Notre Dame Journal of Formal Logic
Volume 34, Number 3, Summer 1993

Significant Parts and Identity of Artifacts

ATHANASSIOS TZOUVARAS

Abstract By assigning numerical values to the atomic parts of a given artifact and, then, assigning the maximum of them to the artifact itself, we get a reasonable notion of *significance* for the parts of artifacts. Using this notion one can define artifact *identity* in a precise way. Namely, the identity is preserved exactly when all the significant parts are preserved. We show that this notion of identity has all the basic properties that one would intuitively expect. A limit case is also considered.

1 Preliminaries In this note we are going to investigate certain aspects of the identity of artifacts using elementary logical means, i.e. some formal predicates modeling the basic relations among artifacts and ordinary predicate calculus. Such a treatment of identity began in Tzouvaras [2].

We shall use a soft formalization only—just what will allow us to be precise and brief. In [2], however, one can find a full formal treatment of everything concerning transformations and identity of artifacts.

Lower case variables x, y, z, dots will denote artifacts (called also simply "objects"). To be more precise, x, y range over *states* of objects, if by "object" we understand something existing in time and thus changing, yet keeping its identity. It is better to think of x, y as *instances* of such identities. Upper case letters $X, Y, Z, \ldots$ denote sets of arbitrary objects. The notation and concepts of intuitive set theory are freely used throughout this article. We have just two fundamental relations among artifacts by which we can express almost everything about them: first, a binary relation called the *fitness relation* and denoted $\mathcal{F}$, and, second, a binary *assembly operation*, denoted $\square$. Their meaning is:

$x\mathcal{F}y$: *The objects x,y fit and may be assembled into a new object.*
$x\square y = z$: *z is the assembly of x,y when the latter fit.*

Thus the first principles governing these relations are the following (we state them in the form of axioms):

Received September 9, 1992; revised March 5, 1993

(O_1) $x\mathfrak{F}y \iff (\exists z)(x\Box y = z)$.

(O_2) $x\Box y = y\Box x$.

(O_3) $x\Box y = x'\Box y' \Rightarrow \{x, y\} = \{x', y'\}$.

Remark 1.1 Concerning O_1 it has to be noted that $x\mathfrak{F}y$ says that the objects x, y *may* be assembled. Therefore O_1 asserts a third object z *could* be the result of the assembly, not that this object is actually made out there. Another option would be to identify the assembled object $x\Box y$ with the "dismantled" object $\{x, y\}$, saying that they both constitute this third object z in two different states. More details can be found in [2].

Remark 1.2 A word of caution is needed for the meaning of O_3. It is connected with the explication given above about what x, y denote. O_3 does *not* say that an object can be decomposed in only one way, i.e., that we cannot make the same artifact in two different ways. Rather O_3 says that we cannot make the same artifact in two different ways *at the same time*.

If $z = x\Box y$, then x, y are said to be *immediate parts* of z and we write $x <_0 z$, $y <_0 z$. x is a *part* of y if there is a finite sequence $(x_0, x_1, \dots, x_n)$ (where n is a natural number) such that

$$x = x_0 <_0 x_1 <_0 \cdots <_0 x_n = y$$

We write that $x \leq y$ for either $x < y$ or $x = y$. If $x < y$, x is a *proper part* of y. x is an *atom* if it has no proper parts. We denote $\Pi(x)$ and $\Pi_0(x)$ the sets of parts and of atomic parts of x, respectively.

(O_4) *Foundation principle: Every object is analyzed into a finite number of atomic parts.*

Foundation allows for inductive treatment of objects in the obvious sense: if a property φ in the language of artifacts holds of all atomic objects and if holding of x, y implies holding of $x\Box y$, then φ holds of every object. Similarly with definitions by recursion.

Two objects x, y are said to be *copies* of one another, or *replicas* or *spare parts*, if each one of them fits precisely wherever the other does, i.e., if the one can replace the other in any assembly of parts. We denote this fact by $x \cong y$. Clearly, $\cong$ is an equivalence relation that can be defined strictly in terms of $\mathfrak{F}$ and $\Box$. If $x < y$ and $x \cong x'$, we write $y[x'/x]$ for the result of replacement of x by x' inside y. The following facts can be proved using the strict definitions of $\cong$ and $<$.

Lemma 1.3 *(i)* $x \leq y \;\&\; x \cong x' \Rightarrow y \cong y[x'/x]$. *(ii)* $x \leq y \leq z \;\&\; x \cong x' \Rightarrow z[x'/x] = z[y[x'/x]/y]$.

Another principle warranting uniformity in the formation of similar objects is the following:

(O_5) *If $x \cong x'$ and $x = y\Box z$, then there are objects y', z' such that $y \cong y'$, $z \cong z'$ and $x' = y'\Box z'$.*

Finally we add the principle below stating that *overlapping objects* (i.e., objects having parts in common) *do not fit:*

(O_6) $(\exists z)(z \leq x \;\&\; z \leq y) \Rightarrow \neg(x\mathfrak{F}y)$.

This seems fairly plausible, since such objects cannot be used *at the same time* for the formation of a third object. As a consequence, $\neg(x\mathcal{F}x)$, i.e., $\mathcal{F}$ is antireflexive. Of course an object x may very well fit a *replica of itself* $x' \cong x$, such that $x \neq x'$, since x, x' are now independent coexisting entities. O_6 will be used to show property I_7 of Proposition 2.4.

Remark 1.4 Axioms O_1–O_6 comprise almost all of the principles one needs for a discussion of the basic aspects of artifact behavior. The full list of axioms contained in [2] contains just one or two more principles which are inessential for the present discussion.

2 Identity and significance of parts The identity problem which we are dealing with here is that arising when an artifact undergoes replacements of parts. No other forms of change, such as those due to decay, damage, etc., are to be taken into account.

We write $I(x, y)$ for the fact that the object-instances x, y are of the same identity. Of course $I(x, y)$ is a relation that has to be specified. In [2], for example, the following particular notion of identity, denoted $\doteq$ has been used:

$$x \doteq y \textit{ iff either } |\Pi_0(x)| \textit{ is standard and } \Pi_0(x) = \Pi_0(y), \textit{ or } \\ |\Pi_0(x)| \textit{ is nonstandard and } |\Pi_0(x)\Delta\Pi_0(y)| \textit{ is standard.}$$

(Here $|\,|$ denotes cardinality and Δ symmetric difference. It is also assumed that our set of natural numbers $\mathbb{N}$ is nonstandard. The interested reader may consult [2] for the role that nonstandardness may play in transformations of artifacts.)

This definition, though it captures a good deal of the actual identity, has a serious drawback: it considers all atomic parts of an object as equally important. Specifically, if the object has few parts, all of them are supposed to be important (in the sense that the replacement of any of them changes the identity), whereas if it has too many parts, then none of them is important.

This is, clearly, unrealistic since we can hardly find artifacts with unclassified parts form the point of view of some notion of importance. A screw, for instance, can never be on par with the frame of, say, a car.

A realistic notion of identity should distinguish the parts of an object into *significant* and *nonsignificant* ones in a nontrivial way. Conversely, given a notion of significance one can define through it a notion of identity. We shall see presently how this can be done.

Let $S(x, y)$ be a new binary predicate symbol intended to mean "x is a significant part of y," and added to our basic language $\mathcal{L} = \{\mathcal{F}, \square\}$. Talking about parts in general and x being a part of itself, one would be absolutely justified to assert that $S(x, x)$ should hold for every x. On the other hand, the reduction to parts is useful only because the reduction reaches atomic parts. This is why we shall be mainly concerned with the significance of *atomic* parts. The significance of the rest can be reduced to that of the latter.

So let *Atom* denote the set of all atomic objects of our domain of discourse, i.e.,

$$Atom = \{x : (\forall y, z)(x \neq y\square z)\}.$$

The identity predicate $I(x, y)$ corresponding to S is first-order definable in the language $\mathcal{L} \cup \{S\}$ as follows.

Definition 2.1 Let $\Pi_s(x) = \{y \in Atom : S(x, y)\}$ be the set of significant atomic parts of x. Then $I(x, y)$ holds if $\Pi_s(x) = \Pi_s(y)$, i.e., if the objects x, y have precisely the same significant parts.

On the other hand, given a predicate $I(x, y)$ for identity, i.e., one intended to mean "x, y are of the same identity," the significance predicate $S(x, y)$ corresponding to I is first-order definable in $\mathcal{L} \cup \{I\}$ as follows.

Definition 2.2 $S(x, y)$ holds if x is a part of y and if we replace x by a copy x' of identity distinct from that of x, then the resulting object $y[x'/x]$ is of identity distinct from that of y. Symbolically,

$$x \leq y \;\&\; (\forall x')(x' \cong x \;\&\; \neg I(x, x') \Rightarrow \neg I(y, y[x'/x])).$$

The above definitions show how the notions of identity and significance can be interwoven and be reduced to one another. Our feeling, however, is that the various decisions on the preservation of artifacts' identity are based on some pre-existent criteria of significance (concerning the replaceable parts) rather than the other way around. That is to say, the idea of significance seems to be more primitive and fundamental. Consequently, we consider as more natural to start with S rather than I. Our intention is to postulate a few principles about S which will imply the basic intuitive, as well as other less obvious, properties of I. We propose as axioms for S the following five principles, S_1–S_5.

(S_1) *Every object has significant atomic parts. In symbols:*

$$(\forall x)(\Pi_s(x) \neq \varnothing).$$

(S_2) *There is no shift of significance, i.e., if an object keeps its parts, then it also keeps its significant ones. In symbols:*

$$\Pi_0(x) = \Pi_0(y) \Rightarrow \Pi_s(x) = \Pi_s(y).$$

(S_3) *Significance is a transitive relation, i.e.,*

$$S(x, y) \;\&\; S(y, z) \Rightarrow S(x, z).$$

(S_4) *The converse of S_3 holds: if an object x is significant in another z, then for any intermediate object y, x is significant in y and y is significant in z. In symbols:*

$$S(x, z) \;\&\; x \leq y \leq z \Rightarrow S(x, y) \;\&\; S(y, z)$$

(S_5) *Copies preserve the significance relation. In symbols:*

$$S(x, y) \;\&\; x \cong x' \Rightarrow S(x', y[x'/x]).$$

The following lemma will be useful below.

Lemma 2.3 *Let $x \leq y$. Then either $S(x, y)$ and $\Pi_s(x) \subseteq \Pi_s(y)$ or $\neg S(x, y)$ and $\Pi_0(x) \cap \Pi_s(y) = \varnothing$. Consequently, $S(x, y) \iff x \leq y \;\&\; \Pi_s(s) \subseteq \Pi_s(y)$.*

Proof: Let $S(x, y)$. Then every significant part of x is a significant part of y by S_3. Suppose $\neg S(x, y)$ and z is an atom of x significant in y. Then x is intermediate between z and y. If $S(z, x)$ then, by S_4, $S(x, y)$, a contradiction. If $\neg S(z, x)$ then, by S_4 again, $\neg S(z, y)$ a contradiction.

Define, then, in $\mathcal{L} \cup \{S\}$ the predicate $I(x, y)$ as in Definition 2.1. The predicate $I(x, y)$ is a reasonable notion of identity, since it satisfies most of the intuitive properties of what we currently understand by "object identity." These are summarized in the next proposition.

Proposition 2.4 *The following properties hold of I:*

(I_1) $I(x, y)$ *is an equivalence.*
(I_2) $\Pi_0(x) = \Pi_0(y) \Rightarrow I(x, y)$.
(I_3) $\Pi_0(x) \cap \Pi_0(y) = \varnothing \Rightarrow \neg I(x, y)$.
(I_4) $x \leq y \;\&\; x \cong x' \;\&\; I(x, x') \Rightarrow I(y, y[x'/x])$.
(I_5) $x \leq y \;\&\; (\exists x')(x' \cong x \;\&\; \neg I(y, y[x'/x])) \Rightarrow S(x, y)$.
(I_6) $x \leq y \;\&\; I(x, y) \Rightarrow S(x, y)$.
(I_7) $I(x, x\Box y) \Rightarrow S(x, x\Box y) \;\&\; \neg S(y, x\Box y)$.

Proof: I_1 is immediate from the definition of I.

I_2: Let $\Pi_0(x) = \Pi_0(y)$. By S_2, $\Pi_s(x) = \Pi_s(y)$, hence $I(x, y)$.

I_3: Equivalently we have to show that $I(x, y) \Rightarrow \Pi_0(x) \cap \Pi_0(y) \neq \varnothing$. If $I(x, y)$ then $\Pi_s(x) = \Pi_s(y) = u$, hence $u \subseteq \Pi_0(x) \cap \Pi_0(y)$. Since, by S_1, $u \neq \varnothing$, the claim follows.

I_4: Let $x \leq y$, $x \cong x'$, and $I(x, x')$, that is, $\Pi_s(x) = \Pi_s(x')$. Put $y' = y[x'/x]$. We have to show that $\Pi_s(y) = \Pi_s(y')$. Suppose first that $\neg S(x, y)$. By Lemma 2.3, no atoms of x belong to $\Pi_s(y)$, therefore obviously $\Pi_s(y) = \Pi_s(y')$. Now let $S(x, y)$. Then by the lemma $\Pi_s(x) \subseteq \Pi_s(y)$. Let $\Pi_s(y) = \Pi_s(x) \cup X$. Then clearly $\Pi_s(y') = \Pi_s(x') \cup X$. Since $\Pi_s(x') = \Pi_s(x)$, it follows that $\Pi_s(y) = \Pi_s(y')$.

I_5: Assume $x \leq y$ and for some $x' \cong x$, $\neg I(y, y')$, that is, $\Pi_s(y) \neq \Pi_s(y')$ (where $y' = y[x'/x]$). We have to show that $S(x, y)$. Suppose $\neg S(x, y)$. By the lemma, $\Pi_0(x) \cap \Pi_s(y) = \varnothing$. It suffices to show that also $\Pi_0(x') \cap \Pi_s(y') = \varnothing$. If this holds, then clearly $\Pi_s(y) = \Pi_s(y')$, reaching a contradiction. Let $\Pi_0(x') \neq \Pi_s(y') \neq \varnothing$. Then by the lemma $S(x', y')$, and by axiom S_5 $S(x, y)$, a contradiction.

I_6: Let $x \leq y$ and $I(x, y)$. Then $\Pi_s(x) = \Pi_s(y)$. Hence by Lemma 2.3 $S(x, y)$.

I_7: Let $I(x, x\Box y)$. It follows by I_6 that $S(x, x\Box y)$ and $\Pi_s(x\Box y) = \Pi_s(x)$. If, moreover, $S(y, x\Box y)$, then also $\Pi_s(y) = \Pi_s(x\Box y)$. Hence $\Pi_s(x) = \Pi_s(y)$. But x, y fit, by assumption, forming $x\Box y$, and so by axiom O_6 they cannot have any parts in common. This shows that $\neg S(y, x\Box y)$.

Remark 2.5 Properties I_2 and I_3, though seemingly natural, do not enjoy universal acceptance, in particular by those philosophers who base their arguments about identity preservation heavily on principles of spatiotemporal continuity. For example, the solution Lowe proposes to the ship of Theseus puzzle in [1] violates both I_2 and I_3, since he declares identical two ships having no part in common and nonidentical two others having exactly the same parts. (Cf. also Section 1 of [2] where the same puzzle is discussed.)

Remark 2.6 The identity $\doteq$ clearly satisfies I_1–I_4.

3 Valuations There is a simple way to obtain significance relations, i.e., relations satisfying the principles S_1–S_5. It will be described in the present section. The idea is simple. Assign to every atomic object a number representing, intuitively, a degree of "ontological importance" for it. To a complex object assign, then, the maximum of the degrees of its parts. Finally, a part of an object is significant in it with respect to a given assignment, if its degree is equal to that of the object itself.

Definition 3.1 A *valuation* is any function v assigning to each atomic object x a natural number $v(x)$, with the only condition being that if $x \cong x'$, then $v(x) = v(x')$. Given a valuation v we extend it to the whole class of objects by putting $v(x) = max\{v(y) : y \in \Pi_0(x)\}$. We denote also by v the extended valuation.

The proof of the following is straightforward using induction on object formation and axiom O_5:

Lemma 3.2 *For all x, y and for any valuation v:*
(i) $v(x \Box y) = max\{v(x), v(y)\}$,
(ii) $x \cong y \Rightarrow v(x) = v(y)$,
(iii) $x \leq y \Rightarrow v(x) \leq v(y)$.

Definition 3.3 Given a valuation v, let $S_v(x, y)$, or simply $S(x, y)$, denote the relation:

$$x \leq y \ \&\ v(x) = v(y).$$

It is quite easy to verify that S has all the properties S_1–S_5, so we obtain the following result.

Proposition 3.4 *For any valuation v, S_v is a significance relation.*

Proof: S_1: Since $v(x) = max\{v(y) : y \in \Pi_0(x)\}$, there is a $y \in \Pi_0(x)$ such that $v(x) = v(y)$. Thus $S(y, x)$, whence it follows that $\Pi_s(x) \neq \varnothing$.

S_2: So long as v is kept fixed, clearly there is no shift in the meaning of S_v.

S_3: Let $S(x, y)$ and $S(y, z)$. Then $x \leq y \leq z$ and $v(x) = v(y) = v(z)$. Thus $S(x, z)$.

S_4: Let $x \leq y \leq z$ and $S(x, z)$. Then by Lemma 3.2, $v(x) \leq v(y) \leq v(z)$ and $v(x) = v(z)$. Therefore $v(x) = v(y) = v(z)$, whence $S(x, y)$ and $S(y, z)$.

S_5: Let $S(x, y)$ and $x \cong x'$. By Lemma 1.3 it is also true that $y \cong y[x'/x]$. By Lemma 3.2, $v(x) = v(x')$ and $v(y) = v(y[x'/x])$. Therefore:

$$S(x, y) \iff v(x) = v(y) \iff v(x') = v(y[x'/x]) \iff S(x', y[x'/x]).$$

Remark 3.5 The referee pointed out that although the notion of the significance of an object x is *relative* to an object y, of which x constitutes a part, the valuations as defined above assign *absolute* degrees of ontological importance. As a consequence, not all significance relations are derivable from valuations. The referee also provided the following specific example of relative significance which cannot be induced from valuations: consider atoms of three kinds, as, bs and cs, such that,

(i) no non-atom fits anything;

(ii) the only possible assemblies are of the forms $a \Box b$, $b \Box c$, and $c \Box a$; and

(iii) only a is significant in $a \Box b$, only b is significant in $b \Box c$, and only c is significant in $c \Box a$.

If v were a valuation producing the preceding relation, then we would have $v(a) < v(b)$, $v(b) < v(c)$, and $v(c) < v(a)$, which is impossible.

Remark 3.6 The valuation as defined above has two aspects: first its definition on the class of atomic objects, and second its extension on the class of all objects. The valuation of atomic objects is "arbitrary." We do not intend here to enter a discussion as to what a "correct valuation" should be like. It is evident that such a notion would be based on extra-logical criteria, such as substance, size, durability, etc., of the objects in question.

Remark 3.7 The second aspect, the way we assign value to a comples object as a specific function of the values of its parts, gives us some insight as to how the reduction of significance to that of the parts could be approached mathematically: *maximization* is such an approach. Whereas other approaches, for example by taking sums or suprema (in a partially ordered set instead of $\mathbb{N}$), do not seem to be appropriate with respect to the principles S_1–S_5. To be specific, by taking sums or suprema instead of maxima, S_1 no longer holds. The value of $v(x)$ is, then, as a rule, strictly greater than the values of all parts of x.

Remark 3.8 On the other hand, as is easily seen from the proof of Proposition 3.4, the remaining principles S_2–S_5 are satisfied if instead of the valuations as defined above, we use arbitrary assignments of v from the class of objects into a partially ordered set $(X, \leq)$ subject only to the constraints:

(i) $x \cong y \Rightarrow v(x) = v(y)$, and

(ii) monotonicity, i.e., $x \leq y \Rightarrow v(x) \leq v(y)$.

However S_1 is crucial for the notion of identity if the latter is to be based on significance. If S_1 fails, there are objects with no significant parts and thus with undecidable identity. If this happens only for a *few* objects, it can be faced as in the limit case examined in the sequel, since possessing only insignificant parts is practically the same as possessing only parts of equal significance. But if this is the rule, significance becomes trivial.

4 A limit case Limit cases always bewilder everybody trying to model real phenomena. In our case, limit cases can even be *constructed ad hoc* in order to falsify as counterexamples any proposed formal view. For instance, what if an object (e.g. a Lego toy) is formed by a large number of indistinguishable small pieces? Since the pieces are all copies of one another, they take the same value in any valuation and this value is obviously assigned also to the object itself. Hence, each one of the pieces forming the object in question is a significant part of it with respect to any valuation. As a result, replacement of even one of them by a copy, should result to change of identity of the object (even if thousands or millions of pieces participate in its formation). This conclusion is rather counterintuitive.

A remedy for such situations could be to base identity on a double criterion using alternatively either the significance of the parts or the relation $\doteq$, i.e. the *number* of common parts, whenever all parts are of equal significance. Of course this presupposes that $\mathbb{N}$ is nonstandard. To be precise to cover the limit case one can put:

$$\begin{aligned} I(x, y) \quad \text{if} \quad & (\Pi_s(x) \neq \Pi_0(x) \;\&\; \Pi_s(x) \neq \Pi_s(y)) \;\; \text{or} \\ & (\Pi_s(x) = \Pi_0(x) \;\&\; \Pi_s(y) = \Pi_0(y) \;\&\; x \doteq y). \end{aligned}$$

Another way to cope with the limit case would be distinguishing between a, so to speak, "strict identity"—where all the parts are significant–and a "near enough" or "approximate identity," that seems to be closer to our intuitions—requiring merely that the vast majority of significant parts be retained. However, "vast majority" is a vague predicate, not entailing transitivity of the corresponding identity, whereas in real life, in spite of vagueness, transitivity is unquestionable. (No one is expected to admit that while a, b are (states of) the same object and b, c are the same object too, nevertheless a and c are distinct!). The problem has to do with the faithful representation of vagueness and exceeds the scope of this article. I would just say that, in my view, the only formal tools which, at present, allow us to preserve both vagueness and transitivity when facing real equivalence relations are the nonstandard sets of naturals numbers (called also "cuts"). Thus we return again to relations like $\doteq$ above. The payoff, however, for using such relations are their inability to be applied to concrete situations. Cuts exist only for external observers of our actual world. The here-and-now inhabitants can hardly perceive them.

Acknowledgment I would like to thank an anonymous referee for constructive and helpful comments.

REFERENCES

[1] Lowe, E. J., "On the identity of artifacts," *Journal of Philosophy*, vol. 80 (1983), pp. 220–232.

[2] Tzouvaras, A., "A formal theory of artifacts," preprint.

Department of Mathematics
University of Thessaloniki
540 06 Thessaloniki
Greece