

**GLOBAL EXPONENTIAL STABILITY AND EXISTENCE
OF ANTI-PERIODIC SOLUTIONS
TO IMPULSIVE COHEN–GROSSBERG NEURAL NETWORKS
ON TIME SCALES**

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ABSTRACT. By using the method of coincidence degree theory and Lyapunov functions, some new criteria are established for the existence and global exponential stability of anti-periodic solutions to impulsive Cohen–Grossberg neural networks on time scales. Our results are new even if the time scale $\mathbb{T} = \mathbb{R}$ or \mathbb{Z} . Finally, an example is given to illustrate our results.

1. Introduction

Since Cohen and Grossberg proposed a class of neural networks in 1983 [10], which include Hopfield neural networks [14], shunting neural networks and other neural networks [23], this model has received increasing interest due to its promising potential for applications in classification, parallel computation, associative memory, especially in solving some optimization problems. Such applications rely on the qualitative properties of stability. Thus, the qualitative analysis of these dynamic behaviors is a prerequisite step for the practical design and application of neural networks. Because of the finite speed of switching and

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transmission of signals in a network, time delays are inevitably present in electronic implementation of neural networks, which may influence the stability of the entire network by creating oscillatory or unstable phenomena. In recent years, the dynamical behaviors of Cohen–Grossberg neural networks with delays have been studied by many researchers (see e.g. [9], [17]–[19], [21], [25]).

In reality, many physical systems undergo abrupt changes at certain moments due to instantaneous perturbations, which lead to impulsive effects. Since the existence of impulses is frequently a source of instability, bifurcation and chaos for neural networks, the impulsive neural networks is an appropriate description of the phenomena of abrupt qualitative dynamical changes of essentially continuous-time systems, see [3], [20], [31], [32] and references therein.

In fact, continuous and discrete systems are very important in implementing and applications. It is well known that the theory of time scales has received a lot of attention which was introduced by Stefan Hilger in order to unify continuous and discrete analysis. Therefore, it is meaningful to study dynamic systems on time scales which can unify differential and difference systems see [20].

Arising from problems in applied sciences, it is well-known that anti-periodic problems of nonlinear differential equations have been extensively studied by many authors during the past twenty years, see [1], [3], [27], [28], [33] and references therein. For example, anti-periodic trigonometric polynomials are important in the study of interpolation problems [11], [12], and anti-periodic wavelets are discussed in [6]. In contrast, however, very few results are available on the existence and exponential stability of anti-periodic solutions for neural networks, while the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations [8], [22], [29], [30].

To the best of our knowledge, the existence of anti-periodic solutions to impulsive neural networks with time-varying delays has seldom been investigated and remains important and challenging.

Motivated by above, in this paper we are concerned with the following impulsive Cohen–Grossberg neural networks on time scales

$$(1.1) \quad \begin{cases} x_i^\Delta(t) = -a_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^n c_{ij}(t) f_j(x_j(t)) \right. \\ \quad \left. - \sum_{j=1}^n d_{ij}(t) g_j(x_j(t - \tau_{ij}(t))) - e_i(t) \right], & t \in \mathbb{T}^+, t \neq t_k, k \in \mathbb{N}, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = I_{ik}(x_i(t_k)), & i = 1, \dots, n, \end{cases}$$

where \mathbb{T} is an $\omega/2$ -periodic time scale which has the subspace topology inherited from the standard topology on \mathbb{R} , $\mathbb{T}^+ = \{t \in \mathbb{T} : t \geq 0\}$, $\mathbb{N} = \{1, 2, \dots\}$; for each interval I of \mathbb{R} , we denote $I_{\mathbb{T}} = I \cap \mathbb{T}$, $x_i(t_k^+)$, $x_i(t_k^-)$ ($i = 1, \dots, n$) represent the right and left limit of $x_i(t_k)$ in the sense of time scales, $\{t_l\}$ is a sequence of

real numbers such that $0 < t_1 < \dots < t_l \rightarrow \infty$ as $l \rightarrow \infty$; there exists a positive integer q such that $t_{l+q} = t_l + \omega/2$, $I_{ik+q} = -I_{ik}$, $l \in \mathbb{N}$, $i = 1, \dots, n$. Without loss of generality, we also assume that $[0, \omega/2)_{\mathbb{T}} \cap \{t_l : l \in \mathbb{N}\} = \{t_1, \dots, t_q\}$.

The initial conditions associated with system (1.1) are of the form

$$x_i(t) = \varphi_i(t), \quad t \in [-\tau, 0]_{\mathbb{T}}, \quad \tau = \max_{1 \leq i, j \leq n} \left\{ \max_{t \in [0, \omega]_{\mathbb{T}}} \tau_{ij}(t) \right\},$$

where $\varphi_i(t)$ ($i = 1, \dots, n$) are continuous functions on $[-\tau, 0]_{\mathbb{T}}$.

The main purpose of this paper is to study the existence and global exponential stability of the anti-periodic solutions of system (1.1) by using the method of coincidence degree theory and Lyapunov functional method.

Throughout this paper, we assume that

- (H₁) $\tau_{ij} \geq 0$, $c_{ij}, d_{ij}, e_i \in C(\mathbb{T}, \mathbb{R})$, $c_{ij}(t + \omega/2) = c_{ij}(t)$, $d_{ij}(t + \omega/2) = d_{ij}(t)$, $e_i(t + \omega/2) = -e_i(t)$, $i, j = 1, \dots, n$.
- (H₂) $a_i \in C(\mathbb{R}, (0, +\infty))$, $a_i(-u) = a_i(u)$ and there exist positive constants a_i^m, a_i^M such that $0 < a_i^m \leq a_i(u) \leq a_i^M$ for all $u \in \mathbb{R}$, $i = 1, \dots, n$.
- (H₃) $b_i \in C(\mathbb{R}, \mathbb{R})$ are delta differentiable, $b_i(0) = 0$, $b_i(-u) = -b_i(u)$ and there exist positive constants ρ_i, δ_i such that $0 < \rho_i \leq b_i^\Delta(u) \leq \delta_i$ for all $u \in \mathbb{R}$, $i = 1, \dots, n$.
- (H₄) $f_j, g_j \in C(\mathbb{R}, \mathbb{R})$, $f_j(-u) = -f_j(u)$, $g_j(-u) = -g_j(u)$ and there exist positive constants L_j and P_j such that

$$|f_j(u) - f_j(v)| \leq L_j|u - v| \quad \text{and} \quad |g_j(u) - g_j(v)| \leq P_j|u - v|$$

for all $u, v \in \mathbb{R}$, $j = 1, \dots, n$.

- (H₅) $I_{ik} \in C(\mathbb{R}, \mathbb{R})$ and there exist positive constants G_{ik} such that

$$|I_{ik}(u) - I_{ik}(v)| \leq G_{ik}|u - v|$$

for all $u, v \in \mathbb{R}$, $k \in \mathbb{N}$, $i = 1, \dots, n$.

For the sake of convenience, we introduce the following notations:

$$h^L = \max_{t \in [0, \omega]_{\mathbb{T}}} |h(t)|, \quad \|h\|_2 = \left(\int_0^\omega |h(t)|^2 \Delta t \right)^{1/2},$$

where h is an ω -periodic function.

The organization of this paper is as follows. In Section 2, we make some preparations. In Section 3, by using the method of coincidence degree, we obtain the existence of the anti-periodic solutions of system (1.1). In Section 4, we give the criteria of global exponential stability of the anti-periodic solutions of system (1.1). In Section 5, an example is also provided to illustrate the effectiveness of the main results in Sections 3 and 4. The conclusions are drawn in Section 6.

2. Preliminaries

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

DEFINITION 2.1 ([5]). A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real set \mathbb{R} with the topology and ordering inherited from \mathbb{R} . The forward and backward jump operators $\sigma, \rho: \mathbb{T} \rightarrow \mathbb{T}$ and the graininess $\mu: \mathbb{T} \rightarrow \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) := \sigma(t) - t.$$

The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense or right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If \mathbb{T} has a left-scattered maximum m , defined $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$.

DEFINITION 2.2 ([5]). A vector function $f: \mathbb{T} \rightarrow \mathbb{R}^n$ is rd-continuous provided it is continuous at each right-dense point in \mathbb{T} and has a left-sided limit at each left-dense point in \mathbb{T} . The set of rd-continuous functions $f: \mathbb{T} \rightarrow \mathbb{R}^n$ will be denoted by $C_{\text{rd}}(\mathbb{T}) = C_{\text{rd}}(\mathbb{T}, \mathbb{R}^n)$.

DEFINITION 2.3 ([5]). For a function $f: \mathbb{T} \rightarrow \mathbb{R}$ (the range \mathbb{R} of f may be actually replaced by Banach space) the (delta) derivative is defined by

$$f^\Delta = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^\Delta = \lim_{s \rightarrow t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}$$

provided this limit exists.

DEFINITION 2.4 ([5]). If $F^\Delta(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s) \Delta s = F(t) - F(a).$$

DEFINITION 2.5 ([5]). A function $p: \mathbb{T} \rightarrow \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$, where $\mu(t) = \sigma(t) - t$ is the graininess function. The set of all regressive rd-continuous functions $f: \mathbb{T} \rightarrow \mathbb{R}$ is denoted by \mathcal{R} while the set \mathcal{R}^+ is given by $\{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0\}$ for all $t \in \mathbb{T}$. Let $p \in \mathcal{R}$. The exponential function is defined by

$$e_p(t, s) = \exp \left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau \right),$$

where $\xi_{h(z)}$ is the so-called cylinder transformation.

DEFINITION 2.6 ([16]). For each $t \in \mathbb{T}$, let N be a neighbourhood of t . Then we defined the generalized derivative (or Dini derivative), $D^+u^\Delta(t)$ to mean that, given $\varepsilon > 0$, there exists a right neighbourhood $N(\varepsilon) \subset N$ of t such that

$$\frac{u(\sigma(t)) - u(s)}{\sigma(t) - s} < D^+u^\Delta(t) + \varepsilon$$

for each $s \in N(\varepsilon)$, $s > t$.

In case t is right-scattered and $u(t)$ is continuous at t , this reduce to

$$D^+u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\sigma(t) - t}.$$

Similar to [15], we shall first give the definition of anti-periodic function on a time scale as following:

DEFINITION 2.7. We say that a time scale \mathbb{T} is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p . We say that the function $f: \mathbb{T} \rightarrow \mathbb{R}$ is $\omega/2$ -anti-periodic if there exists a natural number n such that $\omega/2 = np$, $f(t + \omega/2) = -f(t)$ for all $t \in \mathbb{T}$ and $\omega/2$ is the smallest number such that $f(t + \omega/2) = -f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is $\omega/2$ -anti-periodic if $\omega/2$ is the smallest positive number such that $f(t + \omega/2) = -f(t)$ for all $t \in \mathbb{T}$.

LEMMA 2.8 ([5]). Let $p, q \in \mathcal{R}$. Then:

- (a) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- (b) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (c) $1/e_p(t, s) = e_{\ominus p}(t, s)$, where $\ominus p(t) = -p(t)/(1 + \mu(t)p(t))$;
- (d) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- (e) $e_p^\Delta(\cdot, s) = pe_p(\cdot, s)$.

LEMMA 2.9 ([5]). Assume that $f, g: \mathbb{T} \rightarrow \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$. Then

$$\begin{aligned} (fg)^\Delta(t) &= f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)), \\ f(\sigma(t)) &= f(t) + \mu(t)f^\Delta(t). \end{aligned}$$

The following lemmas can be found in [2], [4], [24], respectively.

LEMMA 2.10. Let $t_1, t_2 \in [0, \omega]_{\mathbb{T}}$. If $x: \mathbb{T} \rightarrow \mathbb{R}$ is ω -periodic, then

$$x(t) \leq x(t_1) + \int_0^\omega |x^\Delta(s)|\Delta s \quad \text{and} \quad x(t) \geq x(t_2) - \int_0^\omega |x^\Delta(s)|\Delta s.$$

LEMMA 2.11 (Cauchy Schwarz inequality on time scale). Let $a, b \in \mathbb{T}$. For rd-continuous functions $f, g: [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ we have

$$\int_a^b |f(t)||g(t)|\Delta t \leq \left(\int_a^b |f(t)|^2\Delta t \right)^{1/2} \left(\int_a^b |g(t)|^2\Delta t \right)^{1/2}.$$

LEMMA 2.12 (Mean value theorem, [13]). *Let function f be continuous on $[a, b]_{\mathbb{T}}$ and delta differentiable on $[a, b)_{\mathbb{T}}$, then there exist $\xi, \varsigma \in [a, b)_{\mathbb{T}}$ such that*

$$f^\Delta(\xi)(b - a) \leq f(b) - f(a) \leq f^\Delta(\varsigma)(b - a).$$

DEFINITION 2.13. The anti-periodic solution $x^*(t) = (x_1^*(t), \dots, x_n^*(t))^T$ of system (1.1) is said to be globally exponentially stable if there exist positive constants λ and $M = M(\lambda) \geq 1$, for any solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ of system (1.1) with the initial value $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))^T \in C([- \tau, 0]_{\mathbb{T}}, \mathbb{R}^n)$, such that

$$\sum_{i=1}^n |x_i(t) - x_i^*(t)| \leq M(\lambda)e_{\ominus\lambda}(t, \alpha)\|\varphi - x^*\|,$$

where

$$\|\varphi - x^*\| = \sum_{i=1}^n \max_{s \in [-\tau, 0]_{\mathbb{T}}} |\varphi_i(s) - x_i^*(s)|, \quad \alpha \in [-\tau, 0]_{\mathbb{T}}.$$

LEMMA 2.14. *Let $f \in C(\mathbb{T}, \mathbb{R})$ is Δ -differentiable at t . Then*

$$\Delta^r |f(t)| \leq \text{sign}(f^\sigma(t))f^\Delta(t), \quad \text{where } f^\sigma(t) = f(\sigma(t)).$$

PROOF. *Case 1.* If t is a right dense point, that is, $\sigma(t) = t$.

$$(2.1) \quad \Delta^r |f(t)| \leq \text{sign}(f(t))f^\Delta(t) = \text{sign}(f^\sigma(t))f^\Delta(t).$$

Case 2. If t is a right scattered point, that is, $\sigma(t) > t$. If $f(t)f^\sigma(t) > 0$, one can easily have $\text{sign}(f(t)) = \text{sign}(f^\sigma(t))$, so we can obtain

$$(2.2) \quad \begin{aligned} \Delta^r |f(t)| &= \frac{|f^\sigma(t)| - |f(t)|}{\mu(t)} = \frac{\text{sign}(f^\sigma(t))f^\sigma(t) - \text{sign}(f(t))f(t)}{\mu(t)} \\ &= \text{sign}(f^\sigma(t)) \left(\frac{f^\sigma(t) - f(t)}{\mu(t)} \right) = \text{sign}(f^\sigma(t))f^\Delta(t). \end{aligned}$$

If $f(t)f^\sigma(t) \leq 0$, then one can get $|f(t)| \geq \text{sign}(f^\sigma(t))f(t)$. Then

$$(2.3) \quad \begin{aligned} \Delta^r |f(t)| &= \frac{|f^\sigma(t)| - |f(t)|}{\mu(t)} = \frac{\text{sign}(f^\sigma(t))f^\sigma(t) - |f(t)|}{\mu(t)} \\ &\leq \frac{\text{sign}(f^\sigma(t))f^\sigma(t) - \text{sign}(f^\sigma(t))f(t)}{\mu(t)} \\ &= \text{sign}(f^\sigma(t)) \left(\frac{f^\sigma(t) - f(t)}{\mu(t)} \right) = \text{sign}(f^\sigma(t))f^\Delta(t). \end{aligned}$$

Therefore, by (2.1)–(2.3), one can get $\Delta^r |f(t)| \leq \text{sign}(f^\sigma(t))f^\Delta(t)$. □

The following fixed point theorem of coincidence degree is crucial in the arguments of our main results.

LEMMA 2.15 ([26]). *Let \mathbb{X}, \mathbb{Y} be two Banach spaces, $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L: D(L) \subset \mathbb{X} \rightarrow \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \bar{\Omega} \neq \emptyset$ and $N: \bar{\Omega} \rightarrow \mathbb{Y}$ is L -compact. Further, we also assume that*

(H) $Lx - Nx \neq \lambda(-Lx - N(-x))$ for all $x \in D(L) \cap \partial\Omega$, $\lambda \in (0, 1]$.

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \bar{\Omega}$.

3. Existence of anti-periodic solutions

THEOREM 3.1. Assume that (H₁)–(H₅) hold. Suppose further that

(H₆) $\Pi = (h_{ij})_{n \times n}$ is a nonsingular M matrix, where

$$h_{ij} = \begin{cases} a_i^m - \delta_i a_i^m a_i^M \omega - \left(\frac{1}{\omega \rho_i} + a_i^m\right) \sum_{k=1}^{2q} G_{ik} \\ \qquad - a_i^M \left(\frac{1}{\rho_i} + a_i^m \omega\right) (c_{ij}^L L_j + d_{ij}^L P_j), & i = j, \\ -a_i^M \left(\frac{1}{\rho_i} + a_i^m \omega\right) (c_{ij}^L L_j + d_{ij}^L P_j), & i \neq j. \end{cases}$$

Then system (1.1) has at least one $\omega/2$ -anti-periodic solution.

PROOF. Let $C^k[0, \omega; t_1, \dots, t_q, t_{q+1}, \dots, t_{2q}]_{\mathbb{T}} = \{x: [0, \omega]_{\mathbb{T}} \rightarrow \mathbb{R} \mid x^k(t)$ is a piecewise continuous map with first-class discontinuous points in $[0, \omega]_{\mathbb{T}} \cap \{t_l : l \in \mathbb{N}\}$ and at each discontinuous point it is continuous on the left}, $k = 0, 1$. Take

$$\mathbb{X} = \{x \in C[0, \omega; t_1, \dots, t_q, t_{q+1}, \dots, t_{2q}]_{\mathbb{T}} : x(t + \omega/2) = -x(t) \text{ for all } t \in [0, \omega/2]_{\mathbb{T}}\}$$

and $\mathbb{Y} = \mathbb{X} \times \mathbb{R}^{n \times q}$, then X and Y are Banach spaces with the norms

$$\|x\|_{\mathbb{X}} = \sum_{i=1}^n |x_i|_0 \quad \text{and} \quad \|z\|_{\mathbb{Y}} = \|x\|_{\mathbb{X}} + \|y\|$$

for all $x \in \mathbb{X}$, $y \in \mathbb{R}^{n \times q}$, respectively, in which $|x_i|_0 = \max_{t \in [0, \omega]_{\mathbb{T}}} |x_i(t)|$, $i = 1, \dots, n$, $\|\cdot\|$ is any norm of $\mathbb{R}^{n \times q}$.

Set $L: \text{Dom } L \cap \mathbb{X} \rightarrow \mathbb{Y}$, $x \rightarrow (x^\Delta, \Delta x(t_1), \dots, \Delta x(t_q))$, where

$$\text{Dom } L = \{x \in C^1[0, \omega; t_1, \dots, t_{2q}]_{\mathbb{T}} : x(t + \omega/2) = -x(t) \text{ for all } t \in [0, \omega/2]_{\mathbb{T}}\},$$

and $N: \mathbb{X} \rightarrow \mathbb{Y}$

$$Nx = \left(\left(\begin{array}{c} A_1(t) \\ \vdots \\ A_n(t) \end{array} \right), \left(\begin{array}{c} I_{11}(x_1(t_1)) \\ \vdots \\ I_{n1}(x_n(t_1)) \end{array} \right), \left(\begin{array}{c} I_{12}(x_1(t_2)) \\ \vdots \\ I_{n2}(x_n(t_2)) \end{array} \right), \dots, \left(\begin{array}{c} I_{1q}(x_1(t_q)) \\ \vdots \\ I_{nq}(x_n(t_q)) \end{array} \right) \right),$$

where, for $i = 1, \dots, n$,

$$A_i(t) = -a_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^n c_{ij}(t) f_j(x_j(t)) - \sum_{j=1}^n d_{ij}(t) g_j(x_j(t - \tau_{ij}(t))) - e_i(t) \right].$$

It is easy to see that

$$\text{Ker } L = \{0\} \quad \text{and} \quad \text{Im } L = \left\{ z = (f, C_1, \dots, C_q) \in \mathbb{Y} : \int_0^\omega f(s) \Delta s = 0 \right\} \equiv \mathbb{Y}.$$

Thus $\dim \text{Ker } L = 0 = \text{codim Im } L$, and L is a linear Fredholm operator of index zero.

Define the continuous projector $P: \mathbb{X} \rightarrow \text{Ker } L$ and the averaging projector $Q: \mathbb{Y} \rightarrow \mathbb{Y}$ by

$$Px = \int_0^\omega x(s) \Delta s = 0$$

and

$$Qz = Q(f, C_1, \dots, C_q) = \left(\frac{1}{\omega} \int_0^\omega f(s) \Delta s, 0, \dots, 0 \right).$$

Hence $\text{Im } P = \text{Ker } L$ and $\text{Ker } Q = \text{Im } L = \text{Im } (I - Q)$.

Denoting by $L_P^{-1}: \text{Im } L \rightarrow \text{Dom}(L) \cap \text{Ker } P$ the inverse of $L|_{\text{Dom}(L) \cap \text{Ker } P}$, we have

$$L_P^{-1}z = \int_0^t f(s) \Delta s + \sum_{t_k < t} C_k - \frac{1}{2} \int_0^{\omega/2} f(s) \Delta s - \frac{1}{2} \sum_{k=1}^q C_k,$$

in which $C_{q+i} = -C_i$ for all $1 \leq i \leq q$.

Similar to [20], it is not difficult to show that $QN(\overline{\Omega})$, $L_P^{-1}(I - Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$. Therefore, N is L -compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

In order to apply Lemma 2.7, we need to find an appropriate open bounded subset Ω in \mathbb{X} . Corresponding to the operator equation

$$Lx - Nx = \lambda(-Lx - N(-x)), \quad \lambda \in (0, 1],$$

we have, for $i = 1, \dots, n$,

$$(3.1) \quad \begin{cases} x_i^\Delta(t) = \frac{1}{1+\lambda} G_i(t, x) - \frac{\lambda}{1+\lambda} G_i(t, -x), & t \in \mathbb{T}^+, \\ \Delta x_i(t_k) = \frac{1}{1+\lambda} I_{ik}(x_i(t_k)) - \frac{\lambda}{1+\lambda} I_{ik}(-x_i(t_k)), & t \neq t_k, k \in \mathbb{N}, \\ & i = 1, \dots, n, \end{cases}$$

where

$$G_i(t, x) = -a_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^n c_{ij}(t) f_j(x_j(t)) - \sum_{j=1}^n d_{ij}(t) g_j(x_j(t - \tau_{ij}(t))) - e_i(t) \right]$$

and

$$G_i(t, -x) = -a_i(-x_i(t)) \left[b_i(-x_i(t)) - \sum_{j=1}^n c_{ij}(t) f_j(-x_j(t)) - \sum_{j=1}^n d_{ij}(t) g_j(-x_j(t - \tau_{ij}(t))) - e_i(t) \right].$$

Set $t_0 = t_0^+ = 0$, $t_{2q+1} = \omega$, in view of (3.1), we get from (H₂)–(H₅) and Lemma 2.5 that

$$\begin{aligned} (3.2) \quad & \int_0^\omega |x_i^\Delta(t)| \Delta t = \sum_{k=1}^{2q+1} \int_{t_{k-1}^+}^{t_k} |x_i^\Delta(t)| \Delta t + \sum_{k=1}^{2q} |I_{ik}(x_i(t_k))| \\ & \leq \int_0^\omega \left| \frac{1}{1+\lambda} G_i(t, x) - \frac{\lambda}{1+\lambda} G_i(t, -x) \right| \Delta t + \sum_{k=1}^{2q} |I_{ik}(x_i(t_k))| \\ & \leq \left[\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} \right] \int_0^\omega \max\{|G_i(t, x)|, |G_i(t, -x)|\} \Delta t \\ & \quad + \sum_{k=1}^{2q} |I_{ik}(x_i(t_k))| \\ & \leq a_i^M \left[\int_0^\omega |b_i(x_i(t))| \Delta t + \sum_{j=1}^n c_{ij}^L \int_0^\omega |f_j(x_j(t)) - f_j(0)| \Delta t \right. \\ & \quad \left. + \sum_{j=1}^n d_{ij}^L \int_0^\omega |g_j(x_j(t - \tau_{ij})) - g_j(0)| \Delta t + e_i^L \omega \right. \\ & \quad \left. + \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| \right] \\ & \quad + \sum_{k=1}^{2q} |I_{ik}(x_i(t_k)) - I_{ik}(0)| + \sum_{k=1}^{2q} |I_{ik}(0)| \\ & \leq a_i^M \left[\delta_i \sqrt{\omega} \|x_i\|_2 + \sum_{j=1}^n c_{ij}^L L_j \sqrt{\omega} \|x_j\|_2 + \sum_{j=1}^n d_{ij}^L P_j \sqrt{\omega} \|x_j\|_2 + e_i^L \omega \right. \\ & \quad \left. + \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| \right] + \sum_{k=1}^{2q} G_{ik} |x_i|_0 + \sum_{k=1}^{2q} |I_{ik}(0)| \end{aligned}$$

$$\begin{aligned}
&= a_i^M \delta_i \sqrt{\omega} \|x_i\|_2 + \sum_{k=1}^{2q} G_{ik} |x_i|_0 + a_i^M \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 \\
&\quad + a_i^M e_i^L \omega + a_i^M \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + a_i^M \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \sum_{k=1}^{2q} |I_{ik}(0)|,
\end{aligned}$$

for $i = 1, \dots, n$. Integrating (3.1) from 0 to ω , we have from (H₂)–(H₅) that

$$\begin{aligned}
&\left| \int_0^\omega \left[\frac{a_i(x_i(t))b_i(x_i(t))}{1+\lambda} - \frac{\lambda a_i(-x_i(t))b_i(-x_i(t))}{1+\lambda} \right] \Delta t \right| \\
&= \left| \int_0^\omega \left[\frac{a_i(x_i(t))b_i(x_i(t))}{1+\lambda} + \frac{\lambda a_i(x_i(t))b_i(x_i(t))}{1+\lambda} \right] \Delta t \right| \\
&= \left| \int_0^\omega a_i(x_i(t))b_i(x_i(t)) \Delta t \right| \\
&= \left| \frac{1}{1+\lambda} \int_0^\omega a_i(x_i(t)) \left[\sum_{j=1}^n c_{ij}(t)f_j(x_j(t)) \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^n d_{ij}(t)g_j(x_j(t - \tau_{ij}(t))) + e_i(t) \right] \Delta t \right. \\
&\quad \left. + \frac{\lambda}{1+\lambda} \int_0^\omega a_i(-x_i(t)) \left[- \sum_{j=1}^n c_{ij}(t)f_j(-x_j(t)) \right. \right. \\
&\quad \left. \left. - \sum_{j=1}^n d_{ij}(t)g_j(-x_j(t - \tau_{ij}(t))) - e_i(t) \right] \Delta t \right. \\
&\quad \left. + \frac{1}{1+\lambda} \sum_{k=1}^{2q} I_{ik}(x_i(t_k)) - \frac{\lambda}{1+\lambda} \sum_{k=1}^{2q} I_{ik}(-x_i(t_k)) \right| \\
&\leq \left[\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} \right] a_i^M \int_0^\omega \max \left\{ \left| \sum_{j=1}^n c_{ij}(t)f_j(x_j(t)) \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^n d_{ij}(t)g_j(x_j(t - \tau_{ij}(t))) + e_i(t) \right|, \right. \\
&\quad \left. \left| \sum_{j=1}^n c_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n d_{ij}(t)g_j(x_j(t - \tau_{ij}(t))) - e_i(t) \right| \right\} \Delta t \\
&\quad + \left[\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda} \right] \max \left\{ \sum_{k=1}^{2q} |I_{ik}(x_i(t_k))|, \sum_{k=1}^{2q} |I_{ik}(-x_i(t_k))| \right\} \\
&\leq a_i^M \int_0^\omega \left[\sum_{j=1}^n c_{ij}^L |f_j(x_j(t)) - f_j(0)| \right. \\
&\quad \left. + \sum_{j=1}^n d_{ij}^L |g_j(x_j(t - \tau_{ij}(t))) - g_j(0)| + e_i^L \right] \Delta t
\end{aligned}$$

$$\begin{aligned}
 & + a_i^M \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + a_i^M \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| \\
 & + \max \left\{ \sum_{k=1}^{2q} |I_{ik}(x_i(t_k)) - I_{ik}(0)|, \sum_{k=1}^{2q} |I_{ik}(-x_i(t_k)) - I_{ik}(0)| \right\} + \sum_{k=1}^{2q} |I_{ik}(0)| \\
 \leq & a_i^M \left[\sum_{j=1}^n c_{ij}^L L_j \sqrt{\omega} \|x_j\|_2 + \sum_{j=1}^n d_{ij}^L P_j \sqrt{\omega} \|x_j\|_2 + e_i^L \omega \right] + a_i^M \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| \\
 & + a_i^M \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \sum_{k=1}^{2q} G_{ik} |x_i|_0 + \sum_{k=1}^{2q} |I_{ik}(0)| \\
 = & a_i^M \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 + \sum_{k=1}^{2q} G_{ik} |x_i|_0 + a_i^M e_i^L \omega \\
 & + a_i^M \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + a_i^M \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \sum_{k=1}^{2q} |I_{ik}(0)|,
 \end{aligned}$$

for $i = 1, \dots, n$, by Lemma 2.5 and (H₃), for $i = 1, \dots, n$, we obtain that

$$\begin{aligned}
 (3.3) \quad & \left| \int_0^\omega a_i(x_i(t))x_i(t)\Delta t \right| \\
 & = \frac{a_i^M}{\rho_i} \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 + \frac{1}{\rho_i} \sum_{k=1}^{2q} G_{ik} |x_i|_0 + \frac{a_i^M}{\rho_i} e_i^L \omega \\
 & \quad + \frac{a_i^M}{\rho_i} \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + \frac{a_i^M}{\rho_i} \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \frac{1}{\rho_i} \sum_{k=1}^{2q} |I_{ik}(0)|,
 \end{aligned}$$

From Lemma 2.3, for any $\zeta_i, \eta_i \in [0, \omega]_{\mathbb{T}}$, $i = 1, \dots, n$, we have

$$\begin{aligned}
 (3.4) \quad & \int_0^\omega a_i(x_i(t))x_i(t)\Delta t \\
 & \leq \int_0^\omega a_i(x_i(t))x_i(\zeta_i)\Delta t + \int_0^\omega a_i(x_i(t)) \left(\int_0^\omega |x_i^\Delta(t)|\Delta t \right) \Delta t,
 \end{aligned}$$

$$\begin{aligned}
 (3.5) \quad & \int_0^\omega a_i(x_i(t))x_i(t)\Delta t \\
 & \geq \int_0^\omega a_i(x_i(t))x_i(\eta_i)\Delta t - \int_0^\omega a_i(x_i(t)) \left(\int_0^\omega |x_i^\Delta(t)|\Delta t \right) \Delta t,
 \end{aligned}$$

where $i = 1, \dots, n$. Dividing by $\int_0^\omega a_i(x_i(t))\Delta t$ on the two sides of (3.4) and (3.5), respectively, we obtain that, for $i = 1, \dots, n$,

$$(3.6) \quad x_i(\zeta_i) \geq \frac{1}{\int_0^\omega a_i(x_i(t))\Delta t} \int_0^\omega a_i(x_i(t))x_i(t)\Delta t - \int_0^\omega |x_i^\Delta(t)|\Delta t,$$

$$(3.7) \quad x_i(\eta_i) \leq \frac{1}{\int_0^\omega a_i(x_i(t))\Delta t} \int_0^\omega a_i(x_i(t))x_i(t)\Delta t + \int_0^\omega |x_i^\Delta(t)|\Delta t.$$

Let $\bar{t}_i, \underline{t}_i \in [0, \omega]_{\mathbb{T}}$ such that $x_i(\bar{t}_i) = \max_{t \in [0, \omega]_{\mathbb{T}}} x_i(t)$, $x_i(\underline{t}_i) = \min_{t \in [0, \omega]_{\mathbb{T}}} x_i(t)$, by the arbitrariness of ζ_i, η_i , we obtain from (3.2), (3.3), (3.6) and (3.7) that

$$(3.8) \quad \begin{aligned} x_i(\underline{t}_i) &\geq \frac{1}{\int_0^\omega a_i(x_i(t))\Delta t} \int_0^\omega a_i(x_i(t))x_i(t)\Delta t - \int_0^\omega |x_i^\Delta(t)|\Delta t \\ &\geq -\frac{1}{\int_0^\omega a_i(x_i(t))\Delta t} \left| \int_0^\omega a_i(x_i(t))x_i(t)\Delta t \right| - \int_0^\omega |x_i^\Delta(t)|\Delta t \\ &\geq -\frac{1}{a_i^m \omega} \left[\frac{a_i^M}{\rho_i} \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 \right. \\ &\quad + \frac{1}{\rho_i} \sum_{k=1}^{2q} G_{ik} |x_i|_0 + \frac{a_i^M}{\rho_i} e_i^L \omega \\ &\quad + \frac{a_i^M}{\rho_i} \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + \frac{a_i^M}{\rho_i} \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \left. \frac{1}{\rho_i} \sum_{k=1}^{2q} |I_{ik}(0)| \right] \\ &\quad - \left[a_i^M \delta_i \sqrt{\omega} \|x_i\|_2 + \sum_{k=1}^{2q} G_{ik} |x_i|_0 \right. \\ &\quad + a_i^M \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 + a_i^M e_i^L \omega \\ &\quad \left. + a_i^M \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + a_i^M \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \sum_{k=1}^{2q} |I_{ik}(0)| \right], \end{aligned}$$

for $i = 1, \dots, n$ and

$$(3.9) \quad \begin{aligned} x_i(\bar{t}_i) &\leq \frac{1}{\int_0^\omega a_i(x_i(t))\Delta t} \int_0^\omega a_i(x_i(t))x_i(t)\Delta t + \int_0^\omega |x_i^\Delta(t)|\Delta t \\ &\leq \frac{1}{\int_0^\omega a_i(x_i(t))\Delta t} \left| \int_0^\omega a_i(x_i(t))x_i(t)\Delta t \right| + \int_0^\omega |x_i^\Delta(t)|\Delta t \\ &\leq \frac{1}{a_i^m \omega} \left[\frac{a_i^M}{\rho_i} \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 \right. \\ &\quad + \frac{1}{\rho_i} \sum_{k=1}^{2q} G_{ik} |x_i|_0 + \frac{a_i^M}{\rho_i} e_i^L \omega + \frac{a_i^M}{\rho_i} \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| \end{aligned}$$

$$\begin{aligned}
 & + \frac{a_i^M}{\rho_i} \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \frac{1}{\rho_i} \sum_{k=1}^{2q} |I_{ik}(0)| \Big] \\
 & + \left[a_i^M \delta_i \sqrt{\omega} \|x_i\|_2 + \sum_{k=1}^{2q} G_{ik} |x_i|_0 \right. \\
 & + a_i^M \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 + a_i^M e_i^L \omega \\
 & \left. + a_i^M \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + a_i^M \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \sum_{k=1}^{2q} |I_{ik}(0)| \right]
 \end{aligned}$$

for $i = 1, \dots, n$. Thus, we have from (3.8) and (3.9) that

$$\begin{aligned}
 (3.10) \quad |x_i|_0 & = \max_{t \in [0, \omega]_{\mathbb{T}}} |x_i(t)| \leq \frac{1}{a_i^m \omega} \left[\frac{a_i^M}{\rho_i} \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 \right. \\
 & + \frac{1}{\rho_i} \sum_{k=1}^{2q} G_{ik} |x_i|_0 + \frac{a_i^M}{\rho_i} e_i^L \omega + \frac{a_i^M}{\rho_i} \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| \\
 & + \frac{a_i^M}{\rho_i} \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \frac{1}{\rho_i} \sum_{k=1}^{2q} |I_{ik}(0)| \Big] \\
 & + \left[a_i^M \delta_i \sqrt{\omega} \|x_i\|_2 + \sum_{k=1}^{2q} G_{ik} |x_i|_0 \right. \\
 & + a_i^M \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 + a_i^M e_i^L \omega \\
 & \left. + a_i^M \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + a_i^M \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \sum_{k=1}^{2q} |I_{ik}(0)| \right],
 \end{aligned}$$

for $i = 1, \dots, n$. In addition, we have that

$$\|x_i\|_2 = \left(\int_0^\omega |x_i(s)|^2 \Delta s \right)^{1/2} \leq \sqrt{\omega} \max_{t \in [0, \omega]_{\mathbb{T}}} |x_i(t)| = \sqrt{\omega} |x_i|_0, \quad i = 1, \dots, n.$$

By (3.10), we have

$$\begin{aligned}
 a_i^m \omega |x_i|_0 & \leq \left[\frac{a_i^M}{\rho_i} \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 + \frac{1}{\rho_i} \sum_{k=1}^{2q} G_{ik} |x_i|_0 + \frac{a_i^M}{\rho_i} e_i^L \omega \right. \\
 & + \frac{a_i^M}{\rho_i} \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + \frac{a_i^M}{\rho_i} \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \frac{1}{\rho_i} \sum_{k=1}^{2q} |I_{ik}(0)| \Big] \\
 & + a_i^m \omega \left[a_i^M \delta_i \sqrt{\omega} \|x_i\|_2 + \sum_{k=1}^{2q} G_{ik} |x_i|_0 \right.
 \end{aligned}$$

$$\begin{aligned}
& + a_i^M \sqrt{\omega} \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) \|x_j\|_2 + a_i^M e_i^L \omega \\
& + a_i^M \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + a_i^M \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \sum_{k=1}^{2q} |I_{ik}(0)| \Big] \\
\leq & \left[\frac{a_i^M}{\rho_i} \omega \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) |x_j|_0 + \frac{1}{\rho_i} \sum_{k=1}^{2q} G_{ik} |x_i|_0 + \frac{a_i^M}{\rho_i} e_i^L \omega \right. \\
& + \frac{a_i^M}{\rho_i} \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + \frac{a_i^M}{\rho_i} \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \frac{1}{\rho_i} \sum_{k=1}^{2q} |I_{ik}(0)| \Big] \\
& + a_i^m \omega \left[a_i^M \delta_i \omega |x_i|_0 + \sum_{k=1}^{2q} G_{ik} |x_i|_0 + a_i^M \omega \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) |x_j|_0 \right. \\
& \left. + a_i^M e_i^L \omega + a_i^M \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| + a_i^M \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + \sum_{k=1}^{2q} |I_{ik}(0)| \right],
\end{aligned}$$

where $i = 1, 2, \dots, n$. That is,

$$\begin{aligned}
(3.11) \quad & \left[a_i^m - \delta_i a_i^m a_i^M \omega - \frac{1}{\omega \rho_i} \sum_{k=1}^{2q} G_{ik} - a_i^m \sum_{k=1}^{2q} G_{ik} \right] |x_i|_0 \\
& - a_i^M \left(\frac{1}{\rho_i} + a_i^m \omega \right) \sum_{j=1}^n (c_{ij}^L L_j + d_{ij}^L P_j) |x_j|_0 \leq \frac{1}{\omega} \left(\frac{1}{\rho_i} + a_i^m \omega \right) \left[a_i^M \sum_{j=1}^n c_{ij}^L \omega |f_j(0)| \right. \\
& \left. + a_i^M \sum_{j=1}^n d_{ij}^L \omega |g_j(0)| + a_i^M e_i^L \omega + \sum_{k=1}^{2q} |I_{ik}(0)| \right] = D_i
\end{aligned}$$

for $i = 1, \dots, n$. Denote $|x|_0 = (|x_1|_0, \dots, |x_n|_0)^T$ and $D = (D_1, \dots, D_n)^T$. Then (3.11) can be rewritten in the matrix form $\Pi |x|_0 \leq D$.

From the conditions of Theorem 3.1, Π is a nonsingular M matrix, hence

$$|x|_0 \leq \Pi^{-1} D \triangleq (M_1, \dots, M_n)^T.$$

Let $M = \sum_{i=1}^n M_i + 1$ (clearly, M is independent of λ). Take $\Omega = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} < M\}$. It is clear that Ω satisfies all the requirements in Lemma 2.7 and the condition (H) is satisfied. In view of all the discussions above, we conclude from Lemma 2.7 that system (1.1) has at least one $\omega/2$ -anti-periodic solution. This completes the proof. \square

4. Global exponential stability of the anti-periodic solution

Suppose that $x^*(t) = (x_1^*(t), \dots, x_n^*(t))^T$ is an $\omega/2$ -anti-periodic solution of system (1.1). In this section, we will construct some suitable Lyapunov functions to study the global exponential stability of this anti-periodic solution.

In view of system (4.1), for $t \in \mathbb{T}^+$, $t \neq t_k$, $k \in \mathbb{N}$, $i = 1, \dots, n$, we have

$$\begin{aligned} (x_i(t) - x_i^*(t))^\Delta &= -[a_i(x_i(t))b_i(x_i(t)) - a_i(x_i^*(t))b_i(x_i^*(t))] \\ &\quad + [a_i(x_i(t)) - a_i(x_i^*(t))] \sum_{j=1}^n c_{ij}(t)f_j(x_j(t)) \\ &\quad + a_i(x_i^*(t)) \sum_{j=1}^n c_{ij}(t)[f_j(x_j(t)) - f_j(x_j^*(t))] \\ &\quad + [a_i(x_i(t)) - a_i(x_i^*(t))] \sum_{j=1}^n d_{ij}(t)g_j(x_j(t - \tau_{ij})) \\ &\quad + a_i(x_i^*(t)) \sum_{j=1}^n d_{ij}(t)[g_j(x_j(t - \tau_{ij})) - g_j(x_j^*(t - \tau_{ij}))] \\ &\quad + [a_i(x_i(t)) - a_i(x_i^*(t))]e_i(t). \end{aligned}$$

Denote $y_i(t) = x_i(t) - x_i^*(t)$, $i = 1, \dots, n$. Hence we can obtain from (H₇)–(H₉) and Lemma 2.6 that

$$\begin{aligned} D^+|y_i(t)|^\Delta &\leq \text{sign}(y_i^\sigma(t))y_i^\Delta(t) \\ &\leq \left(-l_i + \nu_i \left(\sum_{j=1}^n c_{ij}^L f_j^M + \sum_{j=1}^n d_{ij}^L g_j^M + e_i^L \right) \right) |y_i^\sigma(t) - \mu(t)y_i^\Delta(t)| \\ &\quad + a_i^M \sum_{j=1}^n c_{ij}^L L_j |y_j(t)| + a_i^M \sum_{j=1}^n d_{ij}^L P_j |y_j(t - \tau_{ij})| \\ &\leq \left(-l_i + \nu_i \left(\sum_{j=1}^n c_{ij}^L f_j^M + \sum_{j=1}^n d_{ij}^L g_j^M + e_i^L \right) \right) |y_i^\sigma(t)| \\ &\quad + \mu(t) \left(-l_i + \nu_i \left(\sum_{j=1}^n c_{ij}^L f_j^M + \sum_{j=1}^n d_{ij}^L g_j^M + e_i^L \right) \right) |y_i^\Delta(t)| \\ &\quad + a_i^M \sum_{j=1}^n c_{ij}^L L_j |y_j(t)| + a_i^M \sum_{j=1}^n d_{ij}^L P_j |y_j(t - \tau_{ij})| \\ &\leq -\zeta_i |y_i(t)| + 2\mu(t)\zeta_i |y_i^\Delta(t)| \\ &\quad + a_i^M \sum_{j=1}^n c_{ij}^L L_j |y_j(t)| + a_i^M \sum_{j=1}^n d_{ij}^L P_j |y_j(t - \tau_{ij})| \\ &\leq (2\mu(t)\zeta_i^2 - \zeta_i) |y_i(t)| \\ &\quad + (1 + 2\mu(t)\zeta_i) a_i^M \left[\sum_{j=1}^n c_{ij}^L L_j |y_j(t)| + \sum_{j=1}^n d_{ij}^L P_j |y_j(t - \tau_{ij})| \right], \end{aligned}$$

or $i = 1, \dots, n$. And we have from (H₁₁) that

$$|y_i(t_k^+)| = |1 - \gamma_{ik}| |y_i(t_k)| \leq |y_i(t_k)|, \quad i = 1, \dots, n, \quad k \in \mathbb{N}.$$

For any $\alpha \in [-\tau, 0]_{\mathbb{T}}$, we construct the Lyapunov functional

$$\begin{aligned} V(t) &= V_1(t) + V_2(t), \\ V_1(t) &= \sum_{i=1}^n e_\varepsilon(t, \alpha) |y_i(t)|, \\ V_2(t) &= \sum_{i=1}^n \sum_{j=1}^n \int_{t-\tau_{ij}}^t (1 + \varepsilon\mu(s + \tau_{ij})) e_\varepsilon(s + \tau_{ij}, \alpha) (1 + 2\mu(s + \tau_{ij})) \\ &\quad \times \zeta_i a_i^M d_{ij}^L P_j |y_j(s)| \Delta s. \end{aligned}$$

For $t \in \mathbb{T}^+$, $t \neq t_k$, $k \in \mathbb{N}$, calculating the delta derivative $D^+V(t)^\Delta$ of $V(t)$ along system (4.1), we can get

$$\begin{aligned} D^+V_1^\Delta(t) &= \sum_{i=1}^n [\varepsilon e_\varepsilon(t, \alpha) |y_i(t)| + e_\varepsilon(\sigma(t), \alpha) D^+ |y_i(t)|^\Delta] \\ &\leq \sum_{i=1}^n \left\{ \varepsilon e_\varepsilon(t, \alpha) |y_i(t)| + (1 + \varepsilon\mu(t)) e_\varepsilon(t, \alpha) \left((2\mu(t)\zeta_i^2 - \zeta_i) |y_i(t)| \right. \right. \\ &\quad \left. \left. + (1 + 2\mu(t)\zeta_i) a_i^M \left[\sum_{j=1}^n c_{ij}^L L_j |y_j(t)| + \sum_{j=1}^n d_{ij}^L P_j |y_j(t - \tau_{ij})| \right] \right) \right\} \\ &\leq \sum_{i=1}^n [\varepsilon + (1 + \varepsilon\mu(t))(2\mu(t)\zeta_i^2 - \zeta_i)] e_\varepsilon(t, \alpha) |y_i(t)| \\ &\quad + (1 + \varepsilon\mu(t)) e_\varepsilon(t, \alpha) \\ &\quad \times \sum_{i=1}^n \sum_{j=1}^n (1 + 2\mu(t)\zeta_i) a_i^M [c_{ij}^L L_j |y_j(t)| + d_{ij}^L P_j |y_j(t - \tau_{ij})|] \end{aligned}$$

and

$$\begin{aligned} D^+V_2^\Delta(t) &\leq \sum_{i=1}^n \sum_{j=1}^n (1 + \varepsilon\mu(t + \tau_{ij})) (1 + 2\mu(t + \tau_{ij})\zeta_i) e_\varepsilon(t + \tau_{ij}, \alpha) a_i^M d_{ij}^L P_j |y_j(t)| \\ &\quad - (1 + \varepsilon\mu(t)) e_\varepsilon(t, \alpha) \sum_{i=1}^n \sum_{j=1}^n (1 + 2\mu(t)\zeta_i) a_i^M d_{ij}^L P_j |y_j(t - \tau_{ij})|. \end{aligned}$$

By assumption (H_{10}) , it concludes that

$$\begin{aligned} D^+(V^\Delta(t)) &= D^+V_1^\Delta(t) + D^+V_2^\Delta(t) \\ &\leq \sum_{i=1}^n [\varepsilon + (1 + \varepsilon\mu(t))(2\mu(t)\zeta_i^2 - \zeta_i)] e_\varepsilon(t, \alpha) |y_i(t)| \\ &\quad + (1 + \varepsilon\mu(t)) e_\varepsilon(t, \alpha) \sum_{i=1}^n \sum_{j=1}^n (1 + 2\mu(t)\zeta_i) a_i^M c_{ij}^L L_j |y_j(t)| \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^n \sum_{j=1}^n (1 + \varepsilon\mu(t + \tau_{ij}))(1 + 2\mu(t + \tau_{ij})\zeta_i) e_\varepsilon(t + \tau_{ij}, \alpha) a_i^M d_{ij}^L P_j |y_j(t)| \\
= & \sum_{i=1}^n \left\{ \varepsilon + (1 + \varepsilon\mu(t))(2\mu(t)\zeta_i^2 - \zeta_i) + \sum_{j=1}^n a_i^M \left[(1 + 2\mu(t)\zeta_i) c_{ij}^L L_j \right. \right. \\
& \left. \left. + (1 + \varepsilon\mu(t + \tau_{ij}))(1 + 2\mu(t + \tau_{ij})\zeta_i) e_\varepsilon(t + \tau_{ij}, t) d_{ij}^L P_j \right] \right\} e_\varepsilon(t, \alpha) |y_i(t)| \leq 0,
\end{aligned}$$

for $t \in \mathbb{T}^+$, $t \neq t_k$, $k \in \mathbb{N}$. Also, for $k \in \mathbb{N}$,

$$\begin{aligned}
V(t_k^+) & = V_1(t_k^+) + V_2(t_k^+) \\
= & \sum_{i=1}^n e_\varepsilon(t_k^+, \alpha) |y_i(t_k^+)| \\
& + \sum_{i=1}^n \sum_{j=1}^n \int_{t_k^+ - \tau_{ij}}^{t_k^+} (1 + \varepsilon\mu(s + \tau_{ij})) e_\varepsilon(s + \tau_{ij}, \alpha) (1 + 2\mu(s + \tau_{ij})\zeta_i) a_i^M d_{ij}^L P_j |y_j(s)| \Delta s \\
\leq & \sum_{i=1}^n e_\varepsilon(t_k, \alpha) |y_i(t_k)| \\
& + \sum_{i=1}^n \sum_{j=1}^n \int_{t_k - \tau_{ij}}^{t_k} (1 + \varepsilon\mu(s + \tau_{ij})) e_\varepsilon(s + \tau_{ij}, \alpha) (1 + 2\mu(s + \tau_{ij})\zeta_i) a_i^M d_{ij}^L P_j |y_j(s)| \Delta s \\
= & V(t_k).
\end{aligned}$$

It follows that $V(t) \leq V(0)$ for all $t \in \mathbb{T}^+$. On the other hand, we have

$$\begin{aligned}
V(0) & = V_1(0) + V_2(0) \\
= & \sum_{i=1}^n e_\varepsilon(0, \alpha) |y_i(0)| \\
& + \sum_{i=1}^n \sum_{j=1}^n \int_{-\tau_{ij}}^0 (1 + \varepsilon\mu(s + \tau_{ij})) e_\varepsilon(s + \tau_{ij}, \alpha) \\
& \quad \times (1 + 2\mu(s + \tau_{ij})\zeta_i) a_i^M d_{ij}^L P_j |y_j(s)| \Delta s \\
\leq & \sum_{i=1}^n \left\{ e_\varepsilon(0, \alpha) + \sum_{j=1}^n \int_{-\tau_{ji}}^0 (1 + \varepsilon\mu(s + \tau_{ij})) e_\varepsilon(s + \tau_{ij}, \alpha) \right. \\
& \quad \left. \times (1 + 2\mu(s + \tau_{ij})\zeta_j) a_i^M d_{ji}^L P_i \Delta s \right\} \times \max_{s \in [-\tau, 0]_{\mathbb{T}}} |y_i(s)| \\
\leq & \Gamma(\varepsilon) \sum_{i=1}^n \max_{s \in [-\tau, 0]_{\mathbb{T}}} |\varphi_i(s) - x_i^*(s)|,
\end{aligned}$$

where

$$\Gamma(\varepsilon) = \max_{1 \leq i \leq n} \left\{ \max_{\alpha \in [-\tau, 0]_{\mathbb{T}}} \left\{ e_{\varepsilon}(0, \alpha) + \sum_{j=1}^n \int_{-\tau_{ji}}^0 (1 + \varepsilon\mu(s + \tau_{ij})) e_{\varepsilon}(s + \tau_{ij}, \alpha) \right. \right. \\ \left. \left. \times (1 + 2\mu(s + \tau_{ij}) \zeta_j a_i^M d_{ji}^L P_i \Delta s) \right\} \right\}.$$

It is obvious that

$$\sum_{i=1}^n e_{\varepsilon}(t, \alpha) |x_i(t) - x_i^*(t)| \leq V(t) \leq V(0) \leq \Gamma(\varepsilon) \sum_{i=1}^n \max_{s \in [-\tau, 0]_{\mathbb{T}}} |\varphi_i(s) - x_i^*(s)|.$$

So we can finally get

$$\sum_{i=1}^n |x_i(t) - x_i^*(t)| \leq \Gamma(\varepsilon) e_{\Theta\varepsilon}(t, \alpha) \sum_{i=1}^n \max_{s \in [-\tau, 0]_{\mathbb{T}}} |\varphi_i(s) - x_i^*(s)| = \Gamma(\varepsilon) e_{\Theta\varepsilon}(t, \alpha) \|\varphi - x^*\|.$$

Since $\Gamma(\varepsilon) \geq 1$, from Definition 2.8, the $\omega/2$ -anti-periodic solution of system (1.1) is globally exponential stable. This completes the proof. \square

5. An example

EXAMPLE 5.1. Consider the following impulsive Cohen-Grossberg neural networks

$$(5.1) \quad \begin{cases} x_i^{\Delta}(t) = -a_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^n c_{ij}(t) f_j \right. \\ \left. (x_j(t)) - \sum_{j=1}^n d_{ij}(t) g_j(x_j(t - \tau_{ij})) - e_i(t) \right], \\ t \in \mathbb{T}^+, t \neq t_k, k \in \mathbb{N}, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = I_{ik}(x_i(t_k)), \quad i = 1, 2, \end{cases}$$

where

$$(a_i)_{2 \times 1} = \begin{pmatrix} 1.6 + (2/\pi) \arctan |u| \\ 1.5 + (2/\pi) \arctan |u| \end{pmatrix}, \quad (b_i)_{2 \times 1} = \frac{1}{220\pi} \begin{pmatrix} u \\ u \end{pmatrix}, \\ (f_j)_{2 \times 1} = (g_j)_{2 \times 1} = \frac{1}{2} \begin{pmatrix} \sin u \\ \sin u \end{pmatrix}, \\ (c_{ij})_{2 \times 2} = \frac{1}{2640\pi} \begin{pmatrix} \sin 2t & \cos 2t \\ \cos 2t & \sin 2t \end{pmatrix}, \\ (d_{ij})_{2 \times 2} = \frac{1}{2640\pi} \begin{pmatrix} \sin^2 t & \cos^2 t \\ \cos^2 t & \sin^2 t \end{pmatrix},$$

$$\begin{aligned} (e_i)_{2 \times 1} &= \frac{\pi}{2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, & (\tau_{ij})_{2 \times 2} &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ (I_{ik})_{2 \times 2} &= \frac{1}{240} \begin{pmatrix} -u & -u \\ -u & -u \end{pmatrix}, \\ \omega &= 2\pi, & [0, 2\pi]_{\mathbb{T}} \cap \{t_k : k \in \mathbb{N}\} &= \{t_1, t_2\}. \end{aligned}$$

When $\mathbb{T} = \mathbb{R}$ or \mathbb{Z} , system (5.1) has at least one exponentially stable π -anti-periodic solution.

PROOF. By calculation, we have $a_1^m = 1.6$, $a_2^m = 1.5$, $a_1^M = 2.6$, $a_2^M = 2.5$, $\nu_1 = \nu_2 = 2/\pi$, $l_1 = 0.6 - 1/\pi$, $l_2 = 0.5 - 1/\pi$, $\rho_1 = \rho_2 = 1/(220\pi)$, $\delta_1 = \delta_2 = 1/(220\pi)$, $L_1 = L_2 = 1/2$, $P_1 = P_2 = 1/2$, $f_1^M = f_2^M = 1/2$, $g_1^M = g_2^M = 1/2$, $c_{11}^L = c_{12}^L = c_{21}^L = c_{22}^L = 1/(2640\pi)$, $d_{11}^L = d_{12}^L = d_{21}^L = d_{22}^L = 1/(2640\pi)$, $e_1^L = e_2^L = \pi/2$, $G_{11} = G_{12} = G_{21} = G_{22} = 1/240$.

It is obvious that (H₁)–(H₅), (H₇)–(H₉) and (H₁₁) are satisfied. Furthermore, we can easily calculate that

$$\Pi = (h_{ij})_{2 \times 2} \approx \begin{pmatrix} 0.1926 & -0.4396 \\ -0.4392 & 0.0978 \end{pmatrix}$$

is a nonsingular M matrix, thus (H₆) is satisfied.

Case 1. When $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$. Take $\varepsilon = 0.01$, we have that

$$\max_{t \in [0, \omega/2]_{\mathbb{T}}} \Theta_1(\varepsilon, t) \approx -0.2697 < 0 \quad \text{and} \quad \max_{t \in [0, \omega/2]_{\mathbb{T}}} \Theta_2(\varepsilon, t) \approx -0.1697 < 0.$$

Hence (H₁₀) holds. By Theorems 3.1 and 4.1, system (5.1) has at least one exponentially stable π -anti-periodic solution.

Case 2. When $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$. Take $\varepsilon = 0.001$, we have that

$$\max_{t \in [0, \omega/2]_{\mathbb{T}}} \Theta_1(\varepsilon, t) \approx -0.1233 < 0 \quad \text{and} \quad \max_{t \in [0, \omega/2]_{\mathbb{T}}} \Theta_2(\varepsilon, t) \approx -0.1135 < 0.$$

Hence (H₁₀) holds. By Theorems 3.1 and 4.1, system (5.1) has at least one exponentially stable π -anti-periodic solution. \square

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