from each other under P—hence, *at least two members distinct from each other.* Two such members are of course V and Λ; and, because of that, any relative probability function P is sure to have at least two distinct values: 0 and 1.

(b) Of steps in a proof taken by this consequence of A1′

B6 *For any A, B, and C in S, $P(A \cap B, C) \leq P(B \cap A, C)$,* or this one

B7 *For any A, B, and C in S*, $P(A, B \cap C) \leq P(A, C \cap B)$,

we shall say that they are taken *by Commutation*. B6–B7 are the two extra constraints placed on P when the function is of Type II. Any result obtained by B1–B5 and *Commutation* will thus automatically hold of Popper's relative probability functions of Type II.[10] Of the few steps taken by other consequences of A1′–A5′ we shall say that they are taken *by BA*.

(c) Assembled under Lemmas 2 and 3 in the Appendix are various facts about binary real-valued functions that meet constraints B1–B7, hence about the relative probability functions in this section and the next. We shall invoke the two lemmas repeatedly.

(d) RB8′, KB8′, and CB8′ are the characteristic constraints for Rényi's, Kolmogorov's, and Carnap's functions that we used in [12]; the point of them is to be explained shortly. The alternative constraints RB8, KB8, and CB8 that we promised in Section 1 will be introduced in Section 4.

(e) It is common practice to say of a member B of S that it is *P-normal* if $P(A,B) \neq 1$ for at least one A in S, *P-abnormal* otherwise; and we shall occasionally say that it is of *P_V-probability zero* if $P(B, V) = 0$. So, according to B1, at least one member of S is P-normal, one such member being V by Lemma 2(k); according to B4, $P(-A,B) = 1 - P(A,B)$ if B is P-normal;[11] according to RB8′, $B = \Lambda$ if B is P-abnormal; according to KB8′, B is P-abnormal if of P_V-probability zero, and, according to CB8′, $A = \Lambda$ if A is of P_V-probability zero.

In a way it is to division by zero not being allowed that we owe the diversity of relative probability functions on p. 7. In early texts, e.g. Kolmogorov's 1933 paper [6], absolute probability functions had pride of place, and relative ones were defined in terms of them. Given an absolute probability function P' of the sort found in Section 6, Kolmogorov defined a relative one thus: A here an arbitrary member of the set S on which P' is defined and B one such that $P'(B) \neq 0$:

$$P(A,B) = P'(A \cap B)/P'(B).$$

The relative probability functions that issued deserve study, to be sure, but they are simply absolute probability functions gotten up as relative ones. And they are *partial* functions, a bother in practice and a shortcoming in theory.

However, Kolmogorov's partial functions can be extended to *total* ones, and the functions that result are the relative probability functions of p. 7. Suppose first that $P'(B) = 0$ for only one B. Then that B has to be Λ, P' has to be what we called in Section 1 a Carnap function, and because of Lemma 3(a) we have just one choice: setting $P(A,B)$ at 1. The resulting functions, meeting CB8′, are the relative probability functions we name after Carnap (Case 1). Suppose, on the other hand, that $P'(B) = 0$ for more than one B. Then one has several choices. One may set $P(A,B)$ at 1 for every such B: the resulting functions, meeting KB8′, are the relative probability functions we name (for a reason given on p. 10) after Kolmogorov (Case 2). One may also set $P(A,B)$ at 1 for B identical with Λ, but let it vary with A (subject of course to constraints A1′–A5′ and

B1–B5) for every other B such that $P'(B) = 0$: the resulting functions, meeting constraint RB8′, are the relative probability functions we name after Rényi (Case 3). And, where there are at least three B's such that $P'(B) = 0$, one may set $P(A,B)$ at 1 for B identical with Λ and for a second B such that $P'(B) = 0$, let it vary with A for a third B such that $P'(B) = 0$, and set it at 1 or let it vary with A for any other B such that $P'(B) = 0$: the resulting functions are relative probability functions that are neither Kolmogorov functions nor Rényi ones (Case 4).[12]

To us, of most interest among Popper's relative probability functions of Type I are those falling under Cases 3–4. Carnap's absolute probability functions match the functions under Case 1 one-to-one, and the rest of Popper's absolute probability functions match those under Case 2 one-to-one again. But the latter absolute probability functions, i.e. those that are *not* Carnap ones, do *not* match the functions under Case 3 or those under Case 4 one-to-one. P' above, when *not* a Carnap function, generates at least $2^{\aleph_0}$ functions that fall under Case 3 and an equal number that fall under Case 4. It is owing to this that relative probability theory outstrips absolute probability theory.

The relationships between our various families of relative probability functions are studied in [12] and can be portrayed as in Figure 1, which we borrow

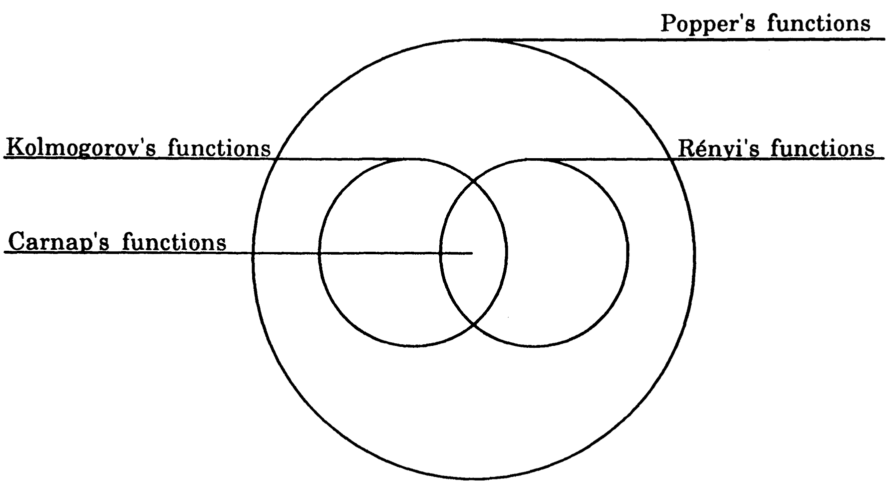

Figure 1.

from [12]. Those among Popper's functions that are neither Kolmogorov nor Rényi functions lie of course outside the two inner circles; Carnap's functions lie, as indicated, at their intersection.

A word concerning our nomenclature is in order. The relative probability functions we name after Popper are *his* in [15] when S there happens to be a standard Boolean algebra. Those we name after Rényi are extensions of *some of his* in [16]. For generality's sake, Rényi defined P there on the cartesian product

$S \times S'$ of a field of sets S and some subset or other S' of $S - \{\Lambda\}$.[13] A closer fit with the present Rényi functions can be had by thinking of S in [16] as a standard Boolean algebra rather than a field of sets and thinking of S' there as a nonempty subset of S. Rényi's own functions would then be the functions we name after him *plus* the results of keeping one or more members of S from serving as B in $P(A,B)$. The relative probability functions we name after Carnap are straightforward extensions of *his* in [4]: we allow Λ to serve as B in $P(A,B)$.[14] Consistency is preserved in either case since Λ is P-abnormal. The relative probability functions we name after Kolmogorov are *not* his. However, with S a standard Boolean algebra rather than a field of sets, they are—we just saw—the *only* extensions of Kolmogorov's relative probability functions in [6] that match one-to-one his absolute ones. So the appellation is not totally inappropriate. The functions in question will play a fascinating role here.

The constraints that Popper placed in [15] on his functions were, *in effect*, B1, B3–B5, and these two:

(*) *For any A, B, and C in S, $P(A \cap B, C) \leq P(A,C)$*

(**) *For any A, B, and D in S, if $P(A,C) = P(B,C)$ for every C in S, then $P(D,A) = P(D,B)$.*

(**) is the constraint we referred to on p. 3 as Popper's Constraint. It is the earliest constraint we know of to address itself to questions of (left- and right-) indiscernibility under a relative probability function. Popper shows in [15] that each of *his* six constraints is independent of the other five. Similarly with us, B1–B5 deliver (*) only with an assist from B6, and they deliver (**) only with an assist from B6–B7. But, unlike Popper who defined his functions on arbitrary sets rather than standard Boolean algebras and who for reasons discussed in [15] and [9] shunned B6–B7 as extra constraints, we have A1′, hence *Commutation*, and hence a proof of (**).[15]

Most of the theorems in this paper feature a probability function P and a set S. S is the set on which P is defined. In this section and the next P is presumed of course to be a Popper function of Type I. However, all the theorems in this section and all but two in the next are proved using just B1–B5 and *Commutation*, and hence hold for functions of Type II as well. The two exceptions in Section 4 will be marked, and the steps taken by BA in each case will stand out.

Theorem 3.1 *If $P(A,C) = P(B,C)$ for every C in S, then $P(C,A) = P(C,B)$ for every such C.*

Proof: Suppose that $P(A,C) = P(B,C)$ for every C in S, and let C be an arbitrary member of S. Then

1.	$P(C,A) = P(A,C \cap A) \times P(C,A)$	(Lemma 2(c))
2.	$= P(B,C \cap A) \times P(C,A)$	(1, Hyp. on P)
3.	$= P(B \cap C,A)$	(2, B5)
4.	$= P(C \cap B,A)$	(3, *Commutation*)
5.	$= P(C,B \cap A) \times P(B,A)$	(4, B5)
6.	$= P(C,B \cap A) \times P(A,A)$	(5, Hyp. on P)
7.	$= P(C,B \cap A)$	(6, B3).

But

$$P(C,B) = P(C,A \cap B)$$

by simply interchanging A and B throughout 1–7. Hence

$$P(C,A) = P(C,B)$$

by *Commutation.*

So, for *every* relative probability function P of Popper's of Type I, *left-indiscernibility under P entails right-indiscernibility under P, and hence is tantamount to indiscernibility tout court under P.* Further, members A and B of S thus prove to be indiscernible *tout court* if $P(A,C) = P(B,C)$ for every C in S and every relative probability function P of Popper's of Type II defined on S. The converse of Theorem 3.1 does not hold true, as we shall establish in Section 4.

We next show that *each of* $P(A,B) = P(B,A) = 1$ *and* $P(A \cap B, A \cup B) = 1$ *also entails right-indiscernibility under P.* The two results will prove handy. The first of them is listed in [15] as an alternative to (**); proof of it, like proof of (**), uses *Commutation.* We also show that A is of P_V-probability 0 if, and only if, $-A$ is right-indiscernible under P from V.

Theorem 3.2
(a) *If* $P(A,B) = P(B,A) = 1$, *then* $P(C,A) = P(C,B)$ *for every* C *in* S;
(b) *If* $P(A \cap B, A \cup B) = 1$, *then* $P(C,A) = P(C,B)$ *for every* C *in* S;
(c) $P(A,\mathrm{V}) = 0$, *if, and only if,* $P(C,-A) = P(C,\mathrm{V})$ *for every* C *in* S.

Proof: (a) Suppose that $P(A,B) = P(B,A) = 1$, and suppose that A is P-abnormal. Then $P(C \cap B,A) = P(B,A) = P(C,A) = 1$ by definition, and hence $P(C,B \cap A) = P(C,A)$ by B5. Suppose on the other hand that A is P-normal, in which case $P(-B,A) = 0$ by B4 and the hypothesis on P. Then $P(C \cap -B,A) = 0$ by Lemma 2(n), and hence $P(C \cap B,A) = P(C,A)$ by Lemma 2(f). But $P(C \cap B,A) = P(C,B \cap A)$ by B5 and the hypothesis on P. Hence $P(C,B \cap A) = P(C,A)$ again. But $P(C,A \cap B) = P(C,B)$ by simply interchanging A and B throughout. Hence $P(C,A) = P(C,B)$ by *Commutation.* (b) Suppose that $P(A \cap B, A \cup B) = 1$. Then $P(A,B \cap (A \cup B))P(B,A \cup B) = 1$ by B5, hence $P(A,B \cap (A \cup B)) = 1$ by Lemma 2(b), and hence $P(A,B) = 1$ by Lemma 3(e) and Theorem 3.1. But, if $P(A \cap B,A \cup B) = 1$, then $P(B \cap A,A \cup B) = 1$ by *Commutation.* Hence $P(B,A) = 1$ by the very same reasoning. Hence by (a) $P(C,A) = P(C,B)$ for every C in S. (c) Suppose *first* that $P(A,\mathrm{V}) = 0$. Then $P(-A,\mathrm{V}) = 1$ by B4 and Lemma 2(k). But $P(\mathrm{V},-A) = 1$ by Lemma 2(i). So, $P(C,-A) = P(C,\mathrm{V})$ for every C in S by (a). Suppose *next* that $P(C,-A) = P(C,\mathrm{V})$ for every C in S. Then $-A$ is P-normal by Lemma 2(k), hence $P(A,-A) = 0$ by B4 and B3, and hence $P(A,\mathrm{V}) = 0$.

Then we show that if A and B are left-indiscernible (and, hence, indiscernible *tout court*) under P, (a) so are $-A$ and $-B$, *the indiscernibility version of* A4, (b) so are $A \cap D$ and $B \cap D$, *the indiscernibility version of* A5, and hence (c) so are $D \cap A$ and $D \cap B$.

Theorem 3.3
(a) *If* $P(A,C) = P(B,C)$ *for every* C *in* S, *then* $P(-A,C) = P(-B,C)$ *for every such* C;
(b) *If* $P(A,C) = P(B,C)$ *for every* C *in* S, *then* $P(A \cap D,C) = P(B \cap D,C)$ *for every* C *and* D *in* S;
(c) *If* $P(A,C) = P(B,C)$ *for every* C *in* S, *then* $P(D \cap A,C) = P(D \cap B,C)$ *for every* C *and* D *in* S.

Proof: (a) By B4. (b) Suppose that $P(A,C) = P(B,C)$ for every C in S and let C and D be arbitrary members of S. Then $P(A,D \cap C) \times P(D,C) = P(B,D \cap C) \times P(D,C)$, and hence by B5 $P(A \cap D,C) = P(B \cap D,C)$, this for every C in S. (c) By (b) and *Commutation.*

The results in Theorem 3.3 readily generalize. As the reader well knows, the unary Boolean function $-$ and the binary one $\cap$ permit definition of every Boolean function on S. Consequently,

$$P(A,C) = P(B,C) \textit{ for every } C \textit{ in } S$$

entails

(1) $$P(f(A),C) = P(f(B),C) \textit{ for every } C \textit{ in } S,$$

no matter the *unary* Boolean function f on S, and

(2) $$P(f(A,D),C) = P(f(B,D),C) \textit{ and } P(f(D,A),C) = P(f(D,B),C) \\ \textit{for every } D \textit{ and every } C \textit{ in } S,$$

no matter the *binary* Boolean function f on S, etc. Hence, thanks to Popper's Constraint:

Theorem 3.4 *Equivalent to*

$$P(A,C) = P(B,C) \textit{ for every } C \textit{ in } S,$$

is each of these:
(a) $P(f(A),C) = P(f(B),C)$ *and* $P(C,f(A)) = P(C,f(B))$ *for every Boolean function* f *on* S *and every* C *in* S, *and*
(b) $P(f(A),g(A)) = P(f(B),g(B))$ *for every Boolean function* f *and every Boolean function* g *on* S.

The two clauses in Theorem 3.4 bring out the full force of

Member A and member B of S are indiscernible
under relative probability function P.

And the counterpart in Section 6 of clause (b), to wit:

$$P(f(A)) = P(f(B)) \textit{ for every Boolean function } f \textit{ on } S,$$

will point to the appropriate rendition there of

Member A and member B of S are indiscernible
under absolute probability function P.[16]

Our last two theorems in this section, though not explicitly dealing with indiscernibility, are lemmas to theorems in Section 4 that do. Theorem 3.5 brings together a number of equivalent formulations of P-abnormality.

Theorem 3.5 *Equivalent to*

B is P-abnormal

is each of these

(a) $P(A,B) = P(A,\Lambda)$ *for every A in S,*
(b) $P(B,A) = P(\Lambda,A)$ *for every A in S,*
(c) $P(-B,A) = P(\mathrm{V},A)$ *for every A in S,*
(d) $P(-B,A) = 1$ *for every A in S, and*
(e) $P(-B,B) = 1$.

Proof: (i) By Lemma 3(a) B is P-abnormal if, and only if, (a). (ii) Suppose that B is P-abnormal, and let A be an arbitrary member of S. Then $P(-B \cap A, B) = P(A,B) = 1$ by definition, hence $P(-B, A \cap B) = 1$ by B5, hence $P(-B, B \cap A) = 1$ by *Commutation*, hence $P(B,A) = P(-B \cap B, A)$ by B5 again, and hence (b) by the definition of Λ. (iii) (b) entails (c) by Theorem 3.3(a) and the definition of V. (iv) (c) entails (d) by Lemma 2(i). (v) (d) entails (e). (vi) (e) entails that B is P-abnormal by B3–B4.

So, P-abnormality is equivalent to *left*-indiscernibility under P from Λ and to *right*-indiscernibility under P from Λ.

Theorem 3.6 *If B is P-abnormal, then* $P(B,\mathrm{V}) = 0$.

Proof: Suppose $P(A,B) = 1$ for every A in S. Then $P(-B,\mathrm{V}) = 1$ by Theorem 3.5(d), and hence $P(B,\mathrm{V}) = 0$ by B4 and Lemma 2(k).

So, if B is P-abnormal, then in the idiom of p. 8 B is of P_{V}-probability 0. So, when P is a Kolmogorov function, B is P-abnormal if, and only if, it is of P_{V}-probability 0.

4 The relative probability functions of Type I (concluded) We provide in Theorem 4.1 renditions of "A and B are *left*-indiscernible under P", and in Theorem 4.2 renditions of "A and B are *right*-indiscernible under P", in terms of $\dot{-}$. Some of the renditions take after $-(A \dot{-} B) = \mathrm{V}$, others after $A \dot{-} B = \Lambda$. Those in Theorem 4.2 may come as a surprise. We also provide in Theorem 4.4 renditions of "A and B or their complements in S are *right*-indiscernible under P" in terms of $\dot{-}$. The renditions in Theorems 4.1–4.2 pave the way for constraints RB8, KB8, and CB8; those in Theorem 4.4 will deliver our last result in Section 6.

Theorem 4.1 *Equivalent to*

$P(A,C) = P(B,C)$ *for every C in S*

is each of these:
(a) $P(-(A \dot{-} B),C) = 1$ *for every C in S,*
(b) $P(-(A \dot{-} B),C) = P(\mathrm{V},C)$ *for every C in S,*

(c) $P(A \dot{-} B, C) = P(\Lambda, C)$ *for every* C *in* S,
(d) $P(C, A \dot{-} B) = 1$ *for every* C *in* S, *and*
(e) $P(C, A \dot{-} B) = P(C, \Lambda)$ *for every* C *in* S.

Proof: (a) Suppose *first* that $P(A,C) = P(B,C)$ for every C in S, and let C be P-normal. Then $P(A \cap -B, C) = P(B \cap -A, C) = 0$ by Theorem 3.3(b) and Lemma 2(h), and hence $P(-(A \dot{-} B), C) = 1$ by Lemma 3(q) and B4. But $P(-(A \dot{-} B), C) = 1$ by definition when C is P-abnormal. So $P(-(A \dot{-} B), C) = 1$ for every C in S. Suppose *next* that $P(-(A \dot{-} B), C) = 1$ for every C in S, and let C be P-normal. Then $P(A \cap -B, C) = P(B \cap -A, C) = 0$ by B4 and Lemma 3(q) again, hence $P(A,C) = P(A \cap B, C)$ and $P(B,C) = P(B \cap A, C)$ by Lemma 2(f), and hence $P(A,C) = P(B,C)$ by *Commutation*. But $P(A,C) = P(B,C)$ by definition when C is P-abnormal. So $P(A,C) = P(B,C)$ for every C in S. (b) By (a) and Theorem 3.5(c). (c) By (a) and Theorem 3.5(b). (d) By (a) and Theorem 3.5(d). (e) By (a) and Theorem 3.5(a).

Note that in view of rendition (d) A and B are *left*-indiscernible under P just when $A \dot{-} B$ is P-abnormal.

Theorem 4.2 *Equivalent to*

$$P(C,A) = P(C,B) \textit{ for every } C \textit{ in } S$$

is each of these:
(a) $P(-(A \dot{-} B), A \cup B) = 1$,
(b) $P(-(A \dot{-} B), A \cup B) = P(\mathrm{V}, A \cup B)$, *and*
(c) $P(A \dot{-} B, A \cup B) = P(\Lambda, A \cup B)$.

Proof: (a) Suppose *first* that $P(C,A) = P(C,B)$ for every C in S, and let A be P-normal, in which case so is $A \cup B$ by Lemma 3(s). Then $P(A \dot{-} B, A \cup B) = P(-B \cap A, A \cup B) + P(-A \cap B, A \cup B)$ by Lemma 3(p) and *Commutation*, hence $P(A \dot{-} B, A \cup B) = P(-B, A \cap (A \cup B)) \times P(A, A \cup B) + P(-A, B \cap (A \cup B)) \times P(A, A \cup B)$ by B5, and hence $P(A \dot{-} B, A \cup B) = P(-B, A) \times P(A, A \cup B) + P(-A, B) \times P(B, A \cup B)$ by Lemmas 3(d)–(e) and Theorem 3.1. But $P(-A, B) = P(-B, A) = 0$ by Lemma 3(t), the hypothesis on P, and that on A. Hence $P(A \dot{-} B, A \cup B) = 0$, and hence $P(-(A \dot{-} B), A \cup B) = 1$ by B4. Let A on the other hand be P-abnormal, in which case so is B by the hypothesis on P. Then $P(-A, C) = P(-B, C) = 1$ for every such C by Theorem 3.5(d), hence $A \cup B$ is P-abnormal by Lemma 3(u), and hence $P(-(A \dot{-} B), A \cup B) = 1$ by definition. So $P(-(A \dot{-} B), A \cup B) = 1$ whether or not A is P-normal. Suppose *next* that $P(-(A \dot{-} B), A \cup B) = 1$, and let $A \cup B$ be P-normal. Then $P(A \cap B, A \cup B) + P(-A \cap -B, A \cup B) = 1$ by Lemmas 3(g) and 3(r). But $P(-A \cap -B, A \cup B) = 0$ by B4 and the definition of $A \cup B$. So $P(A \cap B, A \cup B) = 1$ by B4, and hence $P(C,A) = P(C,B)$ for every C in S by Theorem 3.2(b). Let $A \cup B$ on the other hand be P-abnormal. Then so are A and B by Lemma 3(s), and hence $P(C,A) = P(C,B)$ for every C in S by definition. So $P(C,A) = P(C,B)$ for every C in S whether or not $A \cup B$ is P-normal. (b) By (a) and Lemma 2(i). (c) By (b), B4, and the definition of V.

Note that, $A \cup B$ being a member of S,

If $P(-(A \dot{-} B), C) = 1$ *for every* C *in* S, *then* $P(-(A \dot{-} B), A \cup B) = 1$

is a truth of quantifier logic. But, owing to Theorem 4.1, the antecedent of the conditional is equivalent to

$$P(A,C) = P(B,C) \text{ for every } C \text{ in } S,$$

and owing to Theorem 4.2 its consequent is equivalent to

$$P(C,A) = P(C,B) \text{ for every } C \text{ in } S.$$

So, as remarked in Section 1, Popper's Constraint is, in the context of B1–B7, a truth of quantifier logic. Its converse, however, is not. So *right*-indiscernibility under P can entail *left*-indiscernibility under P for only certain choices of P. We shall identify them shortly.

The following theorem serves as a lemma to Theorems 4.4, 4.6, and 4.7.

Theorem 4.3.
(a) *If* $P(C,A) = P(C,B)$ *for every* C *in* S, *then* $P(A \dot{-} B,\mathrm{V}) = 0$;
(b) *If* $P(A,\mathrm{V}) \neq 0$, *then* $P(C,A) = P(C,B)$ *for every* C *in* S *if, and only if,* $P(A \dot{-} B,\mathrm{V}) = 0$.

Proof: Suppose that $P(C,A) = P(C,B)$ for every C in S, in which case $P(A,B) = P(B,A) = 1$ by B3; and let A *first* be P-normal. Then $P(-B,A \cap \mathrm{V}) = 0$ by B4 and Lemma 2(j), and hence $P(A \cap -B,\mathrm{V}) = 0$ by B5 and *Commutation*. Let A *next* be P-abnormal. Then $P(A,\mathrm{V}) = 0$ by Theorem 3.6, and hence $P(A \cap -B,\mathrm{V}) = 0$ by Lemma 2(n). So $P(A \cap -B,\mathrm{V}) = 0$ whether or not A is P-normal. But $P(B \cap -A,\mathrm{V}) = 0$ by merely interchanging A and B throughout. So $P(A \dot{-} B,\mathrm{V}) = 0$ by Lemma 3(q). (b) Suppose that $P(A,\mathrm{V}) \neq 0$ and $P(A \dot{-} B,\mathrm{V}) = 0$. $P((A \dot{-} B) \cap (A \cup B),\mathrm{V}) = 0$ by the second hypothesis and Lemma 2(n), and hence $P(A \dot{-} B,A \cup B) \times P(A \cup B,\mathrm{V}) = 0$ by B5 and Lemma 2(j). But $P(A \cup B,\mathrm{V}) \neq 0$ by the first hypothesis, Lemma 3(o), and B2. So, $P(A \dot{-} B,A \cup B) = 0$, so $P(-(A \dot{-} B),A \cup B) = 1$ by B4, and hence $P(C,A) = P(C,B)$ for every C in S by Theorem 4.2(a). So, (b) by (a).

Now our renditions of "A and B or their complements are right-indiscernible under P":

Theorem 4.4 *Equivalent to*

$$P(C,A) = P(C,B) \text{ for every } C \text{ in } S \text{ or}$$
$$P(C,-A) = P(C,-B) \text{ for every such } C$$

is each of these:
(a) $P(A \dot{-} B,\mathrm{V}) = 0$, *and*
(b) $P(C,-(A \dot{-} B)) = P(C,\mathrm{V})$ *for every* C *in* S.

Proof: (a) Suppose *first* that $P(C,A) = P(C,B)$ for every C in S or $P(C,-A) = P(C,-B)$ for every C in S. If $P(C,A) = P(C,B)$ for every C in S, then $P(A \dot{-} B,\mathrm{V}) = 0$ by Theorem 4.3(a). So, suppose $P(C,-A) = P(C,-B)$ for every C in S. Then $P(-A \dot{-} -B,\mathrm{V}) = 0$ by Theorem 4.3(a), and hence $P(A \dot{-} B,\mathrm{V}) = 0$ by Lemma 3(f). Suppose *next* that $P(A \dot{-} B,\mathrm{V}) = 0$. Since either $P(A,\mathrm{V}) \neq 0$ or $P(-A,\mathrm{V}) \neq 0$ by B4 and Lemma 2(k), either $P(C,A) = P(C,B)$ for every C in S or $P(C,-A) = P(C,-B)$ for every such C by Theorem 4.3(b). (b) By (a) and Theorem 3.2(c).

So, A and B themselves or their complements are *right*-indiscernible under P just when $A \dot{-} B$ is of P_V-probability 0.

Theorems 4.1–4.4, proved using just B1–B5 and *Commutation*, hold whether P is of Type I or of Type II. So does Theorem 4.6 below. In contrast, Theorems 4.5 and 4.7 are the two that hold only of functions of Type I. A reminder of that is the '(I)' attached to the call number of each.

It is when P is a relative probability function of Rényi's that *left*-indiscernibility (hence, indiscernibility *tout court*) under P entails identity. Equivalent indeed to

RB8′ *For any B in S, if $P(A,B) = 1$ for every A in S, then $B = \Lambda$,*

is

RB8 *For any A and B in S, if $P(A,C) = P(B,C)$ for every C in S, then $A = B$,*

the constraint we claimed in Section 1 to be characteristic of Rényi's functions.

Theorem 4.5(I) *P meets constraint* RB8′ *if, and only if, it meets constraint* RB8.

Proof: Suppose *first* that P meets RB8′, and let $P(A,C) = P(B,C)$ for every C in S. Then $P(C,A \dot{-} B) = 1$ for every such C by Theorem 4.1(d), hence $A \dot{-} B = \Lambda$ by RB8′, and hence $A = B$ by BA. So P meets RB8. Suppose *next* that P meets RB8, and let $P(A,B) = 1$ for every A in S. Then $P(A,-B \dot{-} V) = 1$ for every such A by Lemma 3(h) and Theorem 3.1, hence $P(-B,A) = P(V,A)$ for every such A by Theorem 4.1(d), hence $-B = V$ by RB8, and hence $B = \Lambda$ by BA. So P meets RB8′.

So, *Rényi's relative probability functions of Type I are those, and those only, among Popper's functions of that Type that meet constraint* RB8.[17]

It is when P is a relative probability function of Kolmogorov's—*and, contrary perhaps to expectation, only then*—that *right*-indiscernibility under P entails *left*-indiscernibility (and, hence, indiscernibility *tout court*) under P. Equivalent indeed to

KB8′ *For any B in S, if $P(B,V) = 0$, then $P(A,B) = 1$ for every A in S,*

is

KB8 *For any A and B in S, if $P(C,A) = P(C,B)$ for every C in S, then $P(A,C) = P(B,C)$ for every such C,*

the constraint we claimed in Section 1 to be characteristic of Kolmogorov's functions.

Theorem 4.6 *P meets constraint* KB8′ *if, and only if, it meets constraint* KB8.

Proof: Suppose *first* that P meets KB8′, and let $P(C,A) = P(C,B)$ for every C in S. Then $P(A \dot{-} B,V) = 0$ by Theorem 4.3(a), hence $P(C,A \dot{-} B) = 1$ for every C in S by KB8′, and hence $P(A,C) = P(B,C)$ for every such C by Theorem 4.1(d). So P meets KB8. Suppose *next* that P meets KB8, and let $P(B,V) = 0$. Then $P(A,-B) = P(A,V)$ for every A in S by Theorem 3.2(c), hence $P(-B,A) = 1$ for every such A by KB8 and Lemma 2(i), and hence $P(A,B) = 1$ for every such A by Theorem 3.5(d). So P meets KB8′.

So, *Kolmogorov's relative probability functions of Type I are those, and those only, among Popper's functions of that type that meet constraint* KB8.

Since Carnap's relative probability functions of Type I are those among Popper's that are both Kolmogorov and Rényi functions, they meet both constraints KB8 and RB8, and hence meet

CB8 *For any A and B in S, if $P(C,A) = P(C,B)$ for every C in S, then $A = B$,*

the constraint we claimed in Section 1 to be characteristic of Carnap's functions. On the other hand, since identity implies *left*-indiscernibility and *left*-indiscernibility implies *right*-indiscernibility by Theorem 3.1, any function satisfying CB8 meets both KB8 and RB8. So it is when P is a Carnap function that *right*-indiscernibility under P entails identity. That

CB8′ *For any A in S, if $P(A,\mathrm{V}) = 0$, then $A = \Lambda$,*

is equivalent to CB8 can also be established directly.

Theorem 4.7(I) *P meets constraint* CB8′ *if, and only if, it meets constraint* CB8.

Proof: Suppose *first* that P meets CB8′, and let $P(C,A) = P(C,B)$ for every C in S. Then $P(A \dot{-} B,\mathrm{V}) = 0$ by Theorem 4.3(a), hence $A \dot{-} B = \Lambda$ by CB8′, and hence $A = B$ by BA. So P meets CB8. Suppose *next* that P meets CB8, and let $P(B,\mathrm{V}) = 0$. Then $P(A,-B) = P(A,\mathrm{V})$ for every A in S by Theorem 3.2(c), hence $-B = \mathrm{V}$ by CB8, and hence $B = \Lambda$ by BA. So P meets CB8′.

So, *Carnap's relative probability functions of Type I are those, and those only, among Popper's functions of that type that meet* CB8.

In summary, then, it is true of *all* the relative probability functions of Popper's of Type I that

$$\textit{If } P(A,C) = P(B,C) \textit{ for every } C \textit{ in } S, \textit{ then } P(C,A) = P(C,B) \textit{ for every such } C.$$

And it is true of the *Rényi* ones, *but of them only*, that

$$\textit{If } P(A,C) = P(B,C) \textit{ for every } C \textit{ in } S, \textit{ then } A = B,$$

true of the *Kolmogorov* ones, *but of them only*, that

$$\textit{If } P(C,A) = P(C,B) \textit{ for every } C \textit{ in } S, \textit{ then } P(A,C) = P(B,C) \textit{ for every such } C,$$

and true of the *Carnap* ones, *but of them only*, that

$$\textit{If } P(C,A) = P(C,B) \textit{ for every } C \textit{ in } S, \textit{ then } A = B.$$

We borrow from [12] examples of four relative probability functions of Popper's of Type I that illustrate the foregoing points:

Illustration 1 Let S be this 4-membered set: $\{\Lambda, a, -a, \mathrm{V}\}$; and let $--a = a$, $-\Lambda = \mathrm{V}$, $-\mathrm{V} = \Lambda$, $\mathrm{V} \cap a = a$, $\mathrm{V} \cap -a = -a$, $a \cap -a = \Lambda \cap \mathrm{V} = \Lambda \cap a =$

$\Lambda \cap -a = \Lambda$, so that S constitutes a standard Boolean algebra. The following binary function P on S meets CB8 (and, hence, RB8 and KB8 as well):

	$P(A,B)$	Λ	a	$-a$	V
		B			
A	Λ	1	0	0	0
	a	1	1	0	1/2
	$-a$	1	0	1	1/2
	V	1	1	1	1

Illustration 2 Let S be as in Illustration 1. The following binary function P meets RB8 but violates KB8 (and, hence, CB8 as well):

	$P(A,B)$	Λ	a	$-a$	V
		B			
A	Λ	1	0	0	0
	a	1	1	0	0
	$-a$	1	0	1	1
	V	1	1	1	1

Illustration 3 Let S be as in Illustration 1. The following binary function P meets KB8 but violates RB8 (and, hence CB8 as well):

	$P(A,B)$	Λ	a	$-a$	V
		B			
A	Λ	1	1	0	0
	a	1	1	0	0
	$-a$	1	1	1	1
	V	1	1	1	1

Illustration 4 Let S be the set of subsets of $\{1,2,3\}$, and let $-$ and $\cap$ be set-theoretic complementation and intersection, respectively. S is a field of sets and hence a standard Boolean algebra. The following binary function P is a relative probability function of Popper's of Type I but violates RB8 and KB8 (and, hence, CB8 as well):

	$P(A,B)$	Λ	$\{1\}$	$\{2\}$	$\{3\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2\}$	V
		B							
A	Λ	1	1	0	0	0	0	0	0
	$\{1\}$	1	1	0	0	0	0	0	0
	$\{2\}$	1	1	1	0	0	0	1	0
	$\{3\}$	1	1	0	1	1	1	0	1
	$\{2,3\}$	1	1	1	1	1	1	1	1
	$\{1,3\}$	1	1	0	1	1	1	0	1
	$\{1,2\}$	1	1	1	0	0	0	1	0
	V	1	1	1	1	1	1	1	1

Rényi's relative probability functions want further comment. RB8′, the constraint we initially placed on them, required that *any* P-abnormal member of S

be identical with Λ; hence, that *any* member of S indiscernible under P from Λ be identical with Λ, and by the same token that any one indiscernible under P from V be identical with V. We just showed RB8′ equivalent to RB8, a constraint requiring that *any* member of S indiscernible under P from *a given member of S*—be that member Λ, V, or one other than Λ and V—be identical with it. So Rényi's functions do *not* allow indiscernibles in the sets on which they are defined. By contrast, those among Popper's relative probability functions of Type I that are not Rényi functions *do*. However, being defined on standard Boolean algebras, they allow indiscernibles in a very systematic way. Indeed, if member A and member B of S are distinct *but* indiscernible from each other, then $-A$ and $-B$ will not only be indiscernible from each other *but* distinct as well, and so will $A \cap C$ and $B \cap C$ for many a C in S. We return to this whole matter at the close of Section 5.

5 The relative probability functions of Type II By a *relative probability function of Popper's of Type II* we understand any binary real-valued function defined on a set closed under $-$ and $\cap$, and meeting constraints B1–B5 on p. 6 *plus* these familiar two:

B6 *For any A, B, and C in S, $P(A \cap B, C) \leq P(B \cap A, C)$* (*Commutation*)
B7 *For any A, B, and C in S, $P(A, B \cap C) \leq P(A, C \cap B)$* (*Commutation*)

And by a *relative probability function of Rényi's, Kolmogorov's, or Carnap's of Type II* we understand any relative probability function of Popper's of that type that meets constraint RB8 in Rényi's case, KB8 in Kolmogorov's, and CB8 in Carnap's.

Notes: (a) Lemmas 2-3 hold of course with P a relative probability function of Popper's of Type II, and are invoked in this section as well.

(b) In the absence of A1′, the indiscernibility versions B6–B7 of A1 are independent of B1–B5 and of each other.[18] But, as we shall shortly see, those of A2–A3 follow from B1–B5 *together with* B6–B7.

(c) In Sections 3–4, where S was presumed to be a standard Boolean algebra, we had Λ and $A \cap -A$, hence V and $A \cup -A$, *identical* for any A in S. Here we merely have Λ and $A \cap -A$, hence V and $A \cup -A$, *indiscernible under P* for any such A (see Lemmas 2(h)–(i)).

Since standard Boolean algebras are sets of a certain sort and B6–B7 hold in Section 3 by dint of A1′, any relative probability function of Type I is one of Type II. But *not* vice-versa, a point possibly obscured by Popper and all too often missed by his readers.[19] Consider, for example, the set $\{a, b, c\}$ such that $-a = c$, $-b = -c = a$, $a \cap b = b \cap a = a \cap c = c \cap a = a$, and $b \cap c = c \cap b = c$. This binary function

	$P(A,B)$	B: a	b	c
	a	1	0	0
A	b	1	1	1
	c	1	1	1

constitutes a relative probability function of Popper's of Type II. But $\{a, b, c\}$ violates A3′ and hence does not constitute a standard Boolean algebra. Nor, in point of fact, could $\{a, b, c\}$ constitute such a Boolean algebra since it has three members. P, by the way, is a Kolmogorov function that is not a Rényi one.

Theorems 5.1–5.6 and 5.10 concern relative probability functions of Popper's of *Type II*, a fact which for special reasons we underscore in two cases. A function of Type I being one of Type II, as just noted, the theorems nonetheless hold for all Popper functions. On exactly which sets functions meeting constraints B1–B7 can be defined was ascertained—in large part—by Popper in [15]. Theorems 5.1–5.5, together with constraint B6 and Theorem 3.3, deliver his share of the result. The proofs of Theorems 5.1–5.3 are his in [15].

Theorem 5.1 $P(A \cap (B \cap C), (A \cap B) \cap C) = 1$.

Proof:

1.	$P(A \cap B, (A \cap B) \cap C) = 1$	(Lemma 2(c))
2.	$P(A, (A \cap B) \cap C) = 1$	(1, Lemma 3(b))
3.	$P(B, (A \cap B) \cap C) = 1$	(1, Lemma 3(b))
4.	$P(C, (A \cap B) \cap C) = 1$	(Lemma 2(c))
5.	$P(B \cap C, (A \cap B) \cap C) = 1$	(3,4, Lemma 3(b))
6.	$P(A \cap (B \cap C), (A \cap B) \cap C) = 1$	(1,5, Lemma 3(b)).

Theorem 5.2 $P((A \cap B) \cap C, A \cap (B \cap C)) = 1$.

Proof: Similar to that of Theorem 5.1.

Theorem 5.3 $P(D, A \cap (B \cap C)) = P(D, (A \cap B) \cap C)$ *for every D in S.*

Proof: By Theorems 5.1–5.2 and Theorem 3.2(a).

Theorem 5.4 $P(A \cap (B \cap C), D) = P((A \cap B) \cap C, D)$ *for every D in S.*

Proof:

1.	$P((A \cap B) \cap C, D) = P(A \cap B, C \cap D) \times P(C, D)$	(B5)
2.	$= P(A, B \cap (C \cap D)) \times P(B, C \cap D) \times P(C, D)$	(1, B5)
3.	$= P(A, B \cap (C \cap D)) \times P(B \cap C, D)$	(2, B5)
4.	$= P(A, (B \cap C) \cap D) \times P(B \cap C, D)$	(3, Theorem 5.3)
5.	$= P(A \cap (B \cap C), D)$	(4, B5).

Theorem 5.5 $P(A \cap -B, D) = P(C \cap -C, D)$ *for every D in S if, and only if, $P(A \cap B, D) = P(A, D)$ for every such D.*

Proof: By Lemmas 2(f) and 2(h) when D is P-normal, by definition otherwise.

Now for what we call Popper's Theorem. Let P be a relative probability function of Popper's of Type II. Clearly, A is indiscernible under P from A; B is indiscernible under P from A if A is from B; and A is indiscernible under P from C if A is from B and B is from C. Further, by B6 $A \cap B$ is indiscernible under P from $B \cap A$; by Theorem 5.4 $A \cap (B \cap C)$ is indiscernible under P from $(A \cap B) \cap C$; by Theorem 5.5 $A \cap -B$ is indiscernible under P from $C \cap -C$ if, and only if, $A \cap B$ is from A; and by Theorem 3.3 $-A$ is indiscernible under P from $-B$, and $A \cap C$ indiscernible under P from $B \cap C$ if A is from B. Finally, as shown in Section 3, at least two members of S are sure because of B1 and B3 to be discernible under P. So:

Theorem 5.6 *Let P be a relative probability function of Popper's of Type II, and S be the set on which P is defined. Then*:

(a) *Indiscernibility under P constitutes an equivalence relation on S*;

(b) *P meets the indiscernibility versions of* A1–A5;

(c) *At least two distinct members of S are discernible under P*; *and hence*

(d) *S constitutes a nondegenerate Boolean algebra with respect to indiscernibility under P (Popper's Theorem).*[20]

That the sets on which Popper defines his relative probability functions constitute nondegenerate Boolean algebras is a remarkable finding. It has a number of corollaries. Note, for instance, that if the *identity* versions A1′–A5′ of A1–A5 deliver a certain identity $A = B$, then their *indiscernibility* versions are sure to deliver

$$P(A,C) = P(B,C) \textit{ for every } C \textit{ in } S,$$

this by clause (b) of Theorem 5.6. So, owing to Theorem 3.1:

Theorem 5.7 *Let P be a binary real-valued function meeting constraints* B1–B7; *let S be the set on which P is defined*; *and suppose $A = B$ is provable by* A1′–A5′. *Then*

$$P(A,C) = P(B,C) \textit{ for every } C \textit{ in } S$$

and

$$P(C,A) = P(C,B) \textit{ for every } C \textit{ in } S$$

are consequences of B1–B7.

Various clauses of Lemma 3 in the Appendix are proved by appeal to this corollary.

Other corollaries of Popper's Theorem are of theoretical significance. Because of it, indiscernibility *tout court* in the sense of p. 11 is one more equivalence relation with respect to which the sets that Popper's relative probability functions are defined on constitute Boolean algebras. Further, suppose P is a Rényi function, and let A and B be arbitrary members of the set S on which P is defined. Since P meets constraint RB8, A will be identical with B if indiscernible under P from B. So S, a Boolean algebra *with respect to indiscernibility under P* by Popper's Theorem, is one *with respect to identity* as well. So:

Theorem 5.8 *Rényi's relative probability functions of Type II are identical with those of Type I.*

But Carnap's relative probability functions are Rényi ones: meeting constraint CB8, they meet constraint RB8. So:

Theorem 5.9 *Carnap's relative probability functions of Type II are identical with those of Type I.*

So, at this point, there is just one family of relative probability functions in Rényi's sense, and just one in Carnap's sense. But, as reported in Section 1, each family can be characterized in two different ways. In both characterizations one must require of the functions that they meet constraints B1–B5 *plus* whichever of constraints RB8 and CB8 is appropriate. This done, however, one may *either* require as in Section 3 that the sets on which the functions are defined be standard Boolean algebras *or* require as in this section that the functions themselves meet the extra constraints B6–B7. Those extra constraints, remarkably, compel the sets on which the functions are defined to be standard Boolean algebras.

Here as in Sections 3–4 those among Popper's relative probability functions that are *not* Rényi ones allow indiscernibles, but they are less systematic about it than on pp. 18–19. For example, look back at the function displayed on pp. 18–19. Members b and c of the set $\{a, b, c\}$ on which the function is defined are indiscernible *but* distinct of course from each other; their complements, on the other hand, are the same. Our present concern, however, is a related but different one, i.e. noting that, where P is a Popper function that is *not* a Rényi one and S the set on which P is defined, one can sort the indiscernibles that P allows into equivalence sets, each set consisting of all the members of S indiscernible under P from a given one, and turn P into a relative probability function on the set consisting of those equivalence sets. That set is of course a standard Boolean algebra, and the function into which P is turned proves of course to be a Rényi one.

More formally, let P and S be as described. S is sure by Popper's Theorem to be a Boolean algebra *with respect to indiscernibility under P.* So let $[S]$ be what we called in Section 2 *the reduction of S with respect to that equivalence relation*,[21] and let $[P]$ be the following binary function on $[S]$, to be known as *the reduction of P to* $[S]$:

$$[P]([A],[B]) = P(A,B),$$

where A is an arbitrary member of $[A]$ and B an arbitrary one of $[B]$. (Since for any A' in $[A]$ and any B' in $[B]$, $P(A',C) = P(A,C)$ and $P(C,B') = P(C,B)$ for every C in S, the choice of A and B is immaterial.) Clearly, $[P]$ is real-valued just as P is and meets constraints B1–B7 just as P does. But, importantly here, $[P]$ also meets constraint RB8. For suppose that

$$[P]([A],[C]) = [P]([B],[C]) \textit{ for every } [C] \textit{ in } [S].$$

Then, by definition,

$$P(A,C) = P(B,C) \textit{ for every } C \textit{ in } S,$$

and hence, as noted on p. 6,

$$[A] = [B].$$

So:

Theorem 5.10 *Let P be a relative probability function of Popper's of Type II that is not a Rényi one; let S be the set on which P is defined; and let* $[S]$ *be the reduction of S with respect to indiscernibility under P. Then the reduction* $[P]$ *of P to* $[S]$ *constitutes a Rényi function on* $[S]$.

We established in Section 4 that the converse

$$\begin{gathered}\textit{If } P(C,A) = P(C,B) \textit{ for every } C \textit{ in } S,\\ \textit{then } P(A,C) = P(B,C) \textit{ for every such } C\end{gathered}$$

of Popper's Constraint is characteristic of Kolmogorov's relative probability functions and hence does not hold true of other relative probability functions.

A case in point is the Rényi function P on p. 18, Illustration 2: $P(C,-a) = P(C,\mathrm{V})$ for every C there, but $P(-a,a) \neq P(\mathrm{V},a)$. However,

$$P(C,A) = P(C,B) \textit{ for every relative probability function } P \textit{ of Popper's of Type II defined on } S \textit{ and every } C \textit{ in } S$$

entails

$$P(A,C) = P(B,C) \textit{ for every such } P \textit{ and every such } C,$$

as we go on to show. So members A and B of S prove to be indiscernible *tout court* if *right*-indiscernible as well as if *left*-indiscernible under every relative probability function of Popper's of Type II defined on S.[22]

Theorem 5.11 *$P(C,A) = P(C,B)$ for every relative probability function P of Popper's of Type II defined on S and every C in S if, and only if, $P(A,C) = P(B,C)$ for every such P and every such S.*

Proof: Suppose P is a relative probability function of Popper's of Type II defined on S and C is a member of S such that $P(A,C) \neq P(B,C)$. Then by B6 either $P(A \cap B,C) \neq P(A,C)$ or $P(B \cap A,C) \neq P(B,C)$. So suppose *first* that $P(A \cap B,C) \neq P(A,C)$, in which case $P(A,C) \neq 0$ by Lemma 2(n); and let P_C be this function gotten from P by *conditionalizing* on C:

$$P_C(D,E) = P(D,E \cap C).$$

P_C meets B2 by B2, B3 by Lemma 2(c), B5 by B5 and Theorem 5.3, B6 by B6, and B7 by Theorem 5.7. As for B1, since $P(A,C) \neq P(B,C)$, $P(A,C) \neq 1$ or $P(B,C) \neq 1$ or both, hence $P(A,C \cap C) \neq 1$ or $P(B,C \cap C) \neq 1$ or both by Theorem 5.7, and hence $P_C(A,C) \neq 1$ or $P_C(B,C) \neq 1$ or both. And as for B4, suppose E is P_C-normal. Then $E \cap C$ is P-normal by definition, hence $P(-D,E \cap C) = 1 - P(D,E \cap C)$ by B4, and hence $P_C(-D,E) = 1 - P_C(D,E)$. So P_C constitutes a relative probability function of Popper's of Type II, and hence in particular $P_C(B,B) = 1$ by B3. But $P_C(B,A) = P(B \cap A,C)/P(A,C)$ by B5 and $P(A,C) \neq 0$, hence $P_C(B,A) = P(A \cap B,C)/P(A,C)$ by B6, and hence $P_C(B,A) \neq 1$ by the hypothesis on $P(A \cap B,C)$. So $P_C(B,A) \neq P_C(B,B)$. Suppose *next* that $P(B \cap A,C) \neq P(B,C)$. Then $P_C(A,B) \neq P_C(A,A)$ by interchanging A and B throughout. So there exists a relative probability function P of Popper's of Type II defined on S and a member C of S such that $P(C,A) \neq P(C,B)$. So Theorem 5.11 by Theorem 3.1.

We put this result to use in Section 7.

6 ***Absolute probability functions*** By an *absolute probability function of Popper's of Type I* we understand any unary real-valued function P defined on a standard Boolean algebra S and meeting these three constraints, adaptations of constraints of Popper's in [14]:

C1 *For any A in S, $0 \leq P(A)$* (*Nonnegativity*)
C2 *For any A in S, $P(A \cup -A) = 1$* (*Normality*)
C3 *For any A and B in S, $P(A) = P(A \cap B) + P(A \cap -B)$* (*Special Addition*)[23]

And by an *absolute probability function of Carnap's of Type I* we understand any absolute probability function of Popper's of that type that meets this extra constraint:

CC7′ *For any A in S, if $P(A) = 0$, then $A = \Lambda$.*

Notes: (a) Assembled under Lemmas 4–5 in the Appendix are various facts about unary real-valued functions that meet constraints C1–C3 above and C4–C6 on p. 26. The latter are the extra constraints placed on P when the function is of Type II. They trivially follow from A1′–A5′. So the various clauses of Lemmas 4 and 5 hold of all absolute probability functions. In accord with our practice in Sections 4–5 we say of the last step in the proof of Theorem 6.1 that it is taken by *Commutation*; and of the one step in this section taken by a consequence of A1′–A5′ other then C1–C6, we shall say that it is taken *by BA*.

(b) CC7′ is the characteristic constraint for Carnap's absolute probability functions that we used in [12]. The alternative constraint CC7 that we promised in Section 1 will be introduced on p. 25.

(c) According to Lemma 4(e)

$$P(A \cap -A) = 0.$$

So P has at least two distinct values. So S is a nondegenerate Boolean algebra.

The condition

$$P(A, C) = P(B, C) \textit{ for every } C \textit{ in } S,$$

which we have used to say that members A and B of S are indiscernible under the *relative* probability function P, proved equivalent in Section 3 to

$$P(f(A), g(A)) = P(f(B), g(B))$$
$$\textit{for every Boolean function } f \textit{ and every Boolean function } g \textit{ on } S.$$

Wanted here is a condition which would prove equivalent to

$$P(f(A)) = P(f(B)) \textit{ for every Boolean function } f \textit{ on } S,$$

and which, as a result, we could use to say that members A and B of S are indiscernible under the *absolute* probability function P.

This condition:

$$P(A) = P(B),$$

would clearly not do. For suppose $P(A)$ equaled 1/2, and let B be $-A$. Then by Lemma 4(i) $P(A)$ would equal $P(B)$. But by Lemma 4(d) $P(A \cap A)$ would equal $P(A)$ and hence 1/2, while by Lemma 4(e) $P(B \cap A)$ (i.e., $P(-A \cap A)$) would equal 0. So there would be a C in S (i.e., A itself) such that $P(A) = P(B)$ but $P(A \cap C) \neq P(B \cap C)$.[24] However,

$$P(A) = P(B) \textit{ and } P(A) = P(A \cap B),$$

in short,

$$P(A) = P(B) = P(A \cap B),$$

would do, as we next show; and, along of course with $P(A) = P(B) = P(B \cap A)$, it is the simplest condition that would.

Theorems 6.1–6.3, though concerned with functions of Type I, are proved using just constraints C1–C6. So they hold of functions of Type II as well. Not so, however, Theorem 6.4.

Theorem 6.1
(a) *If* $P(A) = P(B) = P(A \cap B)$, *then* $P(-A) = P(-B)$;
(b) *If* $P(A) = P(B) = P(A \cap B)$, *then* $P(A \cap C) = P(B \cap C)$ *for every C in S*;
(c) *If* $P(A) = P(B) = P(A \cap B)$, *then* $P(C \cap A) = P(C \cap B)$ *for every such C.*

Proof: (a) By Lemma 4(i). (b) Suppose $P(A) = P(B) = P(A \cap B)$, and let C be an arbitrary member of S. Then $P(A \cap -B) = P(-A \cap B) = 0$ by C3 and Lemma 4(a); hence $P((A \cap C) \cap -B) = P(-A \cap (B \cap C)) = 0$ by Lemmas 4(g) and 4(j) in one case and Lemmas 4(f) and 4(b) in the other; hence $P(A \cap C) = P(A \cap (C \cap B))$ by C3 and Lemma 4(j), and $P(B \cap C) = P(A \cap (C \cap B))$ by Lemmas 4(a) and 4(k); and hence $P(A \cap C) = P(B \cap C)$. (c) By (b) and *Commutation.*

As Theorem 3.3 generalized to Theorem 3.4, so does the foregoing theorem generalize to this (given again that $-$ and $\cap$ permit definition of every Boolean function on S):

Theorem 6.2 $P(A) = P(B) = P(A \cap B)$ *if, and only if,* $P(f(A)) = P(f(B))$ *for every Boolean function f on S.*

Left- and right-indiscernibility under a relative probability function can be phrased in terms of symmetric difference, as we showed in Section 4. So can indiscernibility under an absolute one, our next theorem shows. Four phrasings are provided there. The first of them will deliver CC7, our new characteristic constraint for Carnap's functions. The last two are as close as one can get in probability theory to the two identities $A \dotminus B = \Lambda$ and $-(A \dotminus B) = \mathrm{V}$ in Section 1.

Theorem 6.3 *Equivalent to*

$$P(A) = P(B) = P(A \cap B)$$

is each of these:
(a) $P(A \dotminus B) = 0$,
(b) $P(-(A \dotminus B)) = 1$,
(c) $P(A \dotminus B) = P(\Lambda)$, *and*
(d) $P(-(A \dotminus B)) = P(\mathrm{V})$.

Proof: (a) By C3 and Lemma 4(a) $P(A) = P(B) = P(A \cap B)$ if, and only if, $P(A \cap -B) = P(B \cap -A) = 0$; and hence by Lemma 5(c) if, and only if, $P(A \dotminus B) = 0$. (b) By (a) and Lemma 4(i). (c) By (a) and Lemma 4(e). (d) By (b) and C2.

It is when P is an absolute probability function of Carnap's that indiscernibility under P entails identity. Equivalent indeed to

CC7′ *For any A in S, if* $P(A) = 0$, *then* $A = \Lambda$,

is

CC7 *For any A and B in S, if* $P(A) = P(B) = P(A \cap B)$, *then* $A = B$, the constraint we claimed in Section 1 to be characteristic of Carnap's functions. The theorem to that effect holds only for Popper functions of Type I. A reminder of that is the '(I)' we attach to the call number of the theorem.

Theorem 6.4(I) *P meets constraint CC7′ if, and only if, it meets constraint CC7.*

Proof: Suppose *first* that P meets CC7′, and let $P(A) = P(B) = P(A \cap B)$, in which case $P(B) = P(B \cap A)$ by *Commutation*. Then $P(A \dot{-} B) = 0$ by Theorem 6.3(a), hence $A \dot{-} B = \Lambda$ by CC7′, and hence $A = B$ by BA. So P meets CC7. Suppose *next* that P meets CC7, and let $P(A) = 0$. Then $P(A) = P(\Lambda)$ by Lemma 4(e), $P(A) = P(A \cap \Lambda)$ by Lemma 4(f), and hence $A = \Lambda$ by CC7. So P meets CC7′.

So, *Carnap's absolute probability functions of Type I are those, and those only, among Popper's functions of that type that meet constraint* CC7.

The following illustration yields $2^{\aleph_0}$ distinct absolute probability functions of Popper's of Type I, one for each choice of a *nonnegative* real not exceeding 1 as r. $2^{\aleph_0}$ of the functions will be Carnap ones, those gotten by choosing r larger than 0. Each member of the set $(\Lambda, a, -a, \mathrm{V})$ on which the functions are defined will then be discernible from the other three.

Illustration 5 Let S be the set $\{\Lambda, a, -a, \mathrm{V}\}$ as in Illustration 1.

A	$P(A)$
Λ	0
a	r
$-a$	$1 - r$
V	1

So much, however, for the functions of Type I. By an *absolute probability function of Popper's of Type II* we understand any unary real-valued function defined on a set closed under $-$ and $\cap$, and meeting constraints C1–C3 on p. 23 *plus* these three, due to Popper in [14]:

C4 *For any A and B in S, $P(A \cap B) \leq P(B \cap A)$* (*Commutation*)
C5 *For any A, B, and C in S, $P(A \cap (B \cap C)) \leq P((A \cap B) \cap C)$* (*Association*)
C6 *For any A in S, $P(A) \leq P(A \cap A)$* (*Idempotence*)

And by an *absolute probability function of Carnap's of Type II* we understand any absolute probability function of Popper's of Type II that meets constraint CC7.

Notes: (a) Versions of C5 and C6 with '=' in place of '≤' can be gotten from C1–C6, as shown under Lemma 4 in the Appendix. We list the *Idempotence* constraint last because $A \cap A \approx A$ is a consequence of A3. All three of C4–C6, by the way, are independent of C1–C3 and of each other.

(b) When as on p. 23 S was a standard Boolean algebra we had Λ and $A \cap -A$, hence V and $A \cup -A$, *identical* for any A in S. Here we merely have them indiscernible under P for any such A (Lemma 4(e) and C3).

Since standard Boolean algebras are sets of a certain sort and C4–C6 hold by dint of A1′–A3′, any absolute probability function of Type I is one of Type II but *not* vice-versa. The unary function P gotten from the binary one on p. 19 by setting $P(A)$ at either $P(A,b)$ or $P(A,c)$ is an absolute probability function of Popper's of Type II that is *not* one of Type I:

A	$P(A)$
a	0
b	1
c	1

It is, by the way, one of those among Popper's functions that are not Carnap functions.

Borrowing from [10], we prove the analogue for absolute probability functions of Type II of Popper's Theorem for relative ones of that type.

Theorem 6.5 *Let P be an absolute probability function of Popper's of Type II, and let S be the set on which P is defined. Then:*

(a) *Indiscernibility under P constitutes an equivalence relation on S;*
(b) *P meets the indiscernibility versions of* A1–A5;
(c) *At least two members of S are discernible under P; and hence*
(d) *S constitutes a nondegenerate Boolean algebra with respect to indiscernibility under P.*

Proof: (a) By Lemma 4(d), C4, and Lemma 5(d), respectively,

$$P(A) = P(A) = P(A \cap A),$$
$$\textit{If } P(A) = P(B) = P(A \cap B), \textit{ then } P(B) = P(A) = P(B \cap A),$$

and

$$\textit{If } P(A) = P(B) = P(A \cap B) \textit{ and } P(B) = P(C) = P(B \cap C),$$
$$\textit{then } P(A) = P(C) = P(A \cap C).$$

So indiscernibility under P is reflexive, symmetrical, and transitive. (b) By C4 and Lemma 5(l) $A \cap B$ is indiscernible under P from $B \cap A$ (Constraint A1). By Lemmas 4(b) and 5(r) $A \cap (B \cap C)$ is indiscernible under P from $(A \cap B) \cap C$ (Constraint A2). By C3 and Lemma 4(e) $P(A \cap -B) = P(C \cap -C)$ if, and only if, $P(A \cap B) = P(A)$; hence, by Lemmas 4(e)–(f) and 5(i) $P(A \cap -B) = P(C \cap -C) = P((A \cap -B) \cap (C \cap -C))$ if, and only if, $P(A \cap B) = P(A) = P((A \cap B) \cap A)$; and, hence, $A \cap -B$ is indiscernible under P from $C \cap -C$ if, and only if, $A \cap B$ is from A (Constraint A3). Next, suppose that $P(A) = P(B) = P(A \cap B)$. Then $P(-A) = P(-B)$ by Theorem 6.1(a). But, since $P(B) = P(A \cap B)$ by the hypothesis, $P(-A \cap B) = 0$ by Lemma 4(a), and hence $P(-A) = P(-A \cap -B)$ by C3. Hence $-A$ is indiscernible under P from $-B$ if A is from B (Constraint A4). Lastly, suppose that $P(A) = P(B) = P(A \cap B)$. Then $P(A \cap C) = P(B \cap C)$ by Theorem 6.1(b). But $P((A \cap C) \cap B) = P(((A \cap C) \cap B) \cap C)$ by C3 and Lemma 5(s); and, since $P(A \cap -B) = 0$ by C3 and the hypothesis, $P((A \cap C) \cap -B) = 0$ by Lemmas 4(g) and 4(j). So $P(A \cap C) = P(((A \cap C) \cap B) \cap C)$ by C3 again, and hence $P(A \cap C) = P((A \cap C) \cap (B \cap C))$ by Lemma 4(b). Hence $A \cap C$ is indiscernible under P from $B \cap C$ if A is from B (Constraint A5). (c) By C3 and Lemma 4(e).

So the sets on which Popper defines his absolute probability functions, like those on which he defines his relative ones, constitute nondegenerate Boolean algebras.

Theorem 6.5 has an important corollary. Suppose indeed P is a Carnap function, and let A and B be arbitrary members of the set S on which P is defined. Since P meets constraint CC7, A will be identical with B if indiscernible under P from B. So S, a Boolean algebra *with respect to indiscernibility under* P by Theorem 6.5, is one *with respect to identity* as well. So:

Theorem 6.6 *Carnap's absolute probability functions of Type II are identical with those of Type I.*

So, at this point, there is just one family of absolute probability functions in Carnap's sense. But, as indicated in Section 1, that family can be characterized in two different ways. In both characterizations one must require of the functions that they meet constraints C1–C3 plus constraint CC7. This done, however, one may *either* require that the sets on which the functions are defined be standard Boolean algebras *or* require that the functions themselves meet the extra constraints C4–C6. Those extra constraints, interestingly, compel the sets on which the functions are defined to be standard Boolean algebras.

Popper's absolute probability functions that are *not* Carnap ones allow indiscernibles, those of Type I allowing them in the systematic manner of pp. 18–19 and those of Type II in the less systematic one of p. 22. But, given any such function P that is not a Carnap one, the indiscernibles P allows can of course be gathered into equivalence sets and P made into a Carnap function on the standard Boolean algebra consisting of those sets. Instructions are as on p. 22: S being the Boolean algebra on which by clause (d) in Theorem 6.5 P is defined, and $[S]$ being the reduction of S with respect to the equivalence relation in clause (a) of that theorem, let $[P]$ be this reduction of P to $[S]$:

$$[P]([A]) = P(A).$$

$[P]$ is easily verified to meet constraints C1–C6 and, importantly, constraint CC7 as well. So:

Theorem 6.7 *Let P be an absolute probability function of Popper's of Type II that is not a Carnap one; let S be the set on which P is defined; and let $[S]$ be the reduction of S with respect to indiscernibility under P. Then the reduction $[P]$ of P to $[S]$ constitutes a Carnap function on $[S]$.*

In view of Theorems 5.10 and 6.7 Carnap's absolute probability functions are to the absolute ones of Popper's in [14] what Rényi's relative probability functions are to the relative ones of Popper's in [15]. The foregoing also suggests a way of reducing those among Rényi's relative probability functions that are *not* Carnap functions to Carnap ones. Rényi functions, to be sure, do not allow indiscernibles under themselves; but, when not Carnap functions, they allow indiscernibles under their so-called restrictions to V. Indeed any Popper relative probability function that is not a Carnap one allows indiscernibles in this sense. Formally, let P be a relative probability function of Popper's; let S be the standard Boolean algebra on which it is defined; and let *the restriction of P to V* be the following function, shown in [12] to be an absolute probability function of Popper's:

$$P_{\mathrm{V}}(A) = P(A, \mathrm{V}).$$

We shall then say that members A and B of S are *indiscernible under* P_{V} if

$$P_{\mathrm{V}}(A) = P_{\mathrm{V}}(B) = P_{\mathrm{V}}(A \cap B).$$

Owing to Theorem 6.5(a), indiscernibility under P_{V} constitutes an equivalence relation on S. And, due to Theorem 6.3,

$$P_{\mathrm{V}}(A) = P_{\mathrm{V}}(B) = P_{\mathrm{V}}(A \cap B)$$

is equivalent to

$$P_{\mathrm{V}}(A \dot{-} B) = 0,$$

hence equivalent to

$$P(A \dot{-} B, \mathrm{V}) = 0,$$

and hence equivalent by Theorem 4.4 to

$$P(C,A) = P(C,B) \text{ for every } C \text{ in } S \text{ or}$$
$$P(C,-A) = P(C,-B) \text{ for every such } C.$$

So A and B are indiscernible under P_{V} if, and only if, they themselves are right-indiscernible under P or their complements are.

Let $[S]$ then be the reduction of S with respect to that equivalence relation[25]; and let $[P]$ be the following function on $[S]$:

$$[P]([A],[B]) = \begin{cases} P(A,B) & \textit{if } P_{\mathrm{V}}(B) \neq 0 \\ 1 & \textit{otherwise.} \end{cases}$$

Members of the same equivalence set are not necessarily indiscernible under P. Nevertheless, by Theorem 4.3(b) and Lemma 3(v), $P(A',B') = P(A,B)$ for any A' in $[A]$ and B' in $[B]$, when $P_{\mathrm{V}}(B) \neq 0$. So the definition of $[P]$ is sound. But $[P]$ readily proves to meet constraints B1–B7 *plus* constraints CB8. Hence $[P]$ constitutes a relative probability function of Carnap's.

So, when a relative probability function P of Popper's is not a Carnap one, the indiscernibles under P_{V} that P allows can be gathered into equivalence sets and P made into a Carnap function on the standard Boolean algebra consisting of those sets. This reduction can be thought of as achieved in two steps. First, a Rényi function P' is obtained by reduction with respect to indiscernibility under P, and then a Carnap function by reduction with respect to indiscernibility under P'_{V}.[26]

A last, and possibly unexpected, result is in store. Heeding the precedent of the first paragraph of this paper, we could declare members A and B of a set S indiscernible *tout court* if A and B are indiscernible under every absolute probability function P of Popper's of Type II defined on S, i.e. if $P(A) = P(B) = P(A \cap B)$ for every such P. But, even though

$$P(A) = P(B)$$

and

$$P(A) = P(B) = P(A \cap B)$$

are not equivalent, we have this:

Theorem 6.8 *$P(A) = P(B) = P(A \cap B)$ for every absolute probability function P of Popper's of Type II defined on S if, and only if, $P(A) = P(B)$ for every such P.*

Proof: Suppose P is an absolute probability function of Popper's of Type II defined on S such that $P(A) = P(B)$ but $P(A) \neq P(A \cap B)$. Then by Lemma 4(c) $P(A \cap B) < P(A)$ and hence by C1 $P(A) \neq 0$. Now consider this function, known as *the restriction of P to A*:

$$P_A(C) = P(C \cap A)/P(A).$$

P_A constitutes an absolute probability function of Popper's of Type II since it meets (i) C1 by C1, (ii) C2 by C2, and Lemmas 4(i), 4(f), and 4(a), (iii) C3 by Lemmas 4(a)–(b) and 4(j), and (iv) C4–C6 by Theorem 6.5(a), C4, and Lemmas 4(b) and 4(d). But $P_A(A) = 1$ by Lemma 4(d), while $P_A(B) < 1$ by *Commutation*. So there exists an absolute probability function P of Popper's of Type II such that $P(A) \neq P(B)$.

So we may declare A and B indiscernible *tout court* if, more simply, $P(A) = P(B)$ for every absolute probability function of Popper's of Type II defined on S. It follows from Theorems 6.5 and 6.8 that indiscernibility *tout court* in the present sense is another equivalence relation with respect to which the sets that Popper's absolute probability functions are defined on constitute Boolean algebras. And it will follow from Theorem 7.6 that indiscernibility *tout court* in the present sense and indiscernibility *tout court* in the sense of p. 11 are the same.

7 Rounding off Popper's probability theory According to Popper's Theorem any set (closed under $-$ and $\cap$) on which a function meeting constraints B1–B7 can be defined constitutes a nondegenerate Boolean algebra with respect to indiscernibility, and hence a nondegenerate Boolean algebra *tout court*. The converse of the theorem is also true, as we proceed to show. We use to that effect what algebraists call *ultra-filters*, and we appeal to results about ultra-filters, proofs of which are in Sikorski [20], Chapter 1, §3 and §6.

Let S be a nondegenerate Boolean algebra with respect to some equivalence relation or other $\approx$. By a *filter on S with respect to* $\approx$ we understand any subset S' of S that meets these two constraints:

(i) For any A and B in S, $A \cap B$ belongs to S' if, and only if, A and B both do;
(ii) For any A and B in S, if $A \approx B$, then A belongs to S' if, and only if, B does.

And by an *ultra-filter on S with respect to* $\approx$ we understand any filter on S with respect to $\approx$ that meets this extra constraint:

(iii) For any A in S, exactly one of A and $-A$ belongs to S'.

Theorem 7.1 *Let S be a nondegenerate Boolean algebra with respect to $\approx$; let S' be an ultrafilter on S with respect to $\approx$; and let P be this function on S:*

$$P(A,B) = \begin{cases} 0 \text{ if } B \text{ belongs to } S' \text{ but } A \text{ does not} \\ 1 \text{ otherwise.} \end{cases}$$

Then P meets constraints B1–B7.

Proof: P obviously meets constraints B2 and B3. As for B1, $A \cap -A$ cannot belong to S' since by (i) both A and $-A$ would then belong to S', contrary to (iii); by the same token $-A \cap --A$ cannot belong to S', and hence $A \cup -A$ must by (iii). So $P(A \cup -A, A \cap -A) = 0$ by the definition of P, and hence P meets B1. As for B4, suppose $P(C,B) \neq 1$ for some C in S. Then $P(C,B) = 0$ by the definition of P, and hence B belongs to S' by that definition. But by (iii) exactly one of A and $-A$ belongs to S'. So *either* $P(A,B) = 0$ and $P(-A,B) = 1$ *or* $P(A,B) = 1$ and $P(-A,B) = 0$ by the definition of P, hence $P(-A,B) = 1 - P(A,B)$, and hence P meets B4. The proof that P meets B5 is by cases; it hinges on the fact that by (i) $A \cap B$ belongs to S' if, and only if, both A and B do, and $B \cap C$ does if, and only if, both B and C do. Finally, $(A \cap B) \approx (B \cap A)$ by A1; hence by (ii) $A \cap B$ belongs to S' if, and only if, $B \cap A$ does; and hence $P(A \cap B, C) = P(B \cap A, C)$ by the definition of P. But $P(A, B \cap C) = P(A, C \cap B)$ by the very same reasoning. So P meets B6 and B7 as well.[27]

According to the theorems in [20], any nondegenerate Boolean algebra S has at least one proper subset that constitutes a filter on S and can be extended to an ultra-filter on S. So, given Theorem 7.1:

Theorem 7.2 *Binary real-valued functions meeting constraints* B1–B7 *are definable on all, and only, those sets that constitute nondegenerate Boolean algebras.*[28]

To attend to the absolute case too, let S and S' be as before, and let P' be the V-restriction of P. Then, by virtue again of a result in [12], P' meets constraints C1–C6. So:

Theorem 7.3 *Unary real-valued functions meeting constraints* C1–C6 *are definable on all, and only, those sets that constitute nondegenerate Boolean algebras.*[29]

It follows from Theorems 5.8–5.9 and 6.6 that (i) binary real-valued functions meeting B1–B7 and either of RB8 and CB8, and (ii) unary ones meeting C1–C6 and CC7, are definable *only* on sets constituting nondegenerate standard Boolean algebras. Whether functions of that sort are definable on *all* sets of that sort is an open question as of this writing. We suspect that the answer to it is in the affirmative.

•

Rényi's and Carnap's relative probability functions of Type II and Carnap's absolute ones of that type proved in Sections 5 and 6 to be of Type I after all, hence old acquaintances. As mentioned in Section 1, however, there are probability functions that are closely related to Rényi's and Carnap's but are defined on Boolean algebras *generally* rather than just *standard* ones. To arrive at them, we recast the account in Section 5 of the relative probability functions of Type II and that in Section 6 of the absolute ones. The new account exploits Theorems 7.2 and 7.3, according to which probability functions of Type II are definable on *every* nondegenerate Boolean algebra; the theorems allow us to give S top billing. It also exploits $\approx^*$, the least equivalence relation on S with respect to which S constitutes a nondegenerate Boolean algebra.

Let S be a nondegenerate Boolean algebra, and let $\approx^*$ be the least equivalence relation on S with respect to which S constitutes such an algebra.

A Let P be a binary real-valued function on S. Then P constitutes
(i) a *relative probability function of Popper's of Type II* if it meets constraints B1–B7, and
(ii) a *relative probability function of Rényi's, Kolmogorov's*, or *Carnap's of Type II* if it constitutes a Popper one and also meets this constraint in Rényi's case:

RB8* *For any A and B in S, if $P(A,C) = P(B,C)$ for every C in S, then $A \approx^* B$,*

constraint KB8 in Kolmogorov's, and this constraint in Carnap's:

CB8* *For any A and B in S, if $P(C,A) = P(C,B)$ for every C in S, then $A \approx^* B$.*

B Let P be a *unary* real-valued function on S. Then P constitutes
(i) an *absolute probability function of Popper's of Type II* if it meets constraints C1–C6, and
(ii) an *absolute probability function of Carnap's of Type II* if it constitutes a Popper one and also meets this constraint:

CC7* *For any A and B in S, if $P(A) = P(B) = P(A \cap B)$, then $A \approx^* B$.*

Equivalent of course to constraints RB8*, CB8*, and CB7* are the results RB8′*, CB8′*, and CC7′* of replacing '=' everywhere in RB8′, CB8′, and CC7′ by '$\approx^*$': the counterparts of Theorems 4.5(I), 4.7(I), and 6.4(I) guarantee that.

As before, the relative and the absolute probability functions of Type I are simply those of Type II that are defined on a *standard* Boolean algebra S: in that case (i) constraints B6–B7 and C4–C6 may be dropped, (ii) constraint B1 may be dropped when P is a Rényi function or a Carnap one (for the reason given in Note 17), and (iii) $\approx^*$ being $=$, RB8*, CB8*, and CC7* are the same as RB8, CB8, and CC7, respectively. As for the probability functions of Type II themselves, the Popper and the Kolmogorov ones are of course the same as in Section 5, but the Rényi and the Carnap ones are *new*. Since $A = B$ entails $A \approx^* B$, any Rényi function of Type I is one of Type II, and so is any Carnap function of Type I, relative or absolute, but *not* vice-versa. Displayed below are a Rényi function of Type II (that is not a Carnap one) and two Carnap ones of that type. Defined on a set of cardinality 5, hence on a set that is not a standard Boolean algebra, the functions are not of Type I.

Illustration 6 Let S be the set $\{\Lambda, a, b, -a, \mathrm{V}\}$, where (i) Λ, a, $-a$, and V are as in Illustration 1 in Section 4, but (ii) $-b = -a$, $a \cap b = b \cap b = \mathrm{V} \cap b = b$

and $-a \cap b = \Lambda \cap b = \Lambda$. S is a nondegenerate Boolean algebra with respect to this equivalence relation $\approx^*$, which is the least such relation on S:

$$A \approx^* B =_{def} A = B, \text{ or } A = a \text{ and } B = b, \text{ or } A = b \text{ and } B = a;$$

and the following binary function P on S meets B1–B7 and RB8*, but not CB8*:

	$P(A,B)$	B: Λ	a	b	$-a$	V
	Λ	1	0	0	0	0
	a	1	1	1	0	0
A	b	1	1	1	0	0
	$-a$	1	0	0	1	1
	V	1	1	1	1	1

Illustration 7 Let S be the same Boolean algebra as in Illustration 6. The following binary function P on S meets B1–B7 and CB8*;

	$P(A,B)$	B: Λ	a	b	$-a$	V
	Λ	1	0	0	0	0
	a	1	1	1	0	1/3
A	b	1	1	1	0	1/3
	$-a$	1	0	0	1	2/3
	V	1	1	1	1	1

Illustration 8 Let S again be the Boolean algebra of Illustration 6. The following unary function P on S meets C1–C6 and CC7*:

A	$P(A)$
Λ	0
a	1/3
b	1/3
$-a$	2/3
V	1

The relationships between our various families of relative probability functions can be portrayed as in Figure 2, to be compared with the one on p. 9. The areas marked I consist in all four cases of the functions that are of Type I and hence are defined on *standard* Boolean algebras.

When defined as on p. 32, the probability functions of Type II match some of their counterparts of Type I in a remarkable manner.

Theorem 7.4 *Let S be a nondegenerate Boolean algebra, let $\approx^*$ be the least equivalence relation on S with respect to which S constitutes such an algebra, and let $[S]^*$ be the reduction of S with respect to $\approx^*$. Then:*

(a) *The relative probability functions of Popper's (Rényi's, Kolmogorov's, Carnap's) of Type II defined on S match one-to-one those of Popper's (Rényi's, Kolmogorov's, Carnap's) of Type I defined on $[S]^*$;*

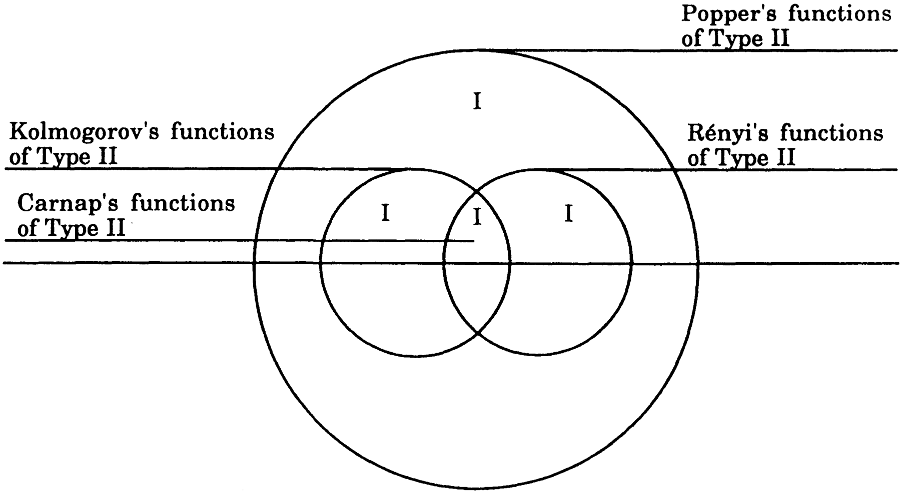

Figure 2.

(b) *The absolute probability functions of Popper's (Carnap's) of Type II defined on S match one-to-one those of Popper's (Carnap's) of Type I defined on* $[S]^*$.

Proof: (a) P^{II} being a relative probability function of Popper's of Type II, let P^{I} be this binary function on $[S]^*$:

$$P^{\mathrm{I}}([A]^*,[B]^*) = P^{\mathrm{II}}(A,B);$$

and P^{I} being a relative probability function of Popper's of Type I defined on $[S]^*$, let P^{II} bet his binary function on S:

$$P^{\mathrm{II}}(A,B) = P^{\mathrm{I}}([A]^*,[B]^*),$$

a function under which A and A' are sure to be indiscernible if $A \approx^* A'$. It is easily verified that: (i) P^{I} meets constraints B1–B5 and hence constitutes a relative probability function of Popper's of Type I, and (ii) P^{II} meets constraints B1–B7 and hence constitutes a relative probability function of Popper's of Type II. And it is clear that the identity

$$P^{\mathrm{I}}([A]^*,[B]^*) = P^{\mathrm{II}}(A,B)$$

establishes a one-to-one correlation between the relative probability functions of Type I on $[S]^*$ and those of Type II on S. So, since

$$[A]^* = [B]^* \textit{ if, and only if, } A \approx^* B,$$

P^{I} meets constraint RB8 precisely when P^{II} meets constraint RB8*, and hence P^{I} constitutes a Rényi function of Type I precisely when P^{II} constitutes one of Type II. Similarly, P^{I} meets constraint KB8 precisely when P^{II} does, and hence P^{I} constitutes a Kolmogorov function of Type I precisely when P^{II} constitutes one of Type II. Lastly, P^{I} meets constraint CB8 precisely when P^{II} meets constraint CB8*, and hence P^{I} constitutes a Carnap function of Type I precisely when P^{II} constitutes one of Type II.

(b) In the absolute case the one-to-one correspondence between the functions of Type I and those of Type II is established by the identity

$$P^{\mathrm{I}}([A]^*) = P^{\mathrm{II}}(A).$$

Clearly, if $A \approx^* A'$, then A and A' are indiscernible under P^{II}. It is easily verified that (i) if P^{I} meets constraints C1–C3, then P^{II} meets constraints C1–C6, and (ii) if P^{II} meets C1–C6, then P^{I} meets constraints C1–C3. And P^{I} meets constraint CC7 if, and only if, P^{II} meets constraint CC7*.

Theorem 7.4 provides the justification for defining Rényi's and Carnap's functions of Type II as we did on p. 32.

Unexpectedly perhaps, but obligingly, $\approx^*$ proves to be the same equivalence relation as indiscernibility *tout court* in the sense of p. 11 and indiscernibility *tout court* in the sense of p. 30.

Theorem 7.5 *Let S be a standard nondegenerate Boolean algebra, and let A and B be distinct members of S. Then there exist a two-valued relative probability function P of Popper's of Type I on S and a two-valued absolute probability function P′ of Popper's of Type I on S such that A and B are discernible under both P and P′.*

Proof: Since $A \neq B$, either $A \neq A \cap B$ or $B \neq A \cap B$. So suppose that $A \neq A \cap B$. Then the set $\{C: (A \cap -B) \cap C = A \cap -B\}$ constitutes a filter on S with respect to $=$, and by the second result from [20] reported on p. 31 that filter can be extended to an ultra-filter S' on S with respect to $=$. So, definable on S' is the two-valued relative probability function P of Popper's in Theorem 7.1, a function of Type I in this case since $\approx$ is $=$. But A belongs to S', $-B$ also does, and hence by clause (iii) in the definition of an ultra-filter B does not. So $P(B,A) = 0$, while $P(A,A) = 1$. Hence $P(A,A) \neq P(B,A)$, and hence A and B are discernible under P. As for the V-restriction P' of P, (i) it constitutes as shown in [12] a two-valued absolute probability function of Popper's of Type I, and (ii) $P'(B) = 0$, while $P'(A) = 1$. So A and B are discernible under P' as well as under P. But, by simply interchanging A and B throughout the argument, so are they when $B \neq A \cap A$. So A and B are discernible under both P and P' in either case.

So:

Theorem 7.6 *Let S and* $\approx^*$ *be as in* Theorem 7.4. *Then*:

(a) *For any A and B in S,* $A \approx^* B$ *if, and only if,* $P(A,C) = P(B,C)$ *for every C in S and every relative probability function P of Popper's of Type II defined on S*;

(b) *For any A and B in S,* $A \approx^* B$ *if, and only if,* $P(C,A) = P(C,B)$ *for every C in S and every relative probability function P of Popper's of Type II defined on S*;

(c) *For any A and B in S,* $A \approx^* B$ *if, and only if,* $P(A) = P(B)$ *for every absolute probability function P of Popper's of Type II defined on S.*

Proof: (a) Suppose *first* that $A \approx^* B$, and let P be an arbitrary relative probability function of Popper's of Type II defined on S. Since indiscernibility under P is by Theorem 5.6 one of the equivalence relations on S with respect to which S constitutes a Boolean algebra and since $\approx^*$ is the least such equivalence relation on S, A and B are sure to be indiscernible under P. So, $P(A,C) = P(B,C)$ for every C in S and every relative probability function P of Popper's of Type II defined on S. Suppose *next* that $A \not\approx^* B$. Then $[A]^* \neq [B]^*$ and there exists by Theorem 7.5 a two-valued relative probability function P^{I} of Type I on $[S]^*$ under which $[A]^*$ and $[B]^*$ are discernible. In that case A and B are discernible under the function P^{II} of Type II that according to Theorem 7.4(a) corresponds to P^{I}. So $P(A,C) \neq P(B,C)$ for at least one C in S and at least one relative probability function P of Popper's of Type II defined on S. (b) By (a) and Theorem 5.11. (c) Suppose *first* that $A \approx^* B$. Then by the same reasoning as under (a), but with P an absolute rather than a relative probability function of Popper's of Type II and with Theorem 6.5 substituting for Theorem 5.6, $P(A) = P(B)$ for every absolute probability function of Popper's of Type II defined on S. Suppose *next* that $A \not\approx^* B$. Then by the same reasoning as under (a), but with clause (b) of Theorem 7.4 substituting for clause (a), $P(A) \neq P(B)$ for at least one absolute probability function of Popper's of Type II defined on S.

So the following will do as characteristic constraints for the relative probability functions of Rényi's and Carnap's of Type II, and the absolute ones of Carnap's of that type:

RB8** *For any A and B in S, if $P(A,C) = P(B,C)$ for every C in S, then $P(A,C) = P(B,C)$ for every such C and every relative probability function P of Popper's of Type II defined on S*;

CB8** *For any A and B in S, if $P(C,A) = P(C,B)$ for every C in S, then $P(C,A) = P(C,B)$ for every such C and every relative probability function P of Popper's of Type II defined on S*; and

CC7** *For any A and B in S, if $P(A) = P(B) = P(A \cap B)$, then $P(A) = P(B)$ for every absolute probability function P of Popper's of Type II defined on S.*

The three constraints are *impredicative*, but they are phrased entirely in terms of properties of the functions P, without explicit reference to features of the set S.

It is an immediate consequence of Theorem 7.6 that, one, indiscernibility *tout court* in the sense of p. 11 and indiscernibility *tout court* in the sense of p. 30 are the same, and, two, in the event that S is a standard Boolean algebra, indiscernibility *tout court* and identity are the same. So, to round off the remarks in Sections 4 and 6:

(i) it is when P is a relative probability function of Rényi's that *left*-indiscernibility *under P* entails indiscernibility *tout court*, and hence—for P of Type I—identity;
(ii) it is when P is a relative probability function of Carnap's that *right*-indiscernibility *under P* entails indiscernibility *tout court*, and hence—for P of Type I—identity; and

(iii) it is when P is an absolute probability function of Carnap's that indiscernibility *under* P entails indiscernibility *tout court*, and hence—for P of Type I—identity.

The extra probability functions of Type II that the account on p. 32 yielded are of particular interest, as we remarked in Section 1, when they are defined on sets of statements. For proof, let S consist of all the statements that can be compounded from given atomic ones by means of '$\sim$' and '&'. S does not constitute a standard Boolean algebra. However, it constitutes a Boolean algebra with respect to a variety of equivalence relations, the least of which is the relation of truth-functional equivalence. So probability functions of Type II can be defined on S, the relative ones of Rényi's meeting this version of RB8′*, in which F is the statement counterpart of Λ:

(i) *For any statement B in S, if $P(A,B) = 1$ for every statement A in S, then B is truth-functionally equivalent to* F,[30]

the relative ones of Carnap's meeting this version of CB8′*, in which T is the statement counterpart of V:

(ii) *For any statement A in S, if $P(A,\mathrm{T}) = 0$, then A is truth-functionally equivalent to* F,[31]

and the absolute ones of Carnap's meeting this version of CC7′*:

(iii) *For any statement A in S, if $P(A) = 0$, then A is truth-functionally equivalent to* F.

Wittgenstein in [21] and Carnap in [2] *first* accord absolute probabilities to the so-called *state-descriptions* in S, i.e. to the statements in S of the sort

$$(\ldots(\pm A_1 \;\&\; \pm A_2) \;\&\ldots) \;\&\; \pm A_n,$$

where (a) $A_1, A_2, \ldots, A_n$ are in some prescribed order the first n atomic statements in S and (b) for each i from 1 through n, $\pm A_i$ is either A_i itself or $\sim A_i$. A being an arbitrary statement in S that is not a state-description in S and B_1, $B_2, \ldots, B_m$ ($m \geq 0$) being the various state-descriptions in S in which A holds, they *then* take the absolute probability of A to be 0 when $m = 0$, otherwise the sum of the absolute probabilities of $B_1, B_2, \ldots, B_m$. Constraints featuring the state-descriptions in S and equivalent to (i)–(iii) can be had for the present functions:

(i′) *For any state-description A in S,* $P(\sim A,A) = 0$

for the relative probability functions of Rényi's of Type II;

(ii′) *For any state-description A in S,* $P(A,T) > 0$

for the relative probability functions of Carnap's of Type II; and

(iii′) *For any state-description A in S,* $P(A) > 0$

for the absolute probability functions of Carnap's of Type II.

The first of these constraints, due to Kit Fine, appears in [13][32]; the other two respectively appear as C53-3 on p. 287 and T55-1 on p. 295 of [2].

Some would rather assign probabilities to propositions than to statements. Their preference is easily met. Consider $[A]^*$, where A is a member of the present set S, i.e. where A is a statement compounded from given atomic statements by means of '$\sim$' and '&'. Owing to Theorem 7.6, $[A]^*$ consists of the statements in S that are indiscernible *tout court* from A. Hence, owing to a theorem of probability semantics, $[A]^*$ consists of the statements in S that are truth-functionally equivalent to A. Hence, under one account of a proposition, $[A]^*$ is *the proposition expressed by A*. So $[S]^*$ consists of the propositions expressed by the statements of S. So the unary real-valued functions defined on $[S]^*$ that meet constraints C1–C3 will assign probabilities to those propositions one at a time, and the binary real-valued functions defined on $[S]^*$ that meet constraints B1–B5 will assign them probabilities two at a time.[33]

Appendix Proofs of the fourteen clauses of Lemma 2 and the eleven clauses of Lemma 4 will be found in the Appendix to [12]. Clauses (c)–(l) of Lemma 3 follow by Theorem 5.7 from the Boolean identities $--A = A$, $A \cap (A \cup B) = A$, etc., proofs of which are left to the reader.

Lemma 1 *Let S be a Boolean algebra with respect to* $\approx$.
(a) $A \cap A \approx A$.

Proof: By A3 by taking B and C there to be A.

(b) $\Lambda \approx A \cap -A$, *and* $\mathrm{V} \approx A \cup -A$, *for any A in S.*

Proof: By (a) $A \cap A \approx A$ for any A in S. Hence by A3 $A \cap -A \approx B \cap -B \approx C \cap -C \approx \ldots$. Hence (b) by the definitions of Λ and V.

Lemma 2 *Let P be a binary real-valued function that meets constraints* B1–B7.
(a) $0 \leq P(A,B) \leq 1$;
(b) *If* $P(A,B) \times P(C,D) = 1$, *then* $P(A,B) = P(C,D) = 1$;
(c) $P(A,A \cap B) = P(A,B \cap A) = 1$;
(d) *If A is P-normal, then* $P(-A,A) = 0$;
(e) *If B is P-normal, then* $P(A \cap -B,B) = P(-B \cap A,B) = 0$;
(f) *If C is P-normal, then* $P(A,C) = P(A \cap B,C) + P(A \cap -B,C)$;
(g) $P(A \cap A,B) = P(A,B)$;
(h) *If B is P-normal, then* $P(A \cap -A,B) = P(\Lambda,B) = 0$;
(i) $P(-(A \cap -A),B) = P(\mathrm{V},B) = 1$;
(j) $P(A,\mathrm{V} \cap B) = P(A,B \cap \mathrm{V}) = P(A,B)$;
(k) V *is P-normal*;
(l) *If* $P(B,\mathrm{V}) \neq 0$, *then* $P(A,B) = P(A \cap B,\mathrm{V})/P(B,\mathrm{V})$;
(m) $P(A \cap B,C) \leq P(A,C)$ *and* $P(A \cap B,C) \leq P(B,C)$;
(n) *If* $P(A,C) = 0$, *then* $P(A \cap B,C) = P(B \cap A,C) = 0$.

Lemma 3 *Let P be a binary real-valued function that meets constraints* B1–B7.
(a) $P(A,B \cap -B) = P(A,\Lambda) = 1$.

Proof: $P(B, B \cap -B) = P(-B, B \cap -B) = 1$ by Lemma 2(c). So $P(A, B \cap -B) = 1$ for every A in S by B4, and hence $P(A, \Lambda) = 1$ by the definition of Λ.

(b) $P(A \cap B, C) = 1$ *if, and only if,* $P(A, C) = P(B, C) = 1$.

Proof: If $P(A \cap B, C) = 1$, then $P(A, C) = P(B, C) = 1$ by Lemmas 2(m) and 2(a). So suppose that $P(A, C) = P(B, C) = 1$, and let C be P-normal. Then $P(-B, C) = 0$ by B4, hence $P(A \cap -B, C) = 0$ by Lemma 2(n), hence $P(A \cap B, C) = P(A, C)$ by Lemma 2(f), and hence $P(A \cap B, C) = 1$. But $P(A \cap B, C) = 1$ by definition when C is P-abnormal. Hence $P(A \cap B, C) = 1$ in either case.

(c) $P(--A, B) = P(A, B)$;
(d) $P(A \cap (A \cup B), C) = P(A, C)$;
(e) $P(B \cap (A \cup B), C) = P(B, C)$;
(f) $P(-A \dot{-} -B, C) = P(A \dot{-} B, C)$;
(g) $P(-(A \dot{-} B), C) = P((A \cap B) \cup (-A \cap -B), C)$;
(h) $P(-A \dot{-} \mathrm{V}, B) = P(A, B)$;
(i) $P(\mathrm{V} \cup A, B) = P(\mathrm{V}, B)$;
(j) $P((A \cap -B) \cap (B \cap -A), C) = P(\Lambda, C)$;
(k) $P((A \cap B) \cap (-A \cap -B), C) = P(\Lambda, C)$;
(l) $P((A \cup B) \cap -A, C) = P(B \cap -A, C)$;
(m) $P(A \cup B, C) = P(A, C) + P(B, C) - P(A \cap B, C)$.

Proof: By definition when C is P-abnormal. So suppose C is P-normal. Then $P(A \cup B, C) = P((A \cup B) \cap A, C) + P((A \cup B) \cap -A, C)$ by Lemma 2(f), hence $P(A \cup B, C) = P(A, C) + P((A \cup B) \cap -A, C)$ by (d) and B6, hence $P(A \cup B, C) = P(A, C) + P(B \cap -A, C)$ by (l), and hence $P(A \cup B, C) = P(A, C) + P(B, C) - P(A \cap B, C)$ by Lemma 2(f).

(n) *If* $P(A \cap B, C) = 0$, *then* $P(A \cup B, C) = P(A, C) + P(B, C)$.

Proof: By (m).

(o) $P(A, C) \leq P(A \cup B, C)$.

Proof: By (m) and Lemma 2(m).

(p) *If C is P-normal, then* $P(A \dot{-} B, C) = P(A \cap -B, C) + P(B \cap -A, C)$.

Proof: If C is P-normal, then $P((A \cap -B) \cap (B \cap -A), C) = 0$ by (j) and Lemma 2(h). Hence (p) by (n) and the definition of $A \dot{-} B$.

(q) $P(A \dot{-} B, C) = 0$ *if, and only if,* $P(A \cap -B, C) = P(B \cap -A, C) = 0$.

Proof: By (p) and B2.

(r) *If C is P-normal, then* $P((A \cap B) \cup (-A \cap -B), C) = P(A \cap B, C) + P(-A \cap -B, C)$.

Proof: Like that of (p) but using (k) in place of (j).

(s) *If A or B is P-normal, then so is $A \cup B$.*

Proof: Suppose $A \cup B$ is P-abnormal. Then by definition $P(C \cap A, A \cup B) = P(A, A \cup B) = 1$ for every C in S, hence by B5 $P(C, A \cap (A \cup B)) = 1$, and hence by (d) and Theorem 3.1 A is P-abnormal. But $P(C, B \cap (A \cup B)) = 1$ by the same reasoning. So by (e) and Theorem 3.1 B is P-abnormal as well.

(t) *If* $P(C,A) = P(C,B)$ *for every* C *in* S *and* A *is* P*-normal, then* $P(-A,B) = P(-B,A) = 0$.

Proof: Suppose $P(C,A) = P(C,B)$ for every C in S, and suppose A is P-normal. Then B is P-normal by definition. Hence $P(-A,A) = P(-B,B) = 0$ by Lemma 2(d), and hence $P(-A,B) = P(-B,A) = 0$.

(u) *If* $P(-A,C) = P(-B,C) = 1$ *for every* C *in* S*, then* $A \cup B$ *is* P*-abnormal.*

Proof: Suppose $P(-A,C) = P(-B,C) = 1$ for every C in S. Then $P(-A, -B \cap (A \cup B)) = P(-B, A \cup B) = 1$, hence $P(-A \cap -B, A \cup B) = 1$ by B5, hence $P(-(A \cup B), A \cup B) = 1$ by (c) and the definition of $A \cup B$, and hence $A \cup B$ is P-abnormal by B3–B4.

(v) *If* $P(A \dot{-} B, \mathrm{V}) = 0$ *and* $P(C, \mathrm{V}) \neq 0$*, then* $P(A,C) = P(B,C)$.

Proof: Suppose $P(A \dot{-} B, \mathrm{V}) = 0$ and $P(C, \mathrm{V}) \neq 0$. Then $P(A \dot{-} B, C) = 0$ by B5 and Lemmas 2(n) and 2(j). Hence $P(A \cap -B, C) = P(B \cap -A, C) = 0$ by (q). So $P(A,C) = P(A \cap B, C)$ and $P(B,C) = P(B \cap A, C)$ by B4. Hence $P(A,C) = P(B,C)$ by Commutation.

Lemma 4 *Let* P *be a unary real-valued function that meets constraints* C1–C6.
(a) $P(A) = P(B \cap A) + P(-B \cap A)$;
(b) $P(A \cap (B \cap C)) = P((A \cap B) \cap C)$;
(c) $P(A \cap B) \leq P(A)$ *and* $P(A \cap B) \leq P(B)$;
(d) $P(A \cap A) = P(A)$;
(e) $P(A \cap -A) = P(\Lambda) = 0$;
(f) *If* $P(A) = 0$*, then* $P(A \cap B) = P(B \cap A) = 0$;
(g) *If* $P(B \cap C) = 0$*, then* $P((A \cap B) \cap C) = 0$;
(h) $P(\Lambda \cap A) = 0$;
(i) $P(-A) = 1 - P(A)$;
(j) $P((A \cap B) \cap C) = P((B \cap A) \cap C)$;
(k) $P(A \cap (B \cap C)) = P(A \cap (C \cap B))$.

Lemma 5 *Let* P *be a unary real-valued function that meets constraints* C1–C6.
(a) $P(A) \leq 1$.

Proof: $0 \leq P(-A)$ by C1. So (a) by Lemma 4(i).

(b) $P(A \cap B) = 1$ *if, and only if,* $P(A) = P(B) = 1$.

Proof: Like that of Lemma 3(b) but using Lemma 4(c), (a), Lemma 4(i), Lemma 4(f), and C3 in place of Lemmas 2(m) and 2(a), B4, and Lemmas 2(n) and 2(f).

(c) $P(A \cup B) = 0$ *if, and only if,* $P(A) = P(B) = 0$.

Proof: By (b), Lemma 4(i), and the definition of $A \cup B$.

(d) *If* $P(A) = P(A \cap B)$*, and* $P(B) = P(B \cap C)$*, then* $P(A) = P(A \cap C)$.

Proof: If $P(A) = P(A \cap B)$, then $P(-B \cap A) = 0$ by Lemma 4(a), hence $P((-B \cap A) \cap C) = 0$ by Lemma 4(f), hence $P(-B \cap (A \cap C)) = 0$ by Lemma 4(b), and hence $P(B \cap (A \cap C)) = P(A \cap C)$ by Lemma 4(a). But, if $P(B) = P(B \cap C)$, then $P(B \cap -C) = 0$ by C3, hence $P(A \cap (B \cap -C)) = 0$ by Lemma 4(f), hence $P((A \cap B) \cap -C) = 0$ by Lemma 4(b), hence $P((B \cap A) \cap -C) = 0$ by Lemma 4(j), hence $P((B \cap A) \cap C) = P(B \cap A)$ by C3, and hence $P(B \cap (A \cap C)) = P(A \cap B)$ by Lemma 4(b) and C4. Hence, if $P(A) = P(A \cap B)$ and $P(B) = P(B \cap C)$, then $P(A \cap C) = P(A \cap B)$, and hence $P(A) = P(A \cap C)$.

(e) $P((A \cap B) \cap C) \leq P(B \cap C)$.

Proof: By Lemmas 4(b) and (c).

(f) $P((A \cap B) \cap C) \leq P(A \cap C)$.

Proof: By (e) and Lemma 4(j).

(g) *If* $P(A) \leq P(\Lambda)$, *then* $P(A) = 0$.

Proof: By Lemma 4(e) and C1.

(h) $P((A \cap B) \cap -A) = 0$.

Proof: By (f) and (g).

(i) $P(A \cap B) = P((A \cap B) \cap A)$.

Proof: By C3 and (h).

(j) $P(A \cap B) = P((A \cap B) \cap B)$.

Proof: $P(B \cap A) = P((B \cap A) \cap B)$ by (i). Hence (j) by C4 and Lemma 4(j).

(k) $P(((A \cap B) \cap B) \cap -A) = 0$.

Proof: By (f) and (g).

(l) $P(A \cap B) = P((A \cap B) \cap (B \cap A))$.

Proof: $P(A \cap B) = P(((A \cap B) \cap B) \cap A)$ by C3 and (j)–(k). Hence (l) by Lemma 4(b).

(m) $P((A \cap (B \cap C)) \cap -A) = 0$.

Proof: By (f) and (g).

(n) $P(A \cap (B \cap C)) = P((A \cap (B \cap C)) \cap A)$.

Proof: By C3 and (m).

(o) $P(((A \cap (B \cap C)) \cap A) \cap -B) = 0$.

Proof: By (e)–(g).

(p) $P(A \cap (B \cap C)) = P((A \cap (B \cap C)) \cap (A \cap B))$.

Proof: $P(A \cap (B \cap C)) = P(((A \cap (B \cap C)) \cap A) \cap B)$ by C3 and (n) and (o). Hence (p) by Lemma 4(b).

(q) $P(((A \cap (B \cap C)) \cap (A \cap B)) \cap -C) = 0$.

Proof: By (e)–(g).

(r) $P(A \cap (B \cap C)) = P((A \cap (B \cap C)) \cap ((A \cap B) \cap C)$.

Proof: $P(A \cap (B \cap C)) = P(((A \cap (B \cap C)) \cap (A \cap B)) \cap C)$ by C3 and (p) and (q). Hence (r) by Lemma 4(b).

(s) $P(((A \cap C) \cap B) \cap -C) = 0$.

Proof: By (e)–(g).

NOTES

1. The epithet "conditional" is used by Kolmogorov, Rényi, and others in place of "relative". It has misled some into thinking of *conditional probabilities* as *the probabilities of conditionals*, and we avoid it. So does Popper. '$P(A,B)$' is commonly read 'the probability of A, given B'.

2. It is deliberately that we talk of A and B as *left-* and *right*-indiscernible under P, violating in the process the use/mention distinction. "A and B are indiscernible *qua first* (*qua second*) arguments of P" would be more correct, but it does not yield such handy substantives as "*left*-indiscernibility" and "*right*-indiscernibility".

3. All the sets considered in this paper will be closed under $-$ and $\cap$.

4. A1–A3 are adaptations of postulates of Byrne's in [1], and A4 and A5 are adaptations of postulates of Rosenbloom's in [19]. Popper in [15] uses adaptations of equivalent postulates due to Huntington and featuring '$\cup$' in place of '$\cap$'. See Note 23 concerning the independence of the postulates.

5. Note that any set S whatsoever that is closed under $-$ and $\cap$ constitutes a Boolean algebra with respect to the universal relation; but that algebra is of course a degenerate one.

6. Rosenbloom, on pp. 9 and 13–14 of [19], explicitly talks of a *Boolean algebra with respect to an equivalence relation*, said relation possibly, but not necessarily, the identity relation. Popper on p. 356 of [15] talks of substitutional equivalence as an equivalence relation, but goes on to define

$$A = B$$

as

$$P(A,C) = P(B,C) \textit{ for every } C \textit{ in } S.$$

This has misled many, possibly Popper himself, into thinking of the sets on which his probability functions are defined as *standard* Boolean algebras. *That* they need not be. The set considered on p. 19 constitutes a Boolean algebra in Popper's sense. Yet it has three distinct members, and hence is *not* a standard Boolean algebra.

7. A Boolean algebra that is not degenerate is sure of course to have more than one member.

8. $[S]$ is what is often called the *quotient-algebra* with respect to the ideal $\Delta = \{A \in S : A \approx \Lambda\}$. Though a set of sets, $[S]$ is *not*, by the way, a field of sets: $-[A]$ is not the set-theoretic complement of $[A]$, nor is $[A] \cap [B]$ the set-theoretic intersection of $[A]$ and $[B]$. For proof, $[A] \cap -[A]$ is not the empty set. Identical

with $[A] \cap [-A]$, and hence with $[A \cap -A]$, it consists of every B in S such that $B \approx A \cap -A$, hence of $A \cap -A$ plus possibly other members of S.

9. Popper's own constraints concerned the functions in Section 5; they will be found on p. 332 of [15]. For more about them see Note 10.

10. Proof that B1–B7 are equivalent to Popper's own constraints in [15] will be found in [5], as will be proof that each of B1–B7 is independent of the remaining six. For more about Popper's own constraints and their independence, see p. 10.

11. It is of course to preserve consistency that a restriction is placed on B in B4. Due to B3, B5, and Lemma 2(a)

$$P(A,\Lambda) = P(-A,\Lambda) = 1,$$

as the proof of Lemma 2(c) in [12] bears out. So, were no restriction placed on B in B4, $P(A,\Lambda)$ would equal 0 as well as 1. But Popper requires of B that it be P-normal. Rényi and Carnap require that it be distinct from Λ, a stronger restriction. Proof that Rényi's functions, as understood here, are those meeting B1–B3, B5, and this version of Complementation:

B4′ *For any A and B in S, if $B \neq \Lambda$ then $P(A,B) = 1 - P(A,B)$,*

will be found in [12]. For more on Rényi's and Carnap's functions see pp. 9–10. See also [9] and [10].

12. In each case $P'(B)$ reappears as $P(B,V)$. Note that when $P'(B) \neq 0$, $P(A,B)$ equals $P'(A \cap B)/P'(B)$ by Lemma 2(l). In that case the value of P for any A and B in S is thus determined by the values of P' for $A \cap B$ and B.

13. In his subsequent work [17] Rényi required of S' that $A \cup B$ belong to it if both A and B do. We cannot pursue the matter here.

14. Functions essentially like those in [4], but defined on the statements of languages with the familiar connectives '$\sim$' and '&', appear in the earlier [2]. We shall consider them on pp. 37–38.

15. On p. 332 of [15] Popper stated *in effect* that B1–B7 deliver (**), but gave no proof of it. The one we supply on pp. 9–10 first appeared in [7]. It is, to our knowledge, the earliest and the shortest proof in print of Popper's Constraint. (*) appears in the Appendix as Lemma 2(m).

16. Incidentally,

$$P(C,A) = P(C,B) \textit{ for every } C \textit{ in } S$$

does not entail

$$P(C,-A) = P(C,-B) \textit{ for every such } C,$$

nor as a result

$$P(C,f(A)) = P(C,f(B)) \textit{ for every } C \textit{ in } S \textit{ and for every Boolean function } f \textit{ on } S.$$

17. It follows of course from RB8 that

$$\textit{If } A \neq B, \textit{ then } P(A,C) \neq P(B,C) \textit{ for at least one } C \textit{ in } S.$$

So, if like Rényi and Carnap we required of every Boolean algebra that it be non-degenerate, then at least two members of the standard Boolean algebra on which a

Rényi—and, hence, a Carnap—function P is defined would have to be discernible under P, and like Rényi and Carnap we could dispense with constraint B1.

18. See [5] for proof of that.

19. Popper talked of

$$\{\langle A, B\rangle : P(A, C) = P(B, C) \textit{ for every } C \textit{ in } S\}$$

in [15] *not* as the relation of *identity* but as a relation of *substitutional equivalence.* But, as indicated in Note 6, by abridging

$$P(A, C) = P(B, C) \textit{ for every } C \textit{ in } S$$

as

$$A = B,$$

he blurred the distinction between substitutional equivalence and identity, hence that between a Boolean algebra *in general* and a *standard* one.

20. On pp. 351–352 of [15] Popper supplies a proof of Theorem 5.4 that in effect does not use B7. So an analogue of Popper's Theorem can be had with P a relative probability function meeting just B1–B6 and *left*-indiscernibility under P substituting for indiscernibility *tout court* under P. Intriguingly, Popper was weary of using (**), a constraint we dubbed in Section 3 Popper's Constraint. *Right*-indiscernibility under P also constitutes an equivalence relation on S, but in view of Note 16 S does not constitute a nondegenerate Boolean algebra with respect to it. See Note 22 for more on this matter.

21. $[S]$ is thus the quotient-algebra with respect to the ideal $\Delta = \{A \in S : A \text{ is } P\text{-abnormal}\}$.

22. And *right*-indiscernibility as well as *left*-indiscernibility under *every* such function is one of the equivalence relations with respect to which the set on which the function is defined constitutes a Boolean algebra.

23. Popper's own constraints concerned the functions on p. 26; they will be found in [14]. Proof that C1–C3 and constraints C4–C6 on p. 26 are equivalent to Popper's constraints will be found in [8]. Each of C1–C3 is independent of the remaining five.

24. The illustration is from [18]. Elliott Mendelson pointed out in a letter that it shows A5′ independent of A1′–A4′. As noted in [1], each of A1′–A3′ is independent of the other two. Whether A4′ is independent of A1′–A3′ and A5′ is still an open question, it appears.

25. $[S]$ is thus the quotient-algebra relative to the ideal $\Delta = \{A \in S : A \text{ is of } P_V\text{-probability } 0\}$.

26. The reductions of relative probability functions of Popper's that are *not* Rényi ones match those functions one-to-one. *Not so* the reductions of Popper functions that are *not* Carnap ones. But, given the remarks on pp. 8–9, that was to be expected: those among Popper's absolute probability functions that are *not* Carnap ones do not match one-to-one the relative probability functions under Cases 3 and 4 there. Our proof that Rényi's functions reduce to Carnap ones is in answer to a question by Peter Schotch.

27. Since $V \in S'$, P meets KB8′ as well, and hence is a Kolmogorov function.

28. The analogue of Popper's Theorem that we mentioned in Note 20 makes for an analogue of Theorem 7.2 with B1–B6 in place of B1–B7.

29. It should be noted that *not* all sets closed under $-$ and $\cap$ and of cardinality larger than 1 constitute nondegenerate Boolean algebras. Consider the three-membered set $\{a, b, c\}$, and let $-a = b$, $-b = c$, and $-c = a$. Then, however $\cap$ be defined and whatever equivalence relation on $\{a, b, c\}$ $\approx$ might be, $A \approx B$ is sure to hold for any A and B in S.

30. Equivalently, *only statements in S that are truth-functionally equivalent to* F *are P-abnormal.* The constraint appears as B8 in [5], where the functions meeting it are ascribed to the Carnap of [3], a text in which Carnap and Rényi are in accord.

31. Equivalently, *only statements in S that are truth-functionally equivalent to* F *are of* P_{T}*-probability zero.*

32. The functions meeting (i′) are ascribed in [13] to the Carnap of [2]. They should have been ascribed to the Carnap of [3] or—as here—to Rényi.

33. This closing paragraph is in answer to a question of Nicholas Asher's.

REFERENCES

[1] Byrne, L., "Two brief formulations of Boolean algebra," *Bulletin of the American Mathematical Society*, vol. 52 (1946), pp. 269–272.

[2] Carnap, R., *Logical Foundations of Probability*, University of Chicago Press, Chicago, Illinois, 1950.

[3] Carnap, R., *The Continuum of Inductive Methods*, University of Chicago Press, Chicago, 1952.

[4] Carnap, R. and R. C. Jeffrey (editors), *Studies in Inductive Logic and Probability*, vol. 1, University of California Press, Berkeley, 1971.

[5] Harper, W. L., H. Leblanc, and B. C. van Fraassen, "On characterizing Popper and Carnap probability functions," pp. 140–152 in *Essays in Epistemology and Semantics*, Haven Publications, New York, 1983.

[6] Kolmogorov, A. N., "Grundbegriffe der Wahrscheinlichkeitsrechnung," *Ergebnisse der Mathematik*, vol. 2 (1933), pp. 1–61. Translated as *Foundations of the Theory of Probability* by N. Morrison, Chelsea, New York, 1950.

[7] Leblanc, H., "What price substitutivity: A note on probability theory," *Philosophy of Science*, vol. 48 (1981), pp. 317–322.

[8] Leblanc, H., "Popper's 1955 axiomatization of absolute probability," *Pacific Philosophical Quarterly*, vol. 63 (1982), pp. 133–145.

[9] Leblanc, H., "The autonomy of probability theory: Notes on Kolmogorov, Rényi, and Popper," *The British Journal for the Philosophy of Science*, vol. 39 (1989), pp. 167–181.

[10] Leblanc, H., "Popper's formal contributions to probability theory," pp. 341–367 in *Perspectives on Psychologism*, edited by M. A. Notturno, E. J. Brill, Leyden, 1989.

[11] Leblanc, H. and P. Roeper, "Absolute probability functions: A recursive and autonomous account," pp. 484–494 in *The Tasks of Contemporary Philosophy, Proceedings of the 10th International Wittgenstein Symposium*, Vienna, 1986.

[12] Leblanc, H. and P. Roeper, "On relativizing Kolmogorov's absolute probability functions," *Notre Dame Journal of Formal Logic*, vol. 30 (1989), pp. 485–512.

[13] Leblanc, H. and B. C. van Fraassen, "A note on Popper and Carnap probability functions," *The Journal of Symbolic Logic*, vol. 44 (1979), pp. 369–373.

[14] Popper, K., "Two autonomous axiom systems for the calculus of probabilities," *The British Journal for the Philosophy of Science*, vol. 6 (1955), pp. 51–57.

[15] Popper, K., *The Logic of Scientific Discovery*, Basic Books, New York, 1959.

[16] Rényi, A., "On a new axiomatic theory of probability," *Acta Mathematica Academiae Scientiarum Hungaricae*, vol. 6 (1955), pp. 286–335.

[17] Rényi, A., "Sur les espaces simples des probabilités conditionnelles," *Annales de l'Institut Poincaré*, N Série, Section B1, 1964, pp. 3–21.

[18] Rescher, N., "A probabilistic approach to modal logic," *Modal and Many-Valued Logics, Acta Philosophica Fennica*, vol. 16 (1963), pp. 215–226.

[19] Rosenbloom, P. C., *The Elements of Mathematical Logic*, Dover Publications, New York, 1950.

[20] Sikorski, R., *Boolean Algebras*, Springer-Verlag, Berlin, 1960.

[21] Wittgenstein, L., *Tractatus Logico-Philosophicus*, Routledge & Kegan Paul, London, 1922.

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