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A FOURTH-ORDER EQUATION WITH CRITICAL GROWTH: THE EFFECT OF THE DOMAIN TOPOLOGY

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Abstract. In this paper we prove the existence of multiple classical solutions for the fourth-order problem

$$\begin{cases} \Delta^2 u = \mu u + u^{2_*-1} & \text{in } \Omega, \\ u, \quad -\Delta u > 0 & \text{in } \Omega, \\ u, \quad \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 8$, $2_* = 2N/(N-4)$ and $\mu_1(\Omega)$ is the first eigenvalue of Δ^2 in $H^2(\Omega) \cap H_0^1(\Omega)$. We prove that there exists $0 < \overline{\mu} < \mu_1(\Omega)$ such that, for each $0 < \mu < \overline{\mu}$, the problem has at least $\mathrm{cat}_\Omega(\Omega)$ solutions.

1. Introduction

Brézis and Nirenberg [8] investigated the question about the existence of a classical solution for the second-order problem

$$\text{(BN)} \qquad \begin{cases} -\Delta u = \lambda u + u^{2^*-1}, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where $2^* = 2N/(N-2)$, $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. Let $\lambda_1(\Omega)$ be the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. It was proved in [8] that:

(a) (BN) has no solution for $\lambda \geq \lambda_1(\Omega)$. If Ω is also starshaped, then the Pohožaev identity [22] guarantees that (BN) has no solution for $\lambda \leq 0$.
(b) For $N \geq 4$ the problem (BN) has a solution for every $0 < \lambda < \lambda_1(\Omega)$.
(c) In case $N = 3$, also called the critical dimensional case, the problem is more complex. Indeed, in case Ω is starshaped, (BN) has no solution when the parameter λ is positive and small enough and, in the particular case when Ω is an open ball, (BN) has a solution if, and only if, $\lambda_1(\Omega)/4 < \lambda < \lambda_1(\Omega)$.

In contrast to the case when Ω is starshaped, consider $N \geq 3$ and a ring $\Omega \subset \mathbb{R}^N$. We know that the embedding $H^1_{0,\mathrm{rad}}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is compact; see Ni [21, Radial Lemma]. Hence, (BN) has a radial solution for every $\lambda \in (-\infty, \lambda_1(\Omega))$.

The above description shows that the shape of Ω and the dimension N interfere in the set of solutions for (BN). Rey [23], [25] observed that the number of solutions of (BN) is strongly influenced by the topology of Ω. Indeed, using arguments based on the Lusternik–Schnirelman category, it was proved by Rey [23] for $N \geq 5$, after by Lazzo [17] for $N \geq 4$, that (BN) has at least $\mathrm{cat}_\Omega(\Omega)$ solutions if the parameter $\lambda > 0$ is sufficiently small.

When using the Lusternik–Schnirelman theory to get the existence of multiple solutions for the problem (BN), the topological arguments applied require that λ be positive and close to zero. In particular, such procedure only works for non-critical dimensions.

In this paper, also inspired by the just described results, we study the existence of multiple classical solutions for the fourth-order problem

$$\text{(P)} \qquad \begin{cases} \Delta^2 u = \mu u + u^{2_*-1} & \text{in } \Omega, \\ u, \quad -\Delta u > 0 & \text{in } \Omega, \\ u, \quad \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 8$, $0 < \mu < \mu_1(\Omega)$, $\mu_1(\Omega)$ is the first eigenvalue of $(\Delta^2, E(\Omega))$, $E(\Omega) := H^2(\Omega) \cap H_0^1(\Omega)$, and $2_* = 2N/(N-4)$ is the critical exponent for the embedding of $E(\Omega)$ into $L^{2_*}(\Omega)$.

In [27], van der Vorst proved that if $N \geq 5$, $\mu \geq \mu_1(\Omega)$ or, $\mu \leq 0$ and if the domain Ω is starshaped, then (P) has no solution. In the same paper, assuming that Ω is a general bounded regular domain in $\mathbb{R}^N$, $N \geq 8$ and $\mu \in (0, \mu_1(\Omega))$, it was proved that (P) has a solution. Later, Gazzola et al. [13] proved that $N = 5, 6, 7$ are the critical dimensions for the problem (P) in the sense that (P) has no solution if $\mu > 0$ is small enough and Ω is an open ball in $\mathbb{R}^N$.

Our main contribution in this paper is to present a result on the existence of multiple solutions for (P) for all non-critical dimensions, namely, for all $N \geq 8$.

Theorem 1.1. *If Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 8$, then there exists $0 < \overline{\mu} < \mu_1(\Omega)$ such that, for each $0 < \mu < \overline{\mu}$, the problem* (P) *has at least* $\mathrm{cat}_\Omega(\Omega)$ *classical solutions.*

We mention that El-Mehdi and Selmi [11], inspired by the procedures adopted in [23]–[25] to deal with (BN), proved that for $N > 8$ the problem (P) has at least $\mathrm{cat}_\Omega(\Omega)$ solutions if the parameter $\mu > 0$ is sufficiently small.

More recently, Abdelhedi [1] used similar techniques to those in [11] to prove the existence of multiple solutions for a similar problem.

We stress that the condition $N > 8$ seems essential in the arguments in [1] and [11] as well as $N \geq 5$ was required by Rey in [23]. In particular, it has been left as open problem the influence of the domain topology on the existence of multiple solutions for problem (P) in case $N = 8$; see [11, Remark 1.4].

To prove our result we use a different approach from that in [1, 11], which seems more direct and works for $N \geq 8$. We must also say that instead of projections we employ suitable extensions; for instance compare [11, p. 419] and (4.3) in this paper. In addition, we believe that the extension and symmetrization techniques in this paper for functions in $H^2(\Omega) \cap H_0^1(\Omega)$ will be useful to treat other fourth-order problems. In particular, the proofs of Lemmas 4.4, 4.6 and equation (4.9) exemplify how our extension procedure replaces the standard extension by zero used to deal with second-order problems.

This manuscript is organized as follows. In Section 2 we set the variational framework. In Section 3 we prove some compactness results and then we prove Theorem 1.1 in Section 4. We also include an appendix within we prove some technical results from Sections 3 and 4.

2. Variational framework

We first fix some notations. We consider the space $E(\Omega) := H^2(\Omega) \cap H_0^1(\Omega)$ endowed with the norm $\|u\| := |\Delta u|_2$, induced by the inner product

$$\langle u, v \rangle = \int_\Omega \Delta u \Delta v \, dx, \quad u, v \in E(\Omega).$$

In this part we will consider the following general assumptions: $\Omega \subset \mathbb{R}^N$, $N \geq 5$, is a bounded smooth domain and

$$0 < \mu < \mu_1(\Omega) = \inf_{\substack{u \in E(\Omega) \\ u \neq 0}} \frac{|\Delta u|_2^2}{|u|_2^2} = \inf_{\substack{u \in E(\Omega) \\ |u|_2 = 1}} |\Delta u|_2^2.$$

Consider the Sobolev constant for the embedding $E(\Omega) \hookrightarrow L^{2_*}(\Omega)$, given by

$$S(\Omega) = \inf \left\{ \int_\Omega |\Delta u|^2 \, dx : u \in E(\Omega), \ \int_\Omega |u|^{2_*} dx = 1 \right\}. \tag{2.1}$$

It is known that $S(\Omega)$ does not depend on Ω and $S(\Omega)$ is not achieved except when $\Omega = \mathbb{R}^N$ [26]. Moreover, $S(\Omega) = S$, where

$$S = \inf\bigg\{\int_{\mathbb{R}^N} |\Delta u|^2 dx : u \in \mathcal{D}^{2,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{2_*}\, dx = 1\bigg\}, \tag{2.2}$$

which is attained precisely by the functions $S^{(4-N)/8}\varphi_{\delta,a}$, with

$$\begin{aligned}\varphi_{\delta,a}(x) = \frac{[(N-4)(N-2)N(N+2)]^{(N-4)/8}\delta^{(N-4)/2}}{(\delta^2+|x-a|^2)^{(N-4)/2}} \\ = \frac{C_N\delta^{(N-4)/2}}{(\delta^2+|x-a|^2)^{(N-4)/2}},\end{aligned} \tag{2.3}$$

for varying $a \in \mathbb{R}^N$ and $\delta > 0$ [13, Lemma 1]. We recall that the functions given by (2.3) are precisely the positive regular solutions of

$$\Delta^2 u = u^{2_*-1} \quad \text{in } \mathbb{R}^N.$$

Define, for $\mu \in (0, \mu_1(\Omega))$, the norm

$$\|u\|_\mu := (|\Delta u|_2^2 - \mu|u|_2^2)^{1/2}, \quad \text{for all } u \in E(\Omega), \tag{2.4}$$

and observe the equivalence

$$\|u\|_\mu \le \|u\| \le c(\Omega)\|u\|_\mu, \quad \text{for all } u \in E(\Omega), \tag{2.5}$$

where $c(\Omega) = (1-\mu/(\mu_1(\Omega)))^{-1/2} > 0$.

To study the existence of solutions for the problem (P), we will consider the functional

$$I(u) := \frac{1}{2}\int_\Omega |\Delta u|^2\, dx - \frac{\mu}{2}\int_\Omega (u^+)^2\, dx - \frac{1}{2_*}\int_\Omega (u^+)^{2_*}\, dx, \quad u \in E(\Omega). \tag{2.6}$$

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$, $N \ge 8$, be a bounded smooth domain and $0 < \mu < \mu_1(\Omega)$. We say that $u \in E(\Omega)$ is a weak solution of (P) if u is a critical point of I, that is, $u \in E(\Omega)$ satisfies

$$\int_\Omega \Delta u \Delta v\, dx = \mu\int_\Omega (u^+)v\, dx + \int_\Omega (u^+)^{2_*-1}v\, dx, \quad \text{for all } v \in E(\Omega).$$

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 8$, be a bounded smooth domain and $0 < \mu < \mu_1(\Omega)$. Then the $C^4(\overline{\Omega})$-classical solutions of* (P) *are precisely the nontrivial critical points of the functional I defined by* (2.6).

Proof. The results in [26, Appendix B], [2, Theorem 12.7] and [14, Theorems 2.19 and 2.20] guarantee that the nontrivial critical points of I are precisely the classical solutions of (P). We mention that the arguments in [9, p. 375] can be used to prove that every nontrivial critical point of I satisfies $u, -\Delta u > 0$ in Ω. $\square$

From now on we will turn our attention to study the functional I, or equivalently to study

$$I_\mu(u) := \int_\Omega |\Delta u|^2\,dx - \mu \int_\Omega (u^+)^2\,dx, \tag{2.7}$$

restricted to the manifold

$$V := \{u \in E(\Omega) : \psi(u) = 1\} \quad \text{where } \psi(u) := \int_\Omega (u^+)^{2_*}\,dx. \tag{2.8}$$

We also define

$$m(\mu,\Omega) := \inf\{I_\mu(u); u \in V\} \tag{2.9}$$

and, if $\Omega = B_\rho(0)$, we denote $m(\mu,\rho) := m(\mu, B_\rho(0))$.

We will prove that the functional $I_\mu|_V$ has at least as many critical points as the Lusternik–Schnirelman category of Ω, which up to suitable multiplicatives constants are classical solutions for (P).

3. Compactness

The next lemma describes the lack of compactness of the embedding of $\mathcal{D}^{2,2}(\mathbb{R}^N)$ into $L^{2_*}(\mathbb{R}^N)$. A similar result for the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ is proved in [28, Lemma 1.40]; see also [4], [5], [18].

LEMMA 3.1 (Concentration and compactness). *Let $(u_n) \subset \mathcal{D}^{2,2}(\mathbb{R}^N)$ be a sequence such that*

$$u_n \rightharpoonup u \quad \textit{in } \mathcal{D}^{2,2}(\mathbb{R}^N), \tag{3.1}$$

$$|\Delta(u_n - u)|^2 \overset{*}{\rightharpoonup} \lambda \quad \textit{in the sense of measures on } \mathbb{R}^N, \tag{3.2}$$

$$|u_n - u|^{2_*} \overset{*}{\rightharpoonup} \nu \quad \textit{in the sense of measures on } \mathbb{R}^N, \tag{3.3}$$

$$u_n \to u \quad \textit{a.e. on } \mathbb{R}^N. \tag{3.4}$$

Define

$$\lambda_\infty = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|\ge R} |\Delta u_n|^2\,dx, \qquad \nu_\infty = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|\ge R} |u_n|^{2_*}\,dx.$$

Then it follows that

$$\|\nu\|^{2/2_*} \le S^{-1}\|\lambda\|, \tag{3.5}$$

$$\nu_\infty^{2/2_*} \le S^{-1}\lambda_\infty, \tag{3.6}$$

$$\overline{\lim_{n\to\infty}} |\Delta u_n|_2^2 = |\Delta u|_2^2 + \|\lambda\| + \lambda_\infty, \tag{3.7}$$

$$\overline{\lim_{n\to\infty}} |u_n|_{2_*}^{2_*} = |u|_{2_*}^{2_*} + \|\nu\| + \nu_\infty. \tag{3.8}$$

Moreover, if $u = 0$ and $\|\nu\|^{2/2_*} = S^{-1}\|\lambda\|$, then λ and ν are concentrated at a common single point.

PROOF. See Appendix A. □

Lemma 3.2. *Assume* $0 < \mu < \mu_1(\Omega)$. *Any* (PS)*-sequence for* I *is bounded.*

Proof. It follows from standard arguments, since

$$\|u\|_\mu = \bigg(\int_\Omega |\Delta u|^2\,dx - \mu\int_\Omega (u^+)^2\,dx\bigg)^{1/2}, \quad u \in E(\Omega)$$

is a norm in $E(\Omega)$ and $2_* > 2$. □

Lemma 3.3. *Assume* $0 < \mu < \mu_1(\Omega)$. *Any sequence* $(u_n) \subset E(\Omega)$ *such that*

$$I(u_n) \to d < c^* := \frac{2}{N}\,S^{N/4} \quad \textit{and} \quad I'(u_n) \to 0$$

contains a convergent subsequence.

Proof. By Lemma 3.2 it follows that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } E(\Omega), \quad u_n \to u \quad \text{in } L^2(\Omega) \quad \text{and} \quad u_n \to u \quad \text{a.e. on } \Omega.$$

For every $\varphi \in E(\Omega)$ we have

$$\int_\Omega \Delta u_n \Delta\varphi\,dx - \mu\int_\Omega (u_n^+)\varphi\,dx = \int_\Omega (u_n^+)^{2_*-1}\varphi\,dx + o_n(1). \tag{3.9}$$

From the continuous embedding $E(\Omega) \hookrightarrow L^{2_*}(\Omega)$, (u_n^+) is bounded in $L^{2_*}(\Omega)$ and consequently $((u_n^+)^{2_*-1})$ is bounded in $L^{2_*/(2_*-1)}(\Omega)$; we have also $u_n^+ \to u^+$ almost everywhere on Ω. Hence, as a consequence of the Brézis–Lieb lemma, see for instance [16, Lemma 4.8], $(u_n^+)^{2_*-1} \rightharpoonup (u^+)^{2_*-1}$ in $L^{2_*/(2_*-1)}(\Omega)$, and we obtain

$$\int_\Omega (u_n^+)^{2_*-1}\varphi\,dx \to \int_\Omega (u^+)^{2_*-1}\varphi\,dx, \quad \text{for all } \varphi \in L^{2_*}(\Omega), \tag{3.10}$$

in particular, (3.10) holds for any $\varphi \in E(\Omega)$. From $u_n \to u$ in $L^2(\Omega)$ we get

$$\int_\Omega (u_n^+)\varphi\,dx \to \int_\Omega (u^+)\varphi\,dx, \quad \text{for all } \varphi \in E(\Omega). \tag{3.11}$$

Now, since $u_n \rightharpoonup u$ in $E(\Omega)$, we obtain

$$\int_\Omega \Delta u_n \Delta\varphi\,dx =: \langle u_n, \varphi\rangle \to \langle u, \varphi\rangle := \int_\Omega \Delta u \Delta\varphi\,dx, \quad \text{for all } \varphi \in E(\Omega). \tag{3.12}$$

Thus, taking $n \to \infty$ in (3.9) and using (3.10)–(3.12) we obtain

$$\int_\Omega \Delta u \Delta\varphi\,dx - \mu\int_\Omega (u^+)\varphi\,dx = \int_\Omega (u^+)^{2_*-1}\varphi\,dx, \quad \text{for all } \varphi \in E(\Omega), \tag{3.13}$$

that is, u is a weak solution for the problem

$$\begin{cases} \Delta^2 u = \mu(u^+) + (u^+)^{2_*-1} & \text{in } \Omega, \\ u,\ \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

and $u, -\Delta u$ are nonnegative in Ω. Indeed, since $-\Delta\colon E(\Omega) \to L^2(\Omega)$ is an isomorphism [15], it follows, from (3.13), that

$$\int_\Omega \Delta u(-w)\,dx = \mu\int_\Omega (u^+)[(-\Delta)^{-1}w]\,dx + \int_\Omega (u^+)^{2_*-1}[(-\Delta)^{-1}w]\,dx,$$

for all $w \in L^2(\Omega)$, and, from the weak maximum principle,

$$(-\Delta)^{-1}w \geq 0, \quad \text{for all } w \in L^2(\Omega) \text{ and } w \geq 0,$$

and thus

$$\int_\Omega \Delta u(-w)\,dx \geq 0, \quad \text{if } w \geq 0.$$

Hence $u \in E(\Omega)$ and $-\Delta u \geq 0$ in Ω. Consequently, by the weak maximum principle, $u \geq 0$ in Ω.

With $\varphi = u$ in (3.13) we obtain

$$|\Delta u|_2^2 - \mu|u^+|_2^2 = |u^+|_{2_*}^{2_*} \tag{3.14}$$

and

$$I(u) = \frac{1}{2}\big[|\Delta u|_2^2 - \mu|u^+|_2^2\big] - \frac{1}{2_*}|u^+|_{2_*}^{2_*} = \left(\frac{1}{2} - \frac{1}{2_*}\right)|u^+|_{2_*}^{2_*} \geq 0. \tag{3.15}$$

Writing now $v_n = u_n - u$, see [28, p. 33] the Brézis–Lieb lemma leads to

$$|u_n^+|_{2_*}^{2_*} = |u^+|_{2_*}^{2_*} + |v_n^+|_{2_*}^{2_*} + o_n(1). \tag{3.16}$$

From $u_n \to u$ in $L^2(\Omega)$, we also have

$$|u_n^+|_2^2 = |u^+|_2^2 + |v_n^+|_2^2 + o_n(1). \tag{3.17}$$

Using now (3.16) and (3.17) we have

$$\begin{aligned} I(u_n) &= \frac{1}{2}|\Delta u_n|_2^2 - \frac{\mu}{2}|u_n^+|_2^2 - \frac{1}{2_*}|u_n^+|_{2_*}^{2_*} \\ &= I(u) + \frac{1}{2}|\Delta v_n|_2^2 - \frac{\mu}{2}|v_n^+|_2^2 - \frac{1}{2_*}|v_n^+|_{2_*}^{2_*} + o_n(1), \end{aligned}$$

because $v_n \rightharpoonup 0$ in $E(\Omega)$. Assuming $I(u_n) \to d < c^*$, we obtain

$$I(u) + \frac{1}{2}|\Delta v_n|_2^2 - \frac{\mu}{2}|v_n^+|_2^2 - \frac{1}{2_*}|v_n^+|_{2_*}^{2_*} \to d. \tag{3.18}$$

Using again (3.16) and (3.17)

$$\begin{aligned} I'(u_n)u_n &= |\Delta u_n|_2^2 - \mu|u_n^+|_2^2 - |u_n^+|_{2_*}^{2_*} \\ &= |\Delta v_n|_2^2 + 2\langle v_n, u\rangle + |\Delta u|_2^2 - \mu|u^+|_2^2 - \mu|v_n^+|_2^2 - |u^+|_{2_*}^{2_*} - |v_n^+|_{2_*}^{2_*} + o_n(1) \end{aligned}$$

and since $I'(u_n)u_n \to 0$, we conclude, now using (3.14), that

$$|\Delta v_n|_2^2 - \mu|v_n^+|_2^2 - |v_n^+|_{2_*}^{2_*} \to |\Delta u|_2^2 - \mu|u^+|_2^2 - |u^+|_{2_*}^{2_*} = 0.$$

So, we may assume that $|\Delta v_n|_2^2 - \mu|v_n^+|_2^2 \to b$ and $|v_n^+|_{2_*}^{2_*} \to b$.

Since $v_n \to 0$ in $L^2(\Omega)$, in particular, $v_n^+ \to 0$ in $L^2(\Omega)$. Then it follows that $|\Delta v_n|_2^2 \to b$. By the definition of S we have,

$$|\Delta v_n|_2^2 \geq S|v_n|_{2_*}^2 \geq S|v_n^+|_{2_*}^2$$

which implies $b \geq Sb^{2/2_*} = Sb^{(N-4)/N}$. Thus, either $b = 0$ or $b \geq S^{N/4}$.

From (3.18),

$$I(u) + \left(\frac{1}{2} - \frac{1}{2_*}\right)b = I(u) + \frac{2}{N}b = d$$

and from (3.15), $d \geq 2/Nb$. If $b \geq S^{N/4}$ we obtain

$$c^* = \frac{2}{N}\, S^{N/4} \leq \frac{2}{N}\, b \leq d < c^*,$$

a contradiction. Hence, $b = 0$, and the proof is complete, because

$$\|u_n - u\|^2 = \|v_n\|^2 = |\Delta v_n|_2^2 \to 0, \quad \text{that is,} \quad u_n \to u \quad \text{in } E(\Omega). \qquad \square$$

Lemma 3.4. *Assume $0 < \mu < \mu_1(\Omega)$. Any sequence $(u_n) \subset V$ such that*

$$I_\mu(u_n) \to c < S, \qquad \|I'_\mu(u_n)\|_* \to 0, \tag{3.19}$$

contains a convergent subsequence, where $\|\cdot\|_*$ denotes the norm of the derivative of $I_\mu|_V$, and is given by

$$\|I'_\mu(u)\|_* = \min_{\lambda\in\mathbb{R}} \|I'_\mu(u) - \lambda\psi'(u)\|, \quad \textit{for all } u \in V.$$

Proof. If (u_n) satisfies (3.19), then $0 \leq I_\mu(u_n) \to c$ and

$$\|I'_\mu(u_n)\|_* = \|I'_\mu(u_n) - \overline{\lambda}_n\psi'(u_n)\| \to 0, \quad \text{for } \overline{\lambda}_n \in \mathbb{R}.$$

So, there exists $(\sigma_n) \subset [0, +\infty)$, $\sigma_n \to 0$ such that

$$\left| \int_\Omega \Delta u_n \Delta w\, dx - \mu \int_\Omega (u_n^+) w\, dx - \lambda_n \int_\Omega (u_n^+)^{2_*-1} w\, dx \right| \leq \sigma_n \|w\|,$$

for all $w \in E(\Omega)$, $\lambda_n \in \mathbb{R}$. The sequence (u_n) is bounded in $E(\Omega)$. Indeed,

$$\|u_n\|^2 = \|u_n\|^2 - \mu|u_n^+|_2^2 + \mu|u_n^+|_2^2 = c + o_n(1) + \mu|u_n^+|_2^2,$$

and from the continuous embedding of $L^{2_*}(\Omega)$ into $L^2(\Omega)$, it follows that (u_n) is bounded in $E(\Omega)$. Thus

$$\left| \int_\Omega [|\Delta u_n|^2 - \mu(u_n^+)^2]\, dx - \lambda_n \int_\Omega (u_n^+)^{2_*}\, dx \right| \leq \sigma_n\|u_n\| \ \Rightarrow\ I_\mu(u_n) - \lambda_n \to 0,$$

that is, $\lambda_n \to c \geq 0$.

If $c = 0$, then

$$\begin{aligned} 0 &\leq \left(1 - \frac{\mu}{\mu_1(\Omega)}\right)\|u_n\|^2 = \|u_n\|^2 - \frac{\mu}{\mu_1(\Omega)}\|u_n\|^2 \\ &\leq \|u_n\|^2 - \mu|u_n|^2 \leq \|u_n\|^2 - \mu|u_n^+|^2 = I_\mu(u_n) \to 0, \end{aligned}$$

and (u_n) converges strongly to 0 in $E(\Omega)$.

If $c > 0$ then $\lambda_n > 0$ for n big enough. So, put $v_n = \lambda_n^{1/(2_*-2)} u_n$. Taking I given by (2.6),

$$\begin{aligned} I(v_n) &= \frac{1}{2}\int_\Omega \big[|\Delta(\lambda_n^{1/(2_*-2)}u_n)|^2 - \mu(\lambda_n^{1/(2_*-2)}u_n^+)^2\big]\,dx \\ &\quad - \frac{1}{2_*}\int_\Omega (\lambda_n^{1/(2_*-2)}u_n^+)^{2_*}\,dx \\ &= \frac{1}{2}\,\lambda_n^{2/(2_*-2)} I_\mu(u_n) - \frac{1}{2_*}\,\lambda_n^{2_*/(2_*-2)} \\ &\to \frac{1}{2}\,c^{2/(2_*-2)}c - \frac{1}{2_*}\,c^{2_*/(2_*-2)} = \frac{2}{N}c^{N/4} \end{aligned}$$

and

$$\begin{aligned} |I'(v_n)w| &= \bigg|\int_\Omega [\Delta(\lambda_n^{1/(2_*-2)}u_n)\Delta w - \mu(\lambda_n^{1/(2_*-2)}u_n^+)w]\,dx \\ &\quad - \int_\Omega (\lambda_n^{1/(2_*-2)}u_n^+)^{2_*-1}w\,dx\bigg| \\ &= \lambda_n^{1/(2_*-2)}\bigg|\int_\Omega [\Delta u_n \Delta w - \mu(u_n^+)w - \lambda_n(u_n^+)^{2_*-1}w]\,dx\bigg| \\ &\le \lambda_n^{1/(2_*-2)}\sigma_n\|w\|, \end{aligned}$$

for all $w \in E(\Omega)$. Hence

$$I(v_n) \to \frac{2}{N}\,c^{N/4} < \frac{2}{N}\,S^{N/4} = c^* \quad \text{and} \quad I'(v_n) \to 0.$$

From Lemma 3.3, (v_n) contains a convergent subsequence, and then (u_n) also contains a convergent subsequence. □

4. Multiplicity of solutions

We first recall a classical result in the theory of the Lusternik–Schnirelman category [19].

THEOREM 4.1 ([28, Theorem 5.20]). *Let X be a Banach space, $\varphi \in \mathcal{C}^1(X,\mathbb{R})$, $\psi \in \mathcal{C}^2(X,\mathbb{R})$, $V = \{v \in X : \psi(v) = 1\}$ and for all $v \in V$, $\psi'(v) \neq 0$. If $\varphi|_V$ is bounded from below and satisfies the* $(\mathrm{PS})_c$*-condition for any $c \in \big[\inf_V \varphi, d\big]$, then $\varphi|_V$ has a minimum and the set $\varphi^d := \{v \in V : \varphi(v) \le d\}$ contains at least* $\mathrm{cat}_{\varphi^d}(\varphi^d)$ *critical points of $\varphi|_V$.*

In our context, $X = E(\Omega)$, $\psi(u) = \int_\Omega (u^+)^{2_*}\,dx$ and $\varphi = I_\mu$.

LEMMA 4.2. *Let $N \ge 8$ and $0 < \mu < \mu_1(\Omega)$. There exists $v \in E(\Omega)\backslash\{0\}$, with $v > 0$ in Ω such that*

$$\frac{\|v\|_\mu^2}{|v|_{2_*}^2} = \frac{|\Delta v|_2^2 - \mu|v|_2^2}{|v|_{2_*}^2} < S. \tag{4.1}$$

PROOF. See Appendix B. □

Lemma 4.3. *If* $0 < \mu < \mu_1(\Omega)$ *and* $N \geq 8$, *then* $m(\mu,\Omega) < S$ *and there exists* $u \in V$, *such that* $u, -\Delta u > 0$ *in* Ω *and* $I_\mu(u) = m(\mu,\Omega)$, *with* $m(\mu,\Omega)$ *as defined by* (2.9).

Proof. By Lemma 4.2, there exists $v \in E(\Omega)\backslash\{0\}$ nonnegative such that

$$\frac{|\Delta v|_2^2 - \mu|v|_2^2}{|v|_{2_*}^2} < S.$$

Setting $w = v/|v|_{2_*}$, we have $w \in V$ and

$$I_\mu(w) = |\Delta w|_2^2 - \mu|w^+|_2^2 = |\Delta w|_2^2 - \mu|w|_2^2 = \frac{|\Delta v|_2^2 - \mu|v|_2^2}{|v|_{2_*}^2} < S,$$

and therefore

$$m(\mu,\Omega) = \inf_{u\in V} I_\mu(u) \leq I_\mu(w) < S.$$

By Lemma 3.4, $I_\mu|_V$ satisfies the $(\mathrm{PS})_c$-condition, with $c = m(\mu,\Omega)$. By Theorem 4.1, $I_\mu|_V$ has a minimum, that is, there exists $u \in V$ such that

$$I_\mu(u) = m(\mu,\Omega) = \min_{u\in V} I_\mu(u).$$

Now we show that $u, -\Delta u > 0$ in Ω. Since u is such that

$$|u^+|_{2_*}^{2_*} = 1, \qquad I_\mu(u) = |\Delta u|_2^2 - \mu|u^+|_2^2 = m(\mu,\Omega) > 0,$$

it follows from Lagrange multipliers theorem that u satisfies

$$\int_\Omega \Delta u\Delta v\,dx = \mu\int_\Omega (u^+)v\,dx + m(\mu,\Omega)\int_\Omega (u^+)^{2_*-1}v\,dx, \quad \text{for all } v \in E(\Omega).$$

So,

$$\int_\Omega \Delta u(-w)\,dx = \mu\int_\Omega (u^+)[(-\Delta)^{-1}w]\,dx + m(\mu,\Omega)\int_\Omega (u^+)^{2_*-1}[(-\Delta)^{-1}w]\,dx,$$

for all $w \in L^2(\Omega)$ and $(-\Delta)^{-1}w \geq 0$, for all $w \in L^2(\Omega)$, $w \geq 0$. Thus,

$$\int_\Omega \Delta u(-w)dx \geq 0, \quad \text{for all } w \geq 0,$$

and therefore $-\Delta u \geq 0$ and consequently $u \geq 0$. Since u is nontrivial, it follows by the strong maximum principle that $u, -\Delta u > 0$ in Ω. □

Lemma 4.4. *If* Ω_1 *and* Ω_2 *are regular bounded domains in* $\mathbb{R}^N$, $N \geq 8$, *such that* $\Omega_1 \subset\subset \Omega_2$ *and* $0 < \mu < \mu_1(\Omega_2)$, *then* $m(\mu,\Omega_1) > m(\mu,\Omega_2)$.

Proof. First we recall that $\Omega_1 \subset\subset \Omega_2$ implies that $\mu_1(\Omega_2) < \mu_1(\Omega_1)$. So, let $u \in E(\Omega_1)$ be a function such that $u, -\Delta u > 0$ in Ω_1 and

$$\int_{\Omega_1} (u^+)^{2_*}\,dx = 1, \qquad \int_{\Omega_1} [|\Delta u|^2 - \mu(u^+)^2]\,dx = m(\mu,\Omega_1),$$

and take w as the solution for

$$\begin{cases} -\Delta w = \widetilde{-\Delta u} & \text{in } \Omega_2, \\ w = 0 & \text{on } \partial\Omega_2, \end{cases}$$

where $\sim$ denotes the zero extension outside Ω_1. Note that $w \geq 0$ in Ω_2 and $w > u$ in Ω_1. Set $\overline{w} = w/|w|_{2_*,\Omega_2}$. Then $|\overline{w}^+|_{2_*,\Omega_2} = 1$ and

$$\begin{aligned} m(\mu,\Omega_2) &\leq \int_{\Omega_2} [|\Delta \overline{w}|^2 - \mu(\overline{w}^+)^2]\,dx = \frac{1}{|w|^2_{2_*,\Omega_2}} \int_{\Omega_2} [|\Delta w|^2 - \mu(w^+)^2]\,dx \\ &< \int_{\Omega_2} [|\Delta w|^2 - \mu(w^+)^2]\,dx < \int_{\Omega_1} [|\Delta u|^2 - \mu(u^+)^2]\,dx = m(\mu,\Omega_1). \quad \square \end{aligned}$$

LEMMA 4.5. *If $\Omega = B_\rho(0) \subset \mathbb{R}^N$, $N \geq 8$ and $0 < \mu < \mu_1(\Omega)$, then $m(\mu,\rho)$ is attained by a function u such that $u, -\Delta u > 0$ in $B_\rho(0)$ and $u, -\Delta u$ are radially symmetric. Moreover, such a solution u is unique.*

PROOF. Let u be a function such that $u, -\Delta u > 0$ in $B_\rho(0)$ and that realizes $m(\mu,\rho)$. Denote by u^* and $(-\Delta u)^*$ the Schwarz symmetrization of u and $-\Delta u$, respectively. If v is the solution of

$$\begin{cases} -\Delta v = (-\Delta u)^* & \text{in } B_\rho(0), \\ v = 0 & \text{on } \partial B_\rho(0), \end{cases}$$

then $v = v^*$. We just need to prove that $u = v$. By [3], see also [6, Lemma 2.8], we have $v \geq u^*$ and

$$|v > u^*| = 0 \Leftrightarrow -\Delta u = (-\Delta u)^*.$$

If $|v > u^*| > 0$, set $w = v/|v|_{2_*}$. So $|w^+|_{2_*} = 1$ and

$$\begin{aligned} m(\mu,\rho) &\leq \int_{B_\rho(0)} [|\Delta w|^2 - \mu(w^+)^2]\,dx = \frac{1}{|v|^2_{2_*}} \int_{B_\rho(0)} [|\Delta v|^2 - \mu(v^+)^2]\,dx \\ &< \frac{1}{|v|^2_{2_*}} \int_{B_\rho(0)} [|(-\Delta u)^*|^2 - \mu(u^*)^2]\,dx \\ &< \frac{1}{|u^*|^2_{2_*}} \int_{B_\rho(0)} [|(-\Delta u)^*|^2 - \mu(u^*)^2]\,dx \\ &= \frac{1}{|u^+|^2_{2_*}} \int_{B_\rho(0)} [|-\Delta u|^2 - \mu(u^+)^2]\,dx = m(\mu,\rho), \end{aligned}$$

which is a contradiction. Thus, $-\Delta u = (-\Delta u)^*$ and since u and v are solutions for the problem

$$\begin{cases} -\Delta w = (-\Delta u)^* & \text{in } B_\rho(0), \\ w = 0 & \text{on } \partial B_\rho(0), \end{cases}$$

it follows that $u = v$.

Finally we mention that the uniqueness of u can be proved arguing as in [12, Section 3] by means of comparison principle for radial function [20]. $\square$

Now define $\beta\colon V \to \mathbb{R}^N$ by

$$\beta(u) = \frac{\displaystyle\int_\Omega |\Delta u|^2 x\,dx}{\displaystyle\int_\Omega |\Delta u|^2\,dx}. \tag{4.2}$$

LEMMA 4.6. *If $(u_n) \subset V$ is such that $\|u_n\|^2 = |\Delta u_n|_2^2 \to S$, then*

$$\operatorname{dist}(\beta(u_n), \Omega) \to 0.$$

PROOF. Suppose, by contradiction, that $\operatorname{dist}(\beta(u_n), \Omega) \not\to 0$. So, there exists $r > 0$ such that, up to a subsequence, $\operatorname{dist}(\beta(u_n), \Omega) > r$.

Set $v_n = u_n/|\Delta u_n|_2 \in E(\Omega)$ and w_n as the Newtonian potential of $|\widetilde{-\Delta v_n}| \in L^2(\mathbb{R}^N)$, where $\sim$ denotes the zero extension outside Ω. Then, by [15, Theorem 9.9], we know that $w_n \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ and

$$-\Delta w_n = |\widetilde{-\Delta v_n}| \quad \text{a.e. in } \mathbb{R}^N. \tag{4.3}$$

In particular, (w_n) is a bounded sequence in $\mathcal{D}^{2,2}(\mathbb{R}^N)$. Then, up to a subsequence,

$$\begin{aligned} w_n &\rightharpoonup w && \text{in } \mathcal{D}^{2,2}(\mathbb{R}^N),\\ |\Delta(w_n - w)|^2 &\overset{*}{\rightharpoonup} \lambda && \text{in the sense of measures on } \mathbb{R}^N,\\ |w_n - w|^{2_*} &\overset{*}{\rightharpoonup} \nu && \text{in the sense of measures on } \mathbb{R}^N,\\ w_n &\to w && \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

We have by Lemma 3.1, taking into account that $\lambda_\infty = 0$ and $w_n \geq |\widetilde{v_n}|$ in $\mathbb{R}^N$,

$$1 = |\Delta w|_2^2 + \|\lambda\|, \tag{4.4}$$

$$\frac{1}{S^{2_*/2}} \leq |w|_{2_*}^{2_*} + \|\nu\|, \tag{4.5}$$

and

$$\|\nu\|^{2/2_*} \leq \frac{1}{S}\|\lambda\|, \qquad |w|_{2_*}^2 \leq \frac{1}{S}|\Delta w|_2^2. \tag{4.6}$$

It follows that the pair $(|\Delta w|_2^2, \|\lambda\|) \in \{(1,0),(0,1)\}$. Indeed, from (4.6)

$$\|\nu\| \leq \frac{1}{S^{2_*/2}}\|\lambda\|^{2_*/2}, \qquad |w|_{2_*}^{2_*} = (|w|_{2_*}^2)^{2_*/2} \leq \frac{1}{S^{2_*/2}}|\Delta w|_2^{2_*},$$

and so

$$\frac{1}{S^{2_*/2}} \leq |w|_{2_*}^{2_*} + \|\nu\| \leq \frac{1}{S^{2_*/2}}[|\Delta w|_2^{2_*} + \|\lambda\|^{2_*/2}],$$

that is

$$|\Delta w|_2^{2_*} + \|\lambda\|^{2_*/2} \geq 1. \tag{4.7}$$

From (4.4), (4.7) and since $2_*/2 > 1$, we get that the pair $(|\Delta w|_2^2, \|\lambda\|) \in \{(1,0),(0,1)\}$.

Suppose now that $|\Delta w|_2^2 = 1$ and $\|\lambda\| = 0$. So, by (4.6), $\|\nu\| = 0$ which implies, by (4.5),

$$\frac{1}{S} \leq |w|_{2_*}^2 \leq \frac{1}{S}|\Delta w|_2^2 = \frac{1}{S}$$

and so $|\Delta w|_2^2/|w|_{2_*}^2 = S$. Then, up to a multiple, w is a non-negative non-trivial solution of the equation

$$\Delta^2 w = w^{2_*-1} \quad \text{in } \mathbb{R}^N,$$

and therefore, $w, -\Delta w > 0$ in $\mathbb{R}^N$. But, for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} |\Delta w_n - \Delta w|^2 \varphi \, dx \to \int_{\mathbb{R}^N} \varphi d\lambda = 0$$

which implies, in particular,

$$\int_\Omega |\Delta w_n - \Delta w|^2 \varphi \, dx + \int_{\mathbb{R}^N \setminus \Omega} |\Delta w|^2 \varphi \, dx \to 0, \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N \setminus \Omega),$$

and then $-\Delta w = 0$ in $\mathbb{R}^N \setminus \Omega$, which leads a contradiction.

Thus, $|\Delta w|_2^2 = 0$ (and from (4.6), it follows that $w = 0$) and $\|\lambda\| = 1$. From (4.5) and (4.6), we get $\|\nu\|^{2/2_*} = S^{-1}\|\lambda\|$. Therefore, by Lemma 3.1, it follows that λ concentrates at a single point $y \in \mathbb{R}^N$.

We infer that $y \in \overline{\Omega}$. Indeed, by contradiction suppose $y \in \mathbb{R}^N \setminus \overline{\Omega}$. Take $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $\psi \equiv 1$ in $B_R(y)$, for some $R > 0$, and $\text{supp}(\psi) \cap \overline{\Omega} = \emptyset$. So,

$$1 = \lambda(\{y\}) = \int_{\mathbb{R}^N} \psi \, d\lambda = \lim_{n\to\infty} \int_{\mathbb{R}^N} \psi |\Delta w_n|^2 \, dx = 0,$$

which is clearly a contradiction. Hence, $y \in \overline{\Omega}$ and taking $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, $\eta \equiv 1$ in $\overline{\Omega}$, we have

$$\begin{aligned} \beta(u_n) = \frac{\displaystyle\int_\Omega |\Delta u_n|^2 x \, dx}{\displaystyle\int_\Omega |\Delta u_n|^2 \, dx} &= \int_\Omega |\Delta v_n|^2 x \, dx \\ = \int_{\mathbb{R}^N} |\Delta w_n|^2 x\eta(x) \, dx &\to \int_{\mathbb{R}^N} x\eta(x) \, d\lambda = y\eta(y) = y \in \overline{\Omega}, \end{aligned}$$

which contradicts our initial hypothesis. $\square$

Without loss of generality we can assume that $0 \in \Omega$. Let $r > 0$ be small enough such that

$$\Omega_r^+ := \{u \in \mathbb{R}^N : \text{dist}(u, \overline{\Omega}) \leq r\} \quad \text{and} \quad \Omega_r^- := \{u \in \Omega : \text{dist}(u, \partial\Omega) \geq r\}$$

are homotopically equivalent to Ω and such that $B_r(0) \subset\subset \Omega$. We also set

$$I_\mu^{m(\mu,r)} := \{u \in V : I_\mu(u) \leq m(\mu, r)\},$$

which is nonempty; see Lemma 4.4.

Lemma 4.7. *There exists $0 < \overline{\mu} < \mu_1(\Omega)$ such that, for $0 < \mu < \overline{\mu}$,*

$$u \in I_\mu^{m(\mu,r)} \Rightarrow \beta(u) \in \Omega_r^+.$$

Proof. If $u \in V$, then by the Hölder inequality,

$$|u^+|_2^2 \le |u^+|_{2_*}^2 |\Omega|^{(2_*-2)/2_*} = |\Omega|^{4/N}. \tag{4.8}$$

By Lemma 4.6, there exists $\varepsilon > 0$ such that

$$u \in V, \quad \|u\|^2 \le S + \varepsilon \Rightarrow \beta(u) \in \Omega_r^+.$$

Set $\overline{\mu} := \varepsilon/|\Omega|^{4/N}$, for $\varepsilon > 0$ sufficiently small such that $0 < \overline{\mu} < \mu_1(\Omega)$. Hence, if $0 < \mu < \overline{\mu}$ and $u \in I_\mu^{m(\mu,r)}$, we obtain, from (4.8) and Lemma 4.3,

$$\begin{aligned}\|u\|^2 &= \|u\|^2 - \mu|u^+|_2^2 + \mu|u^+|_2^2 = I_\mu(u) + \mu|u^+|_2^2 \\ &\le m(\mu,r) + \overline{\mu}|u^+|_2^2 < S + \frac{\varepsilon}{|\Omega|^{4/N}}|\Omega|^{4/N} = S + \varepsilon,\end{aligned}$$

so that $\beta(u) \in \Omega_r^+$. $\square$

Let $\overline{\mu}$ as in Lemma 4.7. For each $0 < \mu < \overline{\mu}$ we define $\gamma_\mu \colon \Omega_r^- \to I_\mu^{m(\mu,r)}$ by

$$\gamma_\mu(y) \colon \Omega \to \mathbb{R}, \qquad x \mapsto \gamma_\mu(y)(x) = \frac{w_y(x)}{|w_y|_{2_*}}, \tag{4.9}$$

where w_y is the solution for the problem

$$\begin{cases} -\Delta w_y = z_y & \text{in } \Omega, \\ w_y = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{with} \quad z_y(x) = \begin{cases} -\Delta v_\mu(x-y) & \text{if } x \in B_r(y), \\ 0 & \text{if } x \in \Omega \setminus B_r(y), \end{cases}$$

where, see Lemma 4.5, v_μ is radially symmetric with respect to zero, $v_\mu, -\Delta v_\mu > 0$ in $B_r(0)$ and

$$\int_{B_r(0)} (v_\mu^+)^{2_*}\,dx = 1, \qquad \int_{B_r(0)} \big[|\Delta v_\mu|^2 - \mu(v_\mu^+)^2\big]\,dx = m(\mu,r).$$

Remark 4.8. Arguing as in the proof of Lemma 2.2, we get that $v_\mu \in C^4(\overline{B_r(0)})$.

Lemma 4.9. *Let $0 < \mu < \overline{\mu}$, where $\overline{\mu}$ is given in Lemma 4.7. Then $\gamma_\mu \colon \Omega_r^- \to I_\mu^{m(\mu,r)}$ is well defined, continuous and*

$$(\beta \circ \gamma_\mu)(y) = y, \quad \textit{for all } y \in \Omega_r^-. \tag{4.10}$$

Proof. First observe that [15, Theorem 9.15] guarantees that $\gamma_\mu(y) \in E(\Omega)$ and, by the strong maximum principle, we have $w_y(x) > v_\mu(x-y)$, for all $x \in B_r(y)$ and $y \in \Omega_r^-$. Then

$$\begin{aligned}\int_\Omega |\Delta w_y|^2\,dx &= \int_{B_r(0)} |\Delta v_\mu|^2\,dx, \\ \int_\Omega |w_y|^2\,dx &> \int_{B_r(0)} |v_\mu|^2\,dx,\end{aligned}$$

$$\int_\Omega |w_y|^{2_*}\,dx > \int_{B_r(0)} |v_\mu|^{2_*}\,dx = 1.$$

So,

$$\begin{aligned} I_\mu(\gamma_\mu(y)) = I_\mu\bigg(\frac{w_y(x)}{|w_y|_{2_*}}\bigg) &= \frac{1}{|w_y|_{2_*}^2}\Big[\int_\Omega |\Delta w_y|^2\,dx - \mu\int_\Omega (w_y^+)^2\,dx\Big] \\ &< \int_\Omega |\Delta w_y|^2\,dx - \mu\int_\Omega (w_y^+)^2\,dx \\ &\le \int_{B_r(0)} |\Delta v_\mu|^2\,dx - \mu\int_{B_r(0)} (v_\mu^+)^2\,dx = m(\mu,r), \end{aligned}$$

that is, $\gamma_\mu(y) \in I_\mu^{m(\mu,r)}$ for every $y \in \Omega_r^-$ and so $\gamma_\mu\colon \Omega_r^- \to I_\mu^{m(\mu,r)}$ is well defined.

The continuity of γ_μ is a consequence of the regularity of v_μ. To prove that γ_μ is continuous, it is enough to prove that $\overline{\gamma}_\mu : \Omega_r^- \to E(\Omega)$, defined by $\overline{\gamma}_\mu(y)(x) = w_y(x)$, is continuous. If $y_n \to y$ in Ω_r^-, then

$$\begin{aligned} \|\overline{\gamma}_\mu(y_n) - \overline{\gamma}_\mu(y)\|^2 &= |\Delta(\overline{\gamma}_\mu(y_n) - \overline{\gamma}_\mu(y))|_2^2 \\ &= |\Delta w_{y_n} - \Delta w_y|_2^2 = |z_{y_n} - z_y|_2^2 \\ &= |z_{y_n}|_2^2 - 2\int_\Omega z_{y_n}(x) z_y(x)dx + |z_y|_2^2 \\ &= 2\bigg[\int_{B_r(0)} |\Delta v_\mu(z)|^2 dz - \int_\Omega z_{y_n}(x) z_y(x)\,dx\bigg] \to 0, \end{aligned}$$

because $\Delta v_\mu\colon \overline{B_r(0)} \to \mathbb{R}$ is continuous. Finally, for every $y \in \Omega_r^-$,

$$\begin{aligned} (\beta\circ\gamma_\mu)(y) &= \frac{\displaystyle\int_\Omega \bigg|\Delta\bigg(\frac{w_y}{|w_y|_{2_*}}\bigg)\bigg|^2 x\,dx}{\displaystyle\int_\Omega \bigg|\Delta\bigg(\frac{w_y}{|w_y|_{2_*}}\bigg)\bigg|^2 dx} = \frac{\displaystyle\int_\Omega |\Delta w_y|^2 x\,dx}{\displaystyle\int_\Omega |\Delta w_y|^2\,dx} \\ &= \frac{\displaystyle\int_{B_r(y)} |\Delta v_\mu(x-y)|^2 x\,dx}{\displaystyle\int_{B_r(y)} |\Delta v_\mu(x-y)|^2\,dx} = \frac{\displaystyle\int_{B_r(0)} |\Delta v_\mu(z)|^2 (z+y)\,dz}{\displaystyle\int_{B_r(0)} |\Delta v_\mu(z)|^2\,dz} \\ &= \frac{\displaystyle\int_{B_r(0)} |\Delta v_\mu(z)|^2 z\,dz}{\displaystyle\int_{B_r(0)} |\Delta v_\mu(z)|^2\,dz} + \frac{\displaystyle y\int_{B_r(0)} |\Delta v_\mu(z)|^2\,dz}{\displaystyle\int_{B_r(0)} |\Delta v_\mu(z)|^2\,dz} = y, \end{aligned}$$

because Δv_μ is radially symmetric. □

LEMMA 4.10. *If $N \ge 8$ and $0 < \mu < \overline{\mu}$, where $\overline{\mu}$ is given in Lemma 4.7, then*

$$\operatorname{cat}_{I_\mu^{m(\mu,r)}}(I_\mu^{m(\mu,r)}) \ge \operatorname{cat}_\Omega(\Omega).$$

Proof. If $\operatorname{cat}_{I_\mu^{m(\mu,r)}}(I_\mu^{m(\mu,r)}) = \infty$, then there is nothing to do.

If $\operatorname{cat}_{I_\mu^{m(\mu,r)}}(I_\mu^{m(\mu,r)}) = n$, then $I_\mu^{m(\mu,r)} = A_1 \cup \ldots \cup A_n$, where A_j is closed and contractible in $I_\mu^{m(\mu,r)}$, for all $j = 1, \ldots, n$.

For each $j = 1, \ldots, n$, let $h_j \colon [0,1] \times A_j \to I_\mu^{m(\mu,r)}$ be a continuous map and $w_j \in I_\mu^{m(\mu,r)}$ such that

$$h_j(0,u) = u, \quad h_j(1,u) = w_j, \quad \text{for all } u \in A_j. \tag{4.11}$$

Consider $B_j = \gamma_\mu^{-1}(A_j)$, where γ_μ is given by (4.9). The sets B_j are closed and $\Omega_r^- = B_1 \cup \ldots \cup B_n$. Define, for $0 < \mu < \overline{\mu}$, the deformation

$$g_j \colon [0,1] \times B_j \to \Omega_r^+, \qquad (t,y) \mapsto g_j(t,y) = \beta(h_j(t, \gamma_\mu(y))).$$

By Lemma 4.7, the deformation g_j is well defined, and from (4.10) and (4.11)

$$\begin{aligned} g_j(0,y) &= \beta(h_j(0,\gamma_\mu(y))) = \beta(\gamma_\mu(y)) = y, && \text{for all } y \in B_j, \\ g_j(1,y) &= \beta(h_j(1,\gamma_\mu(y))) = \beta(w_j), && \text{for all } y \in B_j. \end{aligned}$$

Hence, the sets B_j are contractible in Ω_r^+, and so

$$\operatorname{cat}_\Omega(\Omega) = \operatorname{cat}_{\Omega_r^+}(\Omega_r^-) \le n = \operatorname{cat}_{I_\mu^{m(\mu,r)}}(I_\mu^{m(\mu,r)}). \qquad \square$$

Proof of Theorem 1.1 (completed). By Lemmas 3.4 and 4.3, for $c \le m(\mu,\Omega) \le m(\mu,r) < S$, $I_\mu|_V$ satisfies the $(\mathrm{PS})_c$-condition. By Theorem 4.1, with $d = m(\mu,r)$, it follows that $I_\mu^{m(\mu,r)}$ has at least $\operatorname{cat}_{I_\mu^{m(\mu,r)}}(I_\mu^{m(\mu,r)})$ critical points of $I_\mu|_V$. Then, by Lemma 4.10, for $0 < \mu < \overline{\mu}$, we have that $I_\mu|_V$ has at least $n = \operatorname{cat}_\Omega(\Omega)$ different critical points, say $v_1, \ldots, v_n \in V$.

For each $j = 1, \ldots, n$, there exists $\lambda_j \in \mathbb{R}$ such that v_j satisfies

$$\begin{cases} \Delta^2 v_j = \mu(v_j^+) + \lambda_j (v_j^+)^{2_*-1}, & \text{in } \Omega, \\ v_j, \ \Delta v_j = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $v_j \in V$ we have $v_j \ne 0$, and

$$\lambda_j = \lambda_j \int_\Omega (v_j^+)^{2_*}\, dx = I_\mu(v_j) = \int_\Omega [|\Delta v_j|^2 - \mu (v_j^+)^2]\, dx > 0.$$

Hence, for each $j = 1, \ldots, n$, we have that $u_j := \lambda_j^{1/(2_*-2)} v_j$ is a nontrivial solution of

$$\begin{cases} \Delta^2 u = \mu u^+ + (u^+)^{2_*-1} & \text{in } \Omega, \\ u, \ \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

that is, u_j is a critical point of I. Since $v_j \ne v_i$ if $j \ne i$, it follows that $u_j \ne u_i$ if $j \ne i$. Then, we apply Lemma 2.2 to end this proof. $\square$

Appendix A. Proof of Lemma 3.1

PROOF. *Particular case*: *Assume first* $u = 0$.

For every $h \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, we infer from (2.2) that

$$\left(\int_{\mathbb{R}^N} |hu_n|^{2_*}\,dx\right)^{2/2_*} \leq S^{-1}\int_{\mathbb{R}^N} |\Delta(hu_n)|^2\,dx. \tag{A.1}$$

Using (3.2) and (3.3) we get

$$\left(\int_{\mathbb{R}^N} |h|^{2_*}|u_n|^{2_*}\,dx\right)^{2/2_*} \to \left(\int_{\mathbb{R}^N} |h|^{2_*}\,d\nu\right)^{2/2_*} \tag{A.2}$$

and

$$\int_{\mathbb{R}^N} |h|^2|\Delta u_n|^2\,dx \to \int_{\mathbb{R}^N} |h|^2\,d\lambda. \tag{A.3}$$

Note that

$$\Delta(hu_n) - h\Delta u_n = u_n\Delta h + 2\nabla h.\nabla u_n. \tag{A.4}$$

We have

$$|u_n\Delta h|_2^2 = \int_{B_R(0)} |\Delta h|^2|u_n|^2\,dx \leq C\int_{B_R(0)} |u_n|^2\,dx,$$

where $R > 0$ is such that $\operatorname{supp}(h) \subset \overline{B}_R(0)$ and $C = \max_{\overline{B}_R(0)} |\Delta h|^2$. Then

$$|u_n\Delta h|_2^2 \to 0, \tag{A.5}$$

because $u_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. We also have

$$|\nabla h.\nabla u_n|_2^2 \leq \int_{B_R(0)} |\nabla h|^2|\nabla u_n|^2\,dx \leq \overline{C}\int_{B_R(0)} |\nabla u_n|^2\,dx,$$

where $\overline{C} = \max_{\overline{B}_R(0)} |\nabla h|^2$, and consequently

$$|\nabla h.\nabla u_n|_2^2 \to 0, \tag{A.6}$$

because $\nabla u_n \to 0$ in $[L^2_{\mathrm{loc}}(\mathbb{R}^N)]^N$. From (A.4)–(A.6) follows that

$$\begin{aligned} ||\Delta(hu_n)|_2 - |h\Delta u_n|_2| &\leq |\Delta(hu_n) - h\Delta u_n|_2 \\ &= |u_n\Delta h + 2\nabla h.\nabla u_n|_2 \leq |u_n\Delta h|_2 + 2|\nabla h.\nabla u_n|_2 \to 0, \end{aligned}$$

that is,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\Delta(hu_n)|^2\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} |h\Delta u_n|^2\,dx = \int_{\mathbb{R}^N} |h|^2 d\lambda.$$

Hence, from (A.1)–(A.2) we get

$$\left(\int_{\mathbb{R}^N} |h|^{2_*}\,d\nu\right)^{2/2_*} \leq S^{-1}\int_{\mathbb{R}^N} |h|^2\,d\lambda. \tag{A.7}$$

Taking now the sequence $(h_n) \subset \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that

$$h_n \equiv 1 \quad \text{in } B_n(0), \qquad \operatorname{supp}(h_n) \subset B_{n+1}(0), \quad 0 \le h_n \le 1,$$

it follows by dominated convergence theorem that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |h_n|^{2_*}\, d\nu = \int_{\mathbb{R}^N} 1\, d\nu = \|\nu\| \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |h_n|^2\, d\lambda = \int_{\mathbb{R}^N} 1\, d\lambda = \|\lambda\|.$$

Then we obtain (3.5) using (h_n) in (A.7) and taking $n \to \infty$.

Now we proceed to prove (3.6). Fix $R > 0$ and let $\psi_R \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that $\psi_R(x) = 1$ for $|x| \ge R+1$, $\psi_R(x) = 0$ for $|x| \le R$ and $0 \le \psi_R \le 1$ on $\mathbb{R}^N$. By the Sobolev inequality, we have

$$\overline{\lim_{n\to\infty}} \bigg(\int_{\mathbb{R}^N} |\psi_R u_n|^{2_*}\, dx \bigg)^{2/2_*} \le S^{-1} \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} |\Delta(\psi_R u_n)|^2\, dx. \tag{A.8}$$

We have

$$0 \le \int_{\mathbb{R}^N} |u_n \Delta\psi_R|^2\, dx \le \int_{|x|\le R+1} |\Delta\psi_R|^2 |u_n|^2\, dx \le C_R \int_{|x|\le R+1} |u_n|^2\, dx,$$

where $C_R = \max\limits_{\overline{B}_{R+1}(0)} |\Delta\psi_R|^2$, and

$$0 \le \int_{\mathbb{R}^N} |\nabla\psi_R . \nabla u_n|^2\, dx \le \int_{|x|\le R+1} |\nabla\psi_R|^2 |\nabla u_n|^2\, dx \le D_R \int_{|x|\le R+1} |\nabla u_n|^2\, dx,$$

where $D_R = \max\limits_{\overline{B}_{R+1}(0)} |\nabla\psi_R|^2$. Thus

$$||\Delta(\psi_R u_n)|_2 - |\psi_R \Delta u_n|_2| \le |u_n \psi_R|_2 + |2\nabla\psi_R . \nabla u_n|_2 \to 0,$$

because $u_n, \nabla u_n \to 0$ in $L^2_{\rm loc}(\mathbb{R}^N)$, $[L^2_{\rm loc}(\mathbb{R}^N)]^N$, respectively. From (A.8) we conclude

$$\overline{\lim_{n\to\infty}} \bigg(\int_{\mathbb{R}^N} \psi_R^{2_*} |u_n|^{2_*}, dx \bigg)^{2/2_*} \le S^{-1} \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} \psi_R^2 |\Delta u_n|^2\, dx. \tag{A.9}$$

On the another hand, we have

$$\int_{\mathbb{R}^N} |\Delta u_n|^2 \psi_R^2\, dx = \int_{|x|\ge R} |\Delta u_n|^2 \psi_R^2\, dx \le \int_{|x|\ge R} |\Delta u_n|^2\, dx$$

and

$$\int_{|x|\ge R+1} |u_n|^{2_*}\, dx = \int_{|x|\ge R+1} |u_n|^{2_*} \psi_R^{2_*}\, dx \le \int_{\mathbb{R}^N} |u_n|^{2_*} \psi_R^{2_*}\, dx$$

and from (A.9) follows that

$$\begin{aligned} \nu_\infty^{2/2_*} &= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \bigg(\int_{|x|\ge R+1} |u_n|^{2_*}\, dx \bigg)^{2/2_*} \\ &\le S^{-1} \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \bigg(\int_{|x|\ge R} |\Delta u_n|^2\, dx \bigg) = S^{-1} \lambda_\infty, \end{aligned}$$

which proves (3.6).

Assume moreover, that $\|\nu\|^{2/2_*} = S^{-1}\|\mu\|$. We will show that λ and ν are concentrated at a common single point. Given $h \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ we have, from (A.7),

$$\left(\int_{\mathbb{R}^N} |h|^{2_*}\, d\nu\right)^{1/2_*} \le S^{-1/2}\left(\int_{\mathbb{R}^N} |h|^2\, d\lambda\right)^{1/2}, \tag{A.10}$$

and from Hölder inequality we get

$$\int_{\mathbb{R}^N} |h|^{2_*}\, d\nu \le S^{-2_*/2}\|\lambda\|^{4/(N-4)} \int_{\mathbb{R}^N} |h|^{2_*}\, d\lambda, \quad \text{for all } h \in \mathcal{C}_c^\infty(\mathbb{R}^N), \tag{A.11}$$

which implies

$$\nu(\Omega) \le S^{-2_*/2}\|\lambda\|^{4/(N-4)}\lambda(\Omega), \quad \text{for all } \Omega \subset \mathbb{R}^N \text{ measurable.}$$

We prove now that $\nu(\Omega) = S^{-2_*/2}\|\lambda\|^{4/(N-4)}\lambda(\Omega)$, for all $\Omega \subset \mathbb{R}^N$ measurable. Assume that there exists $\Omega_0 \subset \mathbb{R}^N$ such that $\nu(\Omega_0) < S^{-2_*/2}\|\lambda\|^{4/(N-4)}\lambda(\Omega_0)$. By hypothesis, $\|\nu\|^{2/2_*} = S^{-1}\|\lambda\|$, which implies

$$\nu(\mathbb{R}^N) = S^{-2_*/2}\|\lambda\|^{4/(N-4)}\lambda(\mathbb{R}^N). \tag{A.12}$$

Note that

$$\begin{aligned}
\nu(\mathbb{R}^N) &= \nu(\Omega_0) + \nu(\mathbb{R}^N\backslash\Omega_0)\\
&< S^{-2_*/2}\|\lambda\|^{4/(N-4)}\lambda(\Omega_0) + S^{-2_*/2}\|\lambda\|^{4/(N-4)}\lambda(\mathbb{R}^N \setminus \Omega_0)\\
&= S^{-2_*/2}\|\lambda\|^{4/(N-4)}[\lambda(\Omega_0) + \lambda(\mathbb{R}^N \setminus \Omega_0)] = S^{-2_*/2}\|\lambda\|^{4/(N-4)}\lambda(\mathbb{R}^N),
\end{aligned}$$

which contradicts (A.12). It follows from (A.10), $\nu(\Omega) = S^{-2_*/2}\|\lambda\|^{4/(N-4)}\lambda(\Omega)$ and $\|\nu\|^{2/2_*} = S^{-1}\|\lambda\|$ that

$$\left(\int_{\mathbb{R}^N} |h|^{2_*}\, d\nu\right)^{1/2_*}\|\nu\|^{2/N} \le \left(\int_{\mathbb{R}^N} |h|^2\, d\nu\right)^{1/2}, \quad \text{for all } h \in \mathcal{C}_c^\infty(\mathbb{R}^N).$$

Then, for each open set $\Omega \subset \mathbb{R}^N$,

$$\nu(\Omega)^{1/2_*}\nu(\mathbb{R}^N)^{2/N} \le \nu(\Omega)^{1/2}.$$

Since $1/2 - 1/2_* = 2/N$, we have

$$\nu(\Omega) = 0 \quad \text{or} \quad \nu(\Omega) \ge \nu(\mathbb{R}^N), \quad \text{for any open set } \Omega \subset \mathbb{R}^N.$$

Hence, ν is concentrated at a single point, which is the same point where λ concentrates, because $\nu = S^{-2_*/2}\|\lambda\|^{4/(N-4)}\lambda$.

General case: *u is not necessarily zero and we prove* (3.5)–(3.8).

Write $v_n := u_n - u$. So, $v_n \rightharpoonup 0$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, $|\Delta v_n|^2 \overset{*}{\rightharpoonup} \lambda$ and $|v_n|^{2_*} \overset{*}{\rightharpoonup} \nu$ in the sense of measures on $\mathbb{R}^N$, and $v_n \to 0$ almost everywhere on $\mathbb{R}^N$, and thus, from the previous case, (3.5) holds.

We have

$$\begin{aligned}\int_{|x|\geq R}|\Delta u_n|^2\,dx &= \int_{|x|\geq R}|\Delta v_n+\Delta u|^2\,dx\\ &= \int_{|x|\geq R}|\Delta v_n|^2\,dx+2\int_{|x|\geq R}\Delta v_n\Delta u\,dx+\int_{|x|\geq R}|\Delta u|^2\,dx\end{aligned}$$

which implies

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\int_{|x|\geq R}|\Delta u_n|^2\,dx\\ &\quad=\overline{\lim_{n\to\infty}}\int_{|x|\geq R}|\Delta v_n|^2\,dx+2\,\overline{\lim_{n\to\infty}}\int_{|x|\geq R}\Delta v_n\Delta u\,dx+\int_{|x|\geq R}|\Delta u|^2\,dx\end{aligned}$$

and, since $v_n\rightharpoonup 0$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, we conclude that

$$\overline{\lim_{n\to\infty}}\int_{|x|\geq R}|\Delta u_n|^2\,dx=\overline{\lim_{n\to\infty}}\int_{|x|\geq R}|\Delta v_n|^2\,dx+\int_{|x|\geq R}|\Delta u|^2\,dx. \tag{A.13}$$

So, (A.13) implies that

$$\lambda_\infty=\lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{|x|\geq R}|\Delta u_n|^2\,dx=\lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{|x|\geq R}|\Delta v_n|^2\,dx.$$

By the Brézis–Lieb lemma [7],

$$\int_{|x|\geq R}|u|^{2_*}\,dx=\lim_{n\to\infty}\bigg(\int_{|x|\geq R}|u_n|^{2_*}\,dx-\int_{|x|\geq R}|v_n|^{2_*}\,dx\bigg),$$

and therefore

$$\nu_\infty=\lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{|x|\geq R}|u_n|^{2_*}\,dx=\lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{|x|\geq R}|v_n|^{2_*}\,dx.$$

From the previous particular case, it follows (3.6).

Now we proceed to prove (3.7). First we prove that

$$|\Delta u_n|^2\overset{*}{\rightharpoonup}\lambda+|\Delta u|^2. \tag{A.14}$$

Indeed, from the identity $|\Delta u_n|^2=|\Delta v_n+\Delta u|^2=|\Delta v_n|^2+2\Delta v_n\Delta u+|\Delta u|^2$, we have

$$\int_{\mathbb{R}^N}\varphi|\Delta u_n|^2\,dx=\int_{\mathbb{R}^N}\varphi|\Delta v_n|^2\,dx+2\int_{\mathbb{R}^N}\Delta v_n\Delta u\varphi\,dx+\int_{\mathbb{R}^N}\varphi|\Delta u|^2\,dx,$$

for all $\varphi\in\mathcal{C}_0(\mathbb{R}^N)$. Since $v_n\rightharpoonup 0$ in $\mathcal{D}^{2,2}(\mathbb{R}^N)$ and $|\Delta v_n|^2\overset{*}{\rightharpoonup}\lambda$ we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi|\Delta u_n|^2\,dx=\int_{\mathbb{R}^N}\varphi\,d\lambda+\int_{\mathbb{R}^N}\varphi|\Delta u|^2\,dx,$$

for all $\varphi\in\mathcal{C}_0(\mathbb{R}^N)$, which is precisely (A.14).

Fix $R > 0$ and let $\psi_R \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that $\psi_R(x) = 1$ for $|x| \geq R+1$, $\psi_R(x) = 0$ for $|x| \leq R$ and $0 \leq \psi_R \leq 1$ on $\mathbb{R}^N$. From (A.14) we have

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} |\Delta u_n|^2\,dx$$
$$= \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} \psi_R |\Delta u_n|^2\,dx + \int_{\mathbb{R}^N} (1-\psi_R)d\lambda + \int_{\mathbb{R}^N} (1-\psi_R)|\Delta u|^2\,dx.$$

Taking now $R \to \infty$, it follows from the dominated convergence theorem that

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} |\Delta u_n|^2\,dx = \lambda_\infty + \int_{\mathbb{R}^N} 1\,d\lambda + \int_{\mathbb{R}^N} |\Delta u|^2\,dx$$

and thus

$$\overline{\lim_{n\to\infty}} |\Delta u_n|_2^2 = |\Delta u|_2^2 + \|\lambda\| + \lambda_\infty,$$

which is precisely (3.7).

To prove (3.8), first observe that

$$|u_n|^{2_*} \overset{*}{\rightharpoonup} \nu + |u|^{2_*}. \tag{A.15}$$

Indeed, for any $f \in \mathcal{C}_0(\mathbb{R}^N)$ we have, from the Brézis–Lieb [7] lemma applied to f^+ and f^-,

$$\int_{\mathbb{R}^N} f|u|^{2_*}dx = \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} f|u_n|^{2_*}\,dx - \int_{\mathbb{R}^N} f|v_n|^{2_*}\,dx \right),$$

from where (A.15) follows since $|v_n|^{2_*} \overset{*}{\rightharpoonup} \nu$.

Fix $R > 0$ and let $\psi_R \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that $\psi_R(x) = 1$ for $|x| \geq R+1$, $\psi_R(x) = 0$ for $|x| \leq R$ and $0 \leq \psi_R \leq 1$ on $\mathbb{R}^N$. Then

$$\overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} |u_n|^{2_*}\,dx = \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} \psi_R |u_n|^{2_*}\,dx + \int_{\mathbb{R}^N} (1-\psi_R)d\nu + \int_{\mathbb{R}^N} (1-\psi_R)|u|^{2_*}\,dx.$$

Taking $R \to \infty$, it follows from the dominated convergence theorem that

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^N} |u_n|^{2_*}\,dx = \nu_\infty + \int_{\mathbb{R}^N} 1\,d\nu + \int_{\mathbb{R}^N} |u|^{2_*}\,dx$$

and thus

$$\overline{\lim_{n\to\infty}} |u_n|_{2_*}^{2_*} = |u|_{2_*}^{2_*} + \|\nu\| + \nu_\infty. \qquad \square$$

Appendix B. Proof of Lemma 4.2

Proof. Without loss of generality, suppose $0 \in \Omega$. Let $\xi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ be a function such that $0 \leq \xi(x) \leq 1$, for all $x \in \mathbb{R}^N$, $\xi \equiv 1$ in $B(0,\rho/2)$, $\xi \equiv 0$ in $B(0,\rho)^c$, and $B(0,\rho) \subset\subset \Omega$, $\rho > 0$. Set

$$U_\delta(x) := \xi(x)\psi_\delta(x), \quad x \in \mathbb{R}^N,\ 0 < \delta < \rho,$$

where $\psi_\delta = S^{(4-N)/8}\varphi_\delta$ and $\varphi_\delta(x) = \varphi_{\delta,0}(x)$ is given by (2.3). Then

$$\int_{\mathbb{R}^N} |\Delta \psi_\delta|^2\,dx = S \quad \text{and} \quad \int_{\mathbb{R}^N} |\psi_\delta|^{2_*}\,dx = 1,$$

and, see [10, (6.4) and (6.3)] respectively, we have

$$|\Delta U_\delta|^2_{2,\Omega} = S + O(\delta^{N-4}), \tag{B.1}$$

$$|U_\delta|^{2_*}_{2_*,\Omega} = 1 + O(\delta^N). \tag{B.2}$$

In order to get (4.1), we will estimate $|U_\delta|^2_{2,\Omega}$. We have

$$|U_\delta|^2_{2,\Omega} = \int_\Omega |\xi(x)\psi_\delta(x)|^2\,dx = \int_{B(0,\rho)} |\psi_\delta(x)|^2\,dx + \int_{B(0,\rho)} [|\xi(x)|^2 - 1]|\psi_\delta(x)|^2\,dx.$$

Note that

$$\begin{aligned}
&\int_{B(0,\rho)} ||\xi(x)|^2 - 1||\psi_\delta(x)|^2\,dx \\
&= \int_{B(0,\rho)\setminus B(0,\rho/2)} ||\xi(x)|^2 - 1||\psi_\delta(x)|^2\,dx \le \int_{B(0,\rho)\setminus B(0,\rho/2)} |\psi_\delta(x)|^2\,dx \\
&= \int_{B(0,\rho)\setminus B(0,\rho/2)} \frac{C\delta^{N-4}}{(\delta^2+|x|^2)^{N-4}}\,dx \le \int_{B(0,\rho)\setminus B(0,\rho/2)} \frac{C\delta^{N-4}}{|x|^{2(N-4)}}\,dx = O(\delta^{N-4}).
\end{aligned}$$

So, we obtain

$$\int_\Omega |U_\delta(x)|^2\,dx = \int_{B(0,\rho)} |\psi_\delta(x)|^2\,dx + O(\delta^{N-4}). \tag{B.3}$$

Now,

$$\int_{B(0,\rho)} |\psi_\delta(x)|^2\,dx = \int_{B(0,\delta)} |\psi_\delta(x)|^2\,dx + \int_{\delta<|x|<\rho} |\psi_\delta(x)|^2\,dx. \tag{B.4}$$

Note that

$$\begin{aligned}
\int_{B(0,\delta)} |\psi_\delta(x)|^2\,dx &= \int_{B(0,\delta)} \frac{C\delta^{N-4}}{(\delta^2+|x|^2)^{N-4}}\,dx \\
&\ge \int_{B(0,\delta)} \frac{C\delta^{N-4}}{(2\delta^2)^{N-4}}\,dx = C\delta^4,
\end{aligned} \tag{B.5}$$

and

$$\begin{aligned}
\int_{\delta<|x|<\rho} |\psi_\delta(x)|^2\,dx &= \int_{\delta<|x|<\rho} \frac{C\delta^{N-4}}{(\delta^2+|x|^2)^{N-4}}\,dx \ge \int_{\delta<|x|<\rho} \frac{C\delta^{N-4}}{(2|x|^2)^{N-4}}\,dx \\
&= C\delta^{N-4}\int_{\delta<|x|<\rho} \frac{1}{|x|^{2(N-4)}}\,dx = C\delta^{N-4}\int_\delta^\rho\int_{S_r} \frac{1}{r^{2(N-4)}}\,dS\,dr,
\end{aligned}$$

which implies

$$\int_{\delta<|x|<\rho} |\psi_\delta(x)|^2 dx \ge C\delta^{N-4}\begin{cases} \log r|_\delta^\rho & \text{if } N=8, \\ -\dfrac{1}{N-8}\dfrac{1}{r^{N-8}}\Big|_\delta^\rho & \text{if } N>8. \end{cases} \tag{B.6}$$

Finally, combining (B.3)–(B.6), we conclude that

$$|U_\delta|^2_{2,\Omega} \ge \begin{cases} C\delta^4|\log\delta| + O(\delta^4) & \text{if } N=8, \\ C\delta^4 + O(\delta^{N-4}) & \text{if } N>8. \end{cases} \tag{B.7}$$

Then, from (B.1), (B.2) and (B.7), there exists a constant $C = C(N) > 0$ such that

$$\frac{|\Delta U_\delta|_2^2 - \mu |U_\delta|_2^2}{|U_\delta|_{2_*}^2} \leq \left\{ \begin{array}{ll} S - \mu C \delta^4 |\log \delta| + O(\delta^4), & N = 8, \\ S - \mu C \delta^4 + O(\delta^{N-4}), & N > 8, \end{array} \right\} < S$$

for $N \geq 8$ and $\delta > 0$ small. □

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