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Extended Gentzen-type Formulations of Two Temporal Logics Based on Incomplete Knowledge Systems

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Abstract Nakamura proposed two three-valued temporal logics. We present two extended Gentzen-type formulations of these logics. Then we prove the soundness as well as the completeness theorem.

1 Introduction

Two temporal logics based on incomplete knowledge systems were formulated by Nakamura [6]. One of them is a 3-valued temporal logic in which the present knowledge can be changeable. This logic (denoted by **3-TL**) is a kind of three-valued modal logic. In [6] Nakamura left an open problem to axiomatize incompletely valid well-formed formulas where he defined A is incompletely valid if and only if A has the value 1 or 2 in all worlds of all models with reflexive linearly ordered time (Definition 3.1).

Another of them is a three-valued temporal logic in which determined knowledge does not change in the future but only unknown knowledge can come to be determined in the future. This logic (denoted by **L-TL**) is motivated by the concept of completion which was introduced in Lipski [3]. The main purpose of this paper is to present extended Gentzen-type formulations of **3-TL** and **L-TL**. After giving syntax and semantics, we present two formal systems in Gentzen style. Then we prove the soundness as well as the completeness theorem. Following Nakamura, we review here the background of **3-TL** and **L-TL** (see [6]).

Definition 1.1 (Definition of an incomplete temporal information system) An incomplete temporal information system is a system $S = (\mathrm{OB}, \mathrm{AT}, \mathrm{T}, \{\mathrm{VAL}_a\}_{a\in\mathrm{AT}}, f)$ where

1. OB is a set of objects,

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2. AT is a set of attributes,
3. T is a linearly ordered set whose elements are called moments of time,
4. $\mathrm{VAL}_a = \{1, 3\}$ is a set of values of attributes for $a \in \mathrm{AT}$, and VAL is the union of all the sets VAL_a,
5. f is a function from $\mathrm{OB} \times \mathrm{AT} \times \mathrm{T}$ into $\mathrm{VAL} \cup \{2\}$.

Example 1.2 An incomplete temporal information system

$$S = (\mathrm{OB}, \mathrm{AT}, \mathrm{T}, \{\mathrm{VAL}_a\}_{a \in \mathrm{AT}}, f) \quad \mathrm{OB} = \{o_1, o_2, o_3\}, \mathrm{AT} = \{a, b\} \text{ and } f$$

is given in the following table. Here 1 means 'true', 2 means 'indeterminable' or 'unknown', and 3 means 'false'.

$t_1 \in \mathrm{T}$			$t_2 \in \mathrm{T}$		
OB \ AT	a	b	OB \ AT	a	b
o_1	1	3	o_1	1	3
o_2	1	2	o_2	2	3
o_3	2	3	o_3	2	2

Note that $f(o_2, b, t_1) < f(o_2, b, t_2)$ but $f(o_3, b, t_1) > f(o_3, b, t_2)$, that is, the value of attributes to objects depends on time. The value can be different in various moments. Also Lipski discussed the incomplete model such that determined knowledge does not change in the future but only unknown knowledge can come to be determinable in the future.

Example 1.3 An incomplete temporal information system

$$S = (\mathrm{OB}, \mathrm{AT}, \mathrm{T}, \{\mathrm{VAL}_a\}_{a \in \mathrm{AT}}, f) \quad \mathrm{OB} = \{o_1, o_2, o_3\}, \mathrm{AT} = \{a, b\} \text{ and } f$$

is given in the table.

$t_1 \in \mathrm{T}$			$t_2 \in \mathrm{T}$			$t_3 \in \mathrm{T}$		
OB \ AT	a	b	OB \ AT	a	b	OB \ AT	a	b
o_1	1	3	o_1	1	3	o_1	1	3
o_2	1	2	o_2	1	1	o_2	1	1
o_3	2	3	o_3	2	3	o_3	3	3

Note determined values (1 and 3) of attributes to objects don't change in the future but only the undetermined value (2) can come to be the determined value in the future.

We want to formalize the incomplete temporal information systems of Examples 1.2 and 1.3.

2 Matrices

We take $1, 2, 3$ as *truth values*. Let $\mathrm{T} = \{1, 2, 3\}$ be the set of all truth values. Elements of T are denoted by $k, m, \ldots$. Intuitively '1' stands for 'true', '2' stands for 'indeterminable' or 'unknown', and '3' stands for 'false'.

We consider four classes of symbols:

1. Propositional variables: $p, q, r, \ldots$;
2. Propositional connectives: $\neg, \Rightarrow_1, \Rightarrow_2, \Rightarrow_3, \Rightarrow_4$, and $G(*_1, \ldots, *_k)$. With each $G(*_1, \ldots, *_k)$ we associate a function g from T^k into T. We call g the truth function of $G(*_1, \ldots, *_k)$;
3. Modal symbols: [ ] (every time in the future);
4. Auxiliary symbols: (,).

Definition 2.1 (Definition of a formula)

1. A propositional variable is a formula.
2. If A and B are formulas, then $\neg A$, $A \Rightarrow_1 B$, $A \Rightarrow_2 B$, $A \Rightarrow_3 B$, $A \Rightarrow_4 B$, and $[\,]A$ are formulas.
3. If $A_1, \dots, A_{k-1}$ and A_k are formulas, then $G(A_1, \dots, A_k)$ is a formula.

A Gentzen's sequent $A_1, \dots, A_m \to B_1, \dots, B_n$ means intuitively that some formula of $A_1, \dots, A_m$ is false or some formula of $B_1, \dots, B_n$ is true. The truth value 1 corresponds to the succedent and the truth value 3 corresponds to the antecedent. We extend the notion of a sequent to three-valued case.

Definition 2.2 (Definition of a matrix) A *valued formula* is a pair consisting of a formula and a truth value. We call the following finite set of valued formulas a *matrix*: $\{(A_1, m_1), \dots, (A_k, m_k)\}$. We call A_1 or $\cdots$ or A_k the m_1-part of this matrix or $\cdots$ or m_k-part of this matrix, respectively. Intuitively the matrix $\{(A_1, m_1), \dots, (A_k, m_k)\}$ means that A_j has the truth m_j for some $j = 1, \dots, k$.

Abbreviations 2.3 In the following, '$K, L, \dots$' denote matrices, 'Γ, Γ_1' denote finite (possibly empty) sets of formulas, and 'A, B' denote formulas.

1. Let $S \subset T$. The matrix $\{(A, m); A \in \Gamma, m \in S\}$ is abbreviated as (Γ, S), $(\{A\}, S)$ as (A, S), $(\Gamma, \{m\})$ as (Γ, m), $(\Gamma \cup \{A\}, m)$ as $(\{\Gamma, A\}, m)$, $(A, T - \{m\})$ as $(A, \underline{m})$ and $K \cup \{(A, m)\}$ as $K \cup (A, m)$, respectively.
2. We define $K \subset L$, if and only if for all $m \in T$ every formula that occurs in the m-part of K also occurs in the m-part of L.

3 Model

Definition 3.1 (Definition of a 3-TL model) A **3-TL** *model* is a triplet (W, R, φ) where

1. W is a nonempty set,
2. R is a linear order on W, that is, R is a reflexive, transitive, and connected relation on W,
3. φ is a function which assigns a truth value to each pair consisting of a propositional variable and an element of W.

We extend φ to all formulas by induction as follows.

1. $\varphi(\neg A, s) = 4 - \varphi(A, s)$,
2. $\varphi(A \Rightarrow_i B, s) = \varphi(A, s) \to_i \varphi(B, s) \qquad (i = 1, 2, 3, 4)$,
3. $\varphi(G(A_1, \dots, A_k), s) = g(\varphi(A_1, s), \dots, \varphi(A_k, s))$,
4. $\varphi([\,]A, s) = \mathrm{Max}\{\varphi(A, t); sRt\}$, where$\to_i$ is given in the following table.

	$\to_1$				$\to_2$				$\to_3$				$\to_4$		
$A \backslash B$	1	2	3	$A \backslash B$	1	2	3	$A \backslash B$	1	2	3	$A \backslash B$	1	2	3
1	1	2	3	1	1	1	3	1	1	2	3	1	1	1	3
2	1	1	1	2	1	1	3	2	1	2	2	2	1	1	1
3	1	1	1	3	1	1	1	3	1	1	1	3	1	1	1

Definition 3.2 (Definition of an L-TL model) An **L-TL** *model* is obtained from a **3-TL** model by adding the following conditions.

Condition 3.3 If $\varphi(A, s) = 1$ and sRt, then $\varphi(A, t) = 1$.

Condition 3.4 If $\varphi(A, s) = 3$ and sRt, then $\varphi(A, t) = 3$.

Condition 3.5 If $\varphi(A, s) = 2$, then there exists an element $t(sRt)$ in W such that $\varphi(A, t) = 1$ or $\varphi(A, t) = 3$.

Definition 3.6 A matrix L is defined to be *valid* in G if for every G model (W, R, φ) and every s in W, there exists a formula A such that

$$(A, \varphi(A, s)) \in L(G = \textbf{3-TL}, \textbf{L-TL}).$$

In the case of $T = \{1, 3\}$, if we regard truth values $1, 3$ as t(truth), f(false), respectively, and a matrix $(\{A_1, \ldots, A_m\}, 3) \cup (\{B_1, \ldots, B_n\}, 1)$ as a sequent $A_1, \ldots, A_m \to B_1, \ldots, B_n$, this definition is consistent with the usual definition of validity of a sequent (see Takahashi [7].

4 Formal Systems

We introduce the formal system **3-TL** and the formal system **L-TL**. The formal systems are constituted by their axioms and their inference rules.

4.1 3-valued temporal logic (3-TL)

Axioms

1. (beginning matrix) (A, T).
2. $(\{[\,]([\,]A \Rightarrow_1 B), [\,]([\,]B \Rightarrow_1 A)\}, 1)$.
3. $(\{[\,]([\,]A \Rightarrow_2 B), [\,]([\,]B \Rightarrow_2 A)\}, \{1, 2\})$.
4. $([\,]([\,]A \Rightarrow_3 B), 1) \cup ([\,]([\,]B \Rightarrow_4 A), \{1, 2\})$.

Inference Rules

1. Weakening $\dfrac{L}{K}$ where $L \subset K$.
2. Inferences for logical connectives

   (a) Negation $\neg$ $\quad \dfrac{L \cup (A, m)}{L \cup (\neg A, 4 - m).}$

   (b) Implication $\Rightarrow_i$ $(i = 1, 2, 3, 4)$

   $$\frac{L \cup (A, m), \quad L \cup (B, n)}{L \cup (A \Rightarrow_i B, m \to_i n)}$$

   (c) Logical connective $G(*_1, \ldots, *_k)$

   $$\frac{L \cup (A_1, m_1), \ldots, L \cup (A_k, m_k)}{L \cup (G(A_1, \ldots, A_k), g(m_1, \ldots, m_k))}$$

3. Cut $\quad \dfrac{L \cup (A, m), \quad K \cup (A, n)}{L \cup K}$ where $m \neq n$

4. Inferences for modal operations

   In the following by $[\,]\Gamma$ we mean a set of formulas which are formed by prefixing $[\,]$ in front of each formula occuring in Γ.

   (a)
$$\frac{\bigcup_{k\neq m}(\{A,[\,]\Gamma_k\},k)\cup([\,]\Gamma_m,m)}{\bigcup_{k\neq m}(\{[\,]A,[\,]\Gamma_k\},k)\cup([\,]\Gamma_m,m)} \quad \text{where } \varnothing=\Gamma_1\subset\Gamma_2\subset\Gamma_3.$$

   (b)
$$\frac{\bigcup_{k=1}^{3}(\{\Gamma_k,\Delta_k\},k)}{\bigcup_{k=1}^{3}(\{[\,]\Gamma_k,\Delta_k\},k)} \quad \text{where } \varnothing=\Gamma_1\subset\Gamma_2\subset\Gamma_3.$$

   (c)
$$\frac{\bigcup_{k=1}^{m}(\{A,[\,]\Gamma_k\},k)\cup\bigcup_{k=m+1}^{3}([\,]\Gamma_k,k)}{\bigcup_{k=1}^{m}(\{[\,]A,[\,]\Gamma_k\},k)\cup\bigcup_{k=m+1}^{3}([\,]\Gamma_k,k)} \text{ where } \varnothing=\Gamma_1\subset\Gamma_2\subset\Gamma_3.$$

Axioms 2, 3, and 4 are the extension of the axiom $[\,]([\,]A\Rightarrow B)\vee[\,]([\,]B\Rightarrow A)$ in Goldblatt ([1], §4.3).

Inferences for modal operations are inferences in a many-valued **S4**-modal logic (see Morikawa [4] and [5]).

4.2 Lipski's logic (L-TL) **L-TL** is obtained from **3-TL** by adding the following axioms and inference rule.

Axioms

5. $([\,]A,1)\cup(A,\{2,3\})$.
6. $([\,]\neg A,1)\cup(A,\{1,2\})$.

Inference rule

4. Inferences for modal operations

   (d)
$$\frac{(\{A,\Gamma_2\},2)\cup(\{A,\Gamma_3\},3)\quad (A,1)\cup(\{A,\Gamma_2\},2)\cup(\Gamma_3,3)}{(A,1)\cup([\,]\Gamma_2,2)\cup(\{A,[\,]\Gamma_3,3)}$$

   where $\Gamma_2\subset\Gamma_3$.

Definition 4.1 (Definition of provable matrices) A matrix is provable in G if it is obtained from axioms by a finite number of applications of the above inference rules ($G=$ **3-TL**, **L-TL**).

Lemma 4.2

1. *The matrix* $(A,\{1,\ldots,m\})\cup([\,]A,\{m+1,\ldots,3\})$ *is provable in* **3-TL** *and* **L-TL**.
2. *The matrix* $([\,][\,]A,\{1,\ldots,m\})\cup([\,]A,\{m+1,\ldots,3\})$ *is provable in* **3-TL** *and* **L-TL**.
3. *The matrix* $([\,]\neg A,1)\cup(\neg A,\{2,3\})$ *is provable in* **L-TL**.
4. *The matrix* $(A,2)\cup(\{A,\neg A\},3)$ *is provable in* **3-TL** *and* **L-TL**.

5. *The matrix* $(\{A, \neg A\}, 1) \cup (\neg A, 2)$ *is provable in* **3-TL** *and* **L-TL**.
6. *The matrix* $(A, 1) \cup (\neg A, 2) \cup (A, 3)$ *is provable in* **3-TL** *and* **L-TL**.
7. *The matrix* $(\neg A, 1) \cup (A, 2) \cup (\neg A, 3)$ *is provable in* **3-TL** *and* **L-TL**.
8. *The matrix* $(\{A, \neg A\}, 1) \cup (A, 2)$ *is provable in* **3-TL** *and* **L-TL**.
9. *The matrix* $(\neg A, 2) \cup (\{\neg A, A\}, 3)$ *is provable in* **3-TL** *and* **L-TL**.
10. *The matrix* $(B, 1) \cup (\{A, A \Rightarrow_1 B\}, \{2, 3\})$ *is provable in* **3-TL** *and* **L-TL**.
11. *The matrix* $(B, \{1, 2\}) \cup (\{A, A \Rightarrow_2 B\}, 3)$ *is provable in* **3-TL** *and* **L-TL**.
12. *The matrix* $(B, 1) \cup (\{A, B\}, 2) \cup (\{A, A \Rightarrow_3 B\}, 3)$ *is provable in* **3-TL** *and* **L-TL**.
13. *The matrix* $(B, 1) \cup (A \Rightarrow_3 B, 2) \cup (\{A, A \Rightarrow_3 B\}, 3)$ *is provable in* **3-TL** *and* **L-TL**.
14. *The matrix* $(B, 1) \cup (\{A, B\}, 2) \cup (\{A, A \Rightarrow_4 B\}, 3)$ *is provable in* **3-TL** *and* **L-TL**.
15. *The following inference rule is admissible in* **3-TL** *and* **L-TL**.

$$\frac{\bigcup_{k \neq m}(\{A, \Gamma_k\}, k) \cup (\Gamma_m, m)}{\bigcup_{k \neq m}(\{[\,]A, [\,]\Gamma_k\}, k) \cup ([\,]\Gamma_m, m)} \qquad \textit{where } \varnothing = \Gamma_1 \subset \Gamma_2 \subset \Gamma_3.$$

Proof We prove (13). Other cases are similar.

1. $(\{A, B\}, 1) \cup (A, 2) \cup (\{A, B\}, 3)$ is provable in G;
2. $(\{A, B\}, 1) \cup (B, 2) \cup (\{A, B\}, 3)$ is provable in G; so by inference rule $\Rightarrow_3$, [1], and [2],
3. $(\{A, B\}, 1) \cup (A \Rightarrow_3 B, 2) \cup (\{A, B\}, 3)$ is provable in G; also by inference rule $\Rightarrow_3$, [2], and [3],
4. $(B, 1) \cup (A \Rightarrow_3 B, 2) \cup (\{A, B\}, 3)$ is provable in G; hence by inference rule $\Rightarrow_3$, [2], and [3],
5. $(\{A, B\}, 1) \cup (A \Rightarrow_3 B, 2) \cup (A, 3)$ is provable in G; therefore by inference rule $\Rightarrow_3$, [4], and [5], $(B, 1) \cup (A \Rightarrow_3 B, 2) \cup (\{A, A \Rightarrow_3 B\}, 3)$ is provable in G.

□

Theorem 4.3 (Soundness Theorem) *If a matrix is provable in* **3-TL** *or* **L-TL**, *it is valid in* **3-TL** *or* **L-TL**, *respectively.*

Proof It can easily be proved by the induction on the construction of a proof of the given matrix. □

5 Completeness Theorem

Abbreviations 5.1

1. We denote a set of formulas occurring in the m-part of L by 'L_m'. $L_m \cap L_n$ is denoted by 'L_{mn}'. The complement of L_m is denoted by '$L_{\underline{m}}$'.
2. By '$\Gamma^{[\,]}$' we mean a set of formulas A such that $[\,]A$ occurs in Γ. $(L_m)^{[\,]}$ is abbreviated as '$\Gamma_m^{[\,]}$' and $(L_m^{[\,]})^{[\,]}$ as '$L_m^{[\,][\,]}$'.

Lemma 5.2 *If L is unprovable in G, then for any formula A, there exists an $m \in T$ such that $L \cup (A, \underline{m})$ is unprovable in G.*

Proof By using the cut inference rules, it is easily proved (see [7]). □

Definition 5.3 Let the matrix K be fixed. We denote the set of all subformulas of all formulas occurring in K by $FL(K)$. If the matrix L is unprovable in G and for any $A \in FL(K)$there exists an $m \in T$ such that $A \in L_{\underline{m}}$, we say that L is G-complete. We denote the set of all G-complete matrices by $C_G(K)$. We can easily prove the following lemmas (see [4], [5], [7]).

Lemma 5.4 *For any $A \in FL(K)$ and $L \in C_G(K)$, $A \in L_{\underline{m}}$ if and only if $L \cup (A, m)$ is provable in G.*

Lemma 5.5 *If $L \in C_G(K)$, then for any $A \in FL(K)$ there exists one and only one $m \in T$ satisfying $A \in L_{\underline{m}}$.*

Lemma 5.6 *For any $L \in C_G(K)$, $L_{\underline{k}} = L_{mn}$ where k, m and n are distinct.*

Lemma 5.7 (Lindenbaum's Lemma) *If L is unprovable in G, then there exists a matrix M such that $M \in C_G(K)$ and $L \subset M$.*

From Lemma 4.2 we can prove the following lemma.

Lemma 5.8 *For any $L \in C_G(K)$ and $A, B \in FL(K)$,*

1. $L_{23}^{[\,]} \subset L_{23}$.
2. $L_{3}^{[\,]} \subset L_{3}$.
3. $L_{23}^{[\,]} \subset L_{23}^{[\,][\,]}$.
4. $L_{3}^{[\,]} \subset L_{3}^{[\,][\,]}$.
5. *$A \in L_{\underline{1}}$ if and only if $\neg A \in L_{\underline{3}}$.*
6. *$A \in L_{\underline{2}}$ if and only if $\neg A \in L_{\underline{2}}$.*
7. *$A \in L_{\underline{3}}$ if and only if $\neg A \in L_{\underline{1}}$.*
8. *If $A \in L_{\underline{1}}$ and $A \Rightarrow_1 B \in L_{\underline{1}}$, then $B \in L_{\underline{1}}$.*
9. *If $A \in L_3$ and $A \Rightarrow_2 B \in L_3$, then $B \in L_3$.*
10. *If $A \in L_{\underline{1}}$ and $A \Rightarrow_3 B \in L_3$, then $B \in L_3$.*
11. *If $A \in L_3$ and$A \Rightarrow_3 B \in L_{\underline{1}}$, then$B \in L_{\underline{1}}$.*
12. *If $A \in L_{\underline{1}}$ and $A \Rightarrow_4 B \in L_3$, then$B \in L_3$.*

Lemma 5.9 *For any $L \in C_{L-TL}(K)$, $A \in FL(K)$,*

1. *If $\neg A \in L_{\underline{1}}$, then $[\,]\neg A \in L_{\underline{1}}$.*
2. *If $A \in L_{\underline{1}}$, then $[\,]A \in L_{\underline{1}}$.*

We prove the completeness theorem by the powerful method of the canonical model (see [4], [5], [7]).

Definition 5.10 Let a G-complete matrix K be fixed. We define the canonical model $M_G = (W', R', \phi')$ as follows:

1. $W' = \{L \in C_G(K);\ K_{23}^{[\,]} \subset L_{23}$ and $K_{3}^{[\,]} \subset L_3\}$.

2. For any $L, M \in W'$, $LR'M$ iff $L_{23}^{[\,]} \subset M_{23}$ and $L_3^{[\,]} \subset M_3$.
3. For any $L \in W'$ and any propositional variable p, $\phi'(p, L) = m$ iff $p \in L_{\underline{m}}$.

Lemma 5.11 *W' is a nonempty set.*

Proof Let $M_k = K_{k,k+1,\ldots,3}$ for any $k \in T$. Then $\varnothing = M_1 \subset M_2 \subset M_3$ and for any $k \in T$, $M_k \subset K_k$. Since K is unprovable,

$$(\{[\,]A; [\,]A \in M_2\}, 2) \cup (\{[\,]A; [\,]A \in M_3\}, 3)$$

is unprovable. By using inference rule 4.2, $(\{A; A \in M_2^{[\,]}\}, 2) \cup (\{A; A \in M_3^{[\,]}, 3)$ is unprovable. By Lemma 5.7 there exists a complete matrix L such that

$$M_2^{[\,]} = K_{23}^{[\,]} \subset L_2 \text{ and } M_3^{[\,]} \subset L_3.$$

Hence $W' \neq \varnothing$. □

Lemma 5.12 *For any $L, M \in W'$, $LR'M$, or $MR'L$.*

Proof Suppose that neither $LR'M$ nor $MR'L$ holds. We have four cases.

Case 1 There exist a formula A and a formula B which satisfy

$$[\,]A \in L_{23}, A \notin M_{23}, [\,]B \in M_{23},$$

and $B \notin L_{23}$. By Axiom (2) in Section 4.1 $[\,]([\,]A \Rightarrow_1 B) \in K_{23}$ or $[\,]([\,]B \Rightarrow_1 A) \in K_{23}$. If $[\,]([\,]A \Rightarrow_1 B) \in K_{23}$ then by assumption $([\,]A \Rightarrow_1 B) \in L_{23}$. So by Lemma 5.8(8) $B \in L_{23}$. This is a contradiction. If $[\,]([\,]B \Rightarrow_1 A) \in K_{23}$, then by assumption $([\,]B \Rightarrow_1 A) \in M_{23}$. So by Lemma 5.8(8) $A \in M_{23}$. This is a contradiction.

Case 2 There exist a formula A and a formula B which satisfy $[\,]A \in L_{23}$, $A \notin M_{23}$, $[\,]B \in M_3$, and $B \notin L_3$. By Axiom (4) in Section 4.1 $[\,]([\,]A \Rightarrow_4 B) \in K_3$ or $[\,]([\,]B \Rightarrow_3 A) \in K_{23}$. If $[\,]([\,]A \Rightarrow_4 B) \in K_3$ then by assumption $([\,]A \Rightarrow_4 B) \in L_3$. So by Lemma 5.8(12) $B \in L_3$. This is a contradiction. If $[\,]([\,]B \Rightarrow_3 A) \in K_{23}$ then by assumption $([\,]B \Rightarrow_3 A) \in M_{23}$. So by Lemma 5.8(11) $A \in M_{23}$. This is a contradiction.

Case 3 There exist a formula A and a formula B which satisfy$[\,]A \in L_3$, $A \notin M_3$, $[\,]B \in M_{23}$, and $B \notin L_{23}$. We can prove it similarly.

Case 4 There exist a formula A and a formula B which satisfy $[\,]A \in L_3$, $A \notin M_3$, $[\,]B \in M_3$, and $B \notin L_3$. By Axiom (3) in Section 4.1 $[\,]([\,]A \Rightarrow_2 B) \in K_3$ or $[\,]([\,]B \Rightarrow_2 A) \in K_3$. If $[\,]([\,]A \Rightarrow_2 B) \in K_3$ then by assumption $([\,]A \Rightarrow_2 B) \in L_3$. So by Lemma 5.8(9) $B \in L_3$. This is a contradiction.

If $[\,]([\,]B \Rightarrow_2 A) \in K_3$ then by assumption $([\,]B \Rightarrow_2 A) \in M_3$. So by Lemma 5.8(9) $A \in M_3$. This is a contradiction. □

From Lemma 5.8(1), (2) and Lemma 5.12 we can get the following lemma.

Lemma 5.13 *A relation R' on W' is a linear order.*

Lemma 5.14 *For any $L, M \in C_{L-TL}(K)$ and $A \in FL(K)$,*

1. *If $\neg A \in L_{\underline{1}}$, then $[\,]\neg A \in L_{\underline{1}}$.*
2. *If $A \in L_{\underline{1}}$, then $[\,]A \in L_{\underline{1}}$.*
3. *If $A \in L_{\underline{1}}$ and $LR'M$, then $A \in M_{\underline{1}}$.*
4. *If $A \in L_{\underline{3}}$ and $LR'M$, then $A \in M_{\underline{3}}$.*

5. *If $A \in L_{\underline{2}}$, then there exists N such that $LR'N$ either $A \in N_{\underline{1}}$ or $A \in N_{\underline{3}}$.*

Proof By Lemma 5.9(1) and (2) we can easily prove (1) and (2). Therefore we prove (5).

Let $M_k = L_{k,k+1,\dots,3}$ for any $k \in T$. Then $\emptyset = M_1 \subset M_2 \subset M_3$ and for any $k \in T$, $M_k \subset L_k$. Since L is unprovable,

$$(A, 1) \cup (\{[\,]B; [\,]B \in M_2\}, 2) \cup (\{A, [\,]B; [\,]B \in M_3)$$

is unprovable in **L-TL**. By using inference rule 4.4, either

$$(A, 1) \cup (\{A, C; C \in M_2^{[\,]}\}, 2) \cup (\{C; C \in M_3^{[\,]}\}, 3)$$

is unprovable in **L-TL** or

$$(\{A, C; C \in M_2^{[\,]}, 2) \cup (\{A, C; C \in M_3^{[\,]}\}, 3)$$

is unprovable in **L-TL**. By Lemma 5.7 either there exists a complete matrix $N \in C_{L-TL}(K)$ such that $A \in N_{\underline{3}}$, $M_2^{[\,]} = L_{23}^{[\,]} \subset N_2$ and $M_3^{[\,]} = L_3^{[\,]} \subset N_3$ or there exists a complete matrix $N \in C_{L-TL}(K)$ such that $A \in N_{\underline{1}}$, $M_2^{[\,]} = L_{23}^{[\,]} \subset N_2$ and $M_3^{[\,]} = L_3^{[\,]} \subset N_{\underline{3}}$. Hence there exists N such that $LR'N$ either $A \in N_{\underline{1}}$ or $A \in N\underline{3}$. □

Lemma 5.15 *For any $L \in W'$, $A \in FL(K)$ and $m = 1, 2, 3$ if $A \in L_{\underline{m}}$, then $\phi'(A, L) = m$.*

Proof We prove it by induction on the length of A. We consider only the case of $A = []B$. In other cases we can prove it as in [7].

(1) $m = 1$: Suppose $[\,]B \in L_{\underline{1}} = L_{23}$. For any M such that $LR'M$, $B \in M_{\underline{1}} = M_{23}$. By the induction hypothesis, $\phi'(B, L) = 1$. Hence $\phi'([\,]B, L) = 1$.

(2) $m = 2$: Suppose $[\,]B \in L_{\underline{2}} = L_{13}$. Since $L_{13} \subset L_3$,

$$B \in M_3 = M_{13} \cup M_{23} = M_{\underline{2}} \cup M_{\underline{1}}$$

for any M such that $LR'M$. By the induction hypothesis $\phi'(B, M) \leq 2$. Let $M_k = L_{k,k+1,\dots,3}$ for any $k \in T$. Then $\emptyset = M_1 \subset M_2 \subset M_3$ and for any $k \in T$, $M_k \subset L_k$. Since L is unprovable,

$$([\,]B, 1) \cup (\{[\,]C; [\,]C \in M_2\}, 2) \cup (\{[\,]B, \{[\,]C; [\,]C \in M_3\}\}, 3)$$

is unprovable. By Lemma 4.2(3) $(B, 1) \cup (\{C; C \in M_2^{[\,]}\}, 2) \cup (\{B, \{C; C \in M_3^{[\,]}\}\}, 3)$ is unprovable. By Lemma 5.7 there exists a matrix $N \in C(K)$ such that $B \in N_{\underline{2}}$, $M_2^{[\,]} = L_{23}^{[\,]} \subset N_2$ and $M_3^{[\,]} = L_3^{[\,]} \subset N_3$. By the induction hypothesis there exists N such that $LR'N$ and $\phi'(B, N) = 2$. Hence $\phi'([\,]B, N) = 2$.

(3) $m = 3$: We can prove it similarly. □

Therefore we can prove the following lemma and theorem.

Lemma 5.16 *A canonical model M_{L-TL} is an* **L-TL** *model.*

Theorem 5.17 (Main Theorem, Completeness Theorem) *If a matrix is valid in G, it is provable in $G (G =$* **3-TL, L-TL***).*

Remark 5.18 In [2] Hajek gave a formal system (**MTL**) to axiomatize 1-tautologies of a many-valued tense logic with reflexive linearly preorder time, that is, A is provable in **MTL** if and only if A has the value 1 in all worlds of all models with reflexive linearly ordered time (Definition 3.1). In [6] Nakamura left an open problem to axiomatize incompletely valid well-formed formulas, where he defined A is incompletely valid if and only if A has the value 1 or 2 in all worlds of all models with reflexive linearly ordered time (Definition 3.1). The Main Theorem 5.17 is an extension of Hajek's **MTL** and a solution of Nakamura's open problem.

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