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MORE ON C-FRACTION SOLUTIONS TO RICCATI EQUATIONS

K.D. COOPER, S. CLEMENT COOPER*, AND WILLIAM B. JONES*

ABSTRACT. An algorithm due to Merkes and Scott for finding C-fraction solutions of certain Riccati differential equations is generalized to apply to a larger class of Riccati equations. Remarks on computational aspects of the algorithm are made and several examples are presented.

1. Introduction. Continued fractions have been used theoretically to solve Riccati differential equations for many years. Euler [**6, 7**] and Lagrange [**11**] seem to be the originators of this approach. They both concentrated on very special forms of Riccati equations and found C-fraction solutions. More recently, Merkes and Scott [**12**], Stokes [**13**], Fair [**8**], Chisolm [**3**], Cooper [**5**], and Cooper, Jones and Magnus [**4**] have made contributions using various continued fractions.

Riccati equations have the special property that they are invariant (in a sense) under linear fractional transformations (lfts). More precisely, under an lft,

$$y = \frac{\alpha(z)w + \beta(z)}{\gamma(z)w + \delta(z)}, \tag{1.1}$$

a Riccati equation,

$$y' = f_0(z) + f_1(z)y + f_2(z)y^2, \tag{1.2}$$

is transformed into another Riccati equation

$$w' = \tilde{f}_0(z) + \tilde{f}_1(z)w + \tilde{f}_2(z)w^2. \tag{1.3}$$

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The fundamental role played by lfts in the development of continued fractions [**9**] makes a continued fraction approach to Riccati equations quite natural.

Since we will be discussing formal continued fraction solutions, we begin with the definition.

Definition 1. Let D be a (formal) differential operator. A continued fraction with n^{th} approximant $f_n(z)$ is said to be a formal solution of a differential equation $D[W(z)] = 0$ at $z = 0$ if

$$\Lambda_0(D[f_n(z)]) = O(z^{k_n}), \tag{1.4}$$

where $\lim_{n\to\infty} k_n = \infty$. Here $\Lambda_0(f)$ denotes the Laurent series about $z = 0$ for a function f meromorphic in a neighborhood of zero.

The symbol $O(z^{k_n})$ denotes a power series (possibly divergent) whose first nonvanishing term has degree k_n or greater.

Merkes and Scott [**12**] solve Riccati differential equations of the form

$$R[W(z)] := zA(z) + B(z)W(z) + C(z)W^2(z) - z^k W'(z) := 0, \tag{1.5}$$

where $A(z)$, $B(z)$, and $C(z)$ are analytic at $z = 0$ and k is a positive integer, by finding a formal C-fraction solution

$$\frac{d_1 z^{e_1}}{1} \mathop{}_{+} \frac{d_2 z^{e_2}}{1} \mathop{}_{+} \cdots \mathop{}_{+} \frac{d_n z^{e_n}}{1} \mathop{}_{+} \cdots, \tag{1.6}$$

where the numbers d_k are complex constants and the e_k's are positive integers. Notice that these Riccati equations have a singularity at $z = 0$.

Stokes [**13**] considered Riccati equations of the form

$$R[Y] := A(x) + B(x)Y + C(x)Y^2 + xD(x)Y' = 0, \tag{1.7}$$

where $A(x), B(x), C(x)$, and $D(x)$ are real valued polynomials of the real variable x. He obtained regular C-fraction solutions of the form

$$Y = \frac{\alpha_0}{1} \mathop{}_{+} \frac{\alpha_1 x}{1} \mathop{}_{+} \cdots \mathop{}_{+} \frac{\alpha_n x}{1} \mathop{}_{+} \cdots \tag{1.8}$$

or

$$Y = \frac{\alpha_0 x}{1} + \frac{\alpha_1 x}{1} + \cdots + \frac{\alpha_n x}{1} + \cdots, \tag{1.9}$$

where the α_i's are real constants for $i \in \mathbf{Z}^+$, by using a method very similar to that of Merkes and Scott.

From a computational point of view, the C-fraction approach had some advantages over the other continued fraction solutions (including the general T-fraction approach [**4, 5**]), so we have concentrated on C-fractions here. The method used in this paper is an extension of that used by Merkes and Scott. Their algorithm has been modified so that it can also handle Riccati equations without a singularity at zero, such as

$$R[W(z)] := A(z) + B(z)W(z) + C(z)W^2(z) - W'(z) = 0, \tag{1.10}$$

where $A(z), B(z)$, and $C(z)$ are analytic at $z = 0$. This modification is of some importance since such nonsingular equations seem to arise in applications more frequently than singular equations. In the next section we will discuss the algorithm and its theory in some detail.

In discussing the computational aspects of C-fraction solutions, we consider regular C-fractions

$$\frac{d_1 z}{1} + \frac{d_2 z}{1} + \cdots + \frac{d_n z}{1} + \cdots, \tag{1.11}$$

which are C-fractions with $e_k = 1$ for all $k \in \mathbf{Z}^+$, separately from the more general C-fractions in which $e_k > 1$ for some $k \in \mathbf{Z}^+$. There are some desirable consequences of having a regular C-fraction solution, but, unfortunately, not every Riccati equation has such a solution. Section 3 will be devoted entirely to regular C-fraction solutions and the advantages associated with them. In Section 4 we concentrate on the more general C-fractions and some of their drawbacks. We conclude with some examples in Section 5.

2. The algorithm. Merkes and Scott [**12**] developed an algorithm for finding a formal C-fraction solution,

$$\frac{d_1 z^{e_1}}{1} + \frac{d_2 z^{e_2}}{1} + \cdots + \frac{d_n z^{e_n}}{1} + \cdots, \tag{2.1}$$

to Riccati equations of the form

$$R[W(z)] := zA(z) + B(z)W(z) + C(z)W^2(z) - z^k W'(z) = 0, \tag{2.2}$$

where

(i) $A(z), B(z)$ and $C(z)$ are analytic at $z = 0$, and

(ii) k is a positive integer.

We generalize their algorithm to apply to Riccati equations of the form

$$R[W(z)] := A(z) + B(z)W(z) + C(z)W^2(z) - W'(z) = 0, \tag{2.3}$$

where $A(z), B(z)$, and $C(z)$ are analytic at $z = 0$.

For the moment, let us focus our attention on the algorithm due to Merkes and Scott [**12**]. As mentioned in the introduction, Riccati equations have the spectral property that a linear fractional transformation of the dependent variable yields another Riccati equation in the new dependent variable. It is this property that is exploited to generate the C-fraction solution (2.1). We start with

Definition 2. A Riccati equation of the form (2.2) is said to be admissible if the following conditions are satisfied:

(i) $A(z), B(z)$ and $C(z)$ are analytic at $z = 0$;

(ii) $B(0)$ and $C(0)$ are not both zero;

(iii) k is a positive integer;

(iv) if $k = 1$, $B(0)$ is not a positive integer, and, if $k > 1$, $B(0) \neq 0$.

By using the transformations

$$W_n(z) = \frac{d_{n+1} z^{e_{n+1}}}{1 + W_{n+1}}, \quad n = 0, 1, \dots, \tag{2.4}$$

a sequence of Riccati equations

(2.5)

$$\{R_n[W_n(z)] := zA_n(z) + B_n(z)W_n(z) + C_n(z)W_n^2(z) - z^k W_n'(z) = 0\}_{n=0}^{\infty}$$

(where the subscript zero is attached to the original equation) is generated. Note that k is fixed for the sequence and all of the equations have the same form. Let

$$(2.6)\quad \begin{cases} zA_n(z) = a_{\alpha_n-1,n}z^{\alpha_n} + a_{\alpha_n,n}z^{\alpha_n+1} + \cdots, & a_{\alpha_n-1,n} \neq 0, \\ B_n(z) = b_{0,n} + b_{1,n}z + \cdots, \\ C_n(z) = c_{0,n} + c_{1,n}z + \cdots. \end{cases}$$

Merkes and Scott [**12**] proved the following theorem.

Theorem 3. *If*

$$(2.7)\quad R_n[W_n(z)] := zA_n(z) + B_n(z)W_n(z) + C_n(z)W_n^2(z) - z^k W_n'(z) = 0$$

is admissible and $A_n(z), B_n(z)$ *and* $C_n(z)$ *are given by* (2.6), *then* $R_{n+1}[W_{n+1}(z)] = 0$ *is admissible if and only if*

$$(2.8)\quad \begin{cases} e_{n+1} = \alpha_n, \quad d_{n+1} = \frac{a_{\alpha_n-1,n}}{\alpha_n - b_{0,n}} \text{ for } k = 1 \\ e_{n+1} = \alpha_n, \quad d_{n+1} = \frac{-a_{\alpha_n-1,n}}{b_{0,n}} \text{ for } k > 1. \end{cases}$$

A direct consequence of this theorem is

Corollary 4. *If the Riccati differential equation of the form* (2.2) *is admissible, then a* C*-fraction of the form* (2.1) *can be constructed from it with the constants determined uniquely by* (2.8).

Proof. Use the transformations

$$(2.9)\qquad W_n(z) = \frac{d_{n+1}z^{e_{n+1}}}{1 + W_{n+1}(z)}$$

to generate the sequence of Riccati equations. From the sequence of equations, determine $\{e_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ as in Theorem 3. □

Notice that the C-fraction terminates if and only if $A_n(z) \equiv 0$ for some $n \in \mathbf{Z}^+$.

Now let us consider Riccati equations of the form (2.3) where $A(z), B(z)$ and $C(z)$ are analytic at $z = 0$. We can also construct a C-fraction in this case.

Definition 5. A Riccati equation of the form (2.3) is said to be admissible if the functions $A(z), B(z)$ and $C(z)$ are analytic at $z = 0$.

Theorem 6. *If the Riccati equation* (2.3) *is admissible, then it is possible to construct a C-fraction of the form* (2.1) *from it.*

It will be shown later that this C-fraction is the unique formal C-fraction solution that vanishes at $z = 0$.

Proof. Use the transformation

$$W_0(z) = \frac{d_1 z^{e_1}}{1 + W_1(z)} \tag{2.10}$$

to obtain the Riccati equation

$$R_1[W_1(z)] := zA_1(z) + B_1(z)W_1(z) + C_1(z)W_1^2(z) - zW_1'(z) = 0, \tag{2.11}$$

where

$$\begin{cases} zA_1(z) := e_1 - \frac{z^{-e_1+1}}{d_1} A_0(z) - zB_0(z) - d_1 z^{e_1+1} C_0(z), \\ B_1(z) := e_1 - \frac{2z^{-e_1+1}}{d_1} A_0(z) - zB_0(z), \\ C_1(z) := \frac{-z^{e_1+1}}{d_1} A_0(z). \end{cases} \tag{2.12}$$

Let

$$A_0(z) = a_{\alpha_0-1,0} z^{\alpha_0-1} + a_{\alpha_0,0} z^{\alpha_0} + \cdots, \quad a_{\alpha_0-1,0} \neq 0, \ \alpha_0 \geq 1. \tag{2.13}$$

Define

$$e_1 := \alpha_0 \text{ and } d_1 := \frac{a_{\alpha_0-1,0}}{e_1}. \tag{2.14}$$

Then $A_1(z), B_1(z)$ and $C_1(z)$ are analytic at $z = 0$ and $B_1(0) = -e_1 = C_1(0)$, and hence $R_1[W_1(z)] = 0$ is admissible. Therefore, by Corollary 4 we can construct the C-fraction

$$\frac{\tilde{d}_1 z^{\tilde{e}_1}}{1} + \frac{\tilde{d}_2 z^{\tilde{e}_2}}{1} + \cdots + \frac{\tilde{d}_n z^{\tilde{e}_n}}{1} + \cdots \tag{2.15}$$

from it. Thus, we can construct the C-fraction (2.1) from the Riccati equation (2.3), where d_1 and e_1 are given by (2.14) and $d_k = \tilde{d}_{k-1}$ and $e_k = \tilde{e}_{k-1}$ for $k = 2, 3, \dots$. □

Note that the transformation (2.10) introduces a singularity at $z = 0$ which will be retained by the subsequent equations.

The C-fractions constructed are formal continued fraction solutions to their respective differential equations. Furthermore, there is a very nice relationship between the formal continued fraction solution and the formal power series solution. The following lemma will be useful in establishing this.

Lemma 7. (A) *Let the C-fraction* (2.1) *be constructed from the admissible Riccati equation*

$$R_0[W_0(z)] := A_0(z) + B_0(z)W_0(z) + C_0(z)W_0^2(z) - W_0'(z) = 0. \tag{2.16}$$

Let $f_n(z)$ be the n^{th} approximant of (2.1). *Then*

$$\Lambda_0[R_0(f_n)] = O(z^n). \tag{2.17}$$

(B) *Let the C-fraction* (2.1) *be constructed from the admissible Riccati equation*

$$R_0[W_0(z)] := zA_0(z) + B_0(z)W_0(z) + C_0(z)W_0^2(z) - z^kW_0'(z) = 0. \tag{2.18}$$

Let $f_n(z)$ be the n^{th} approximant of (2.1). *Then*

$$\Lambda_0[R_0(f_n)] = O(z^{n+1}). \tag{2.19}$$

Proof. (A). Since $W_0(z), W_1(z), \dots, W_n(z)$ are related to each other by

$$W_0(z) = \frac{d_1z^{e_1}}{1} \underset{+}{} \frac{d_2z^{e_2}}{1} \underset{+}{} \cdots \underset{+}{} \frac{d_nz^{e_n}}{1 + W_n(z)}, \quad n = 1, 2, \dots, \tag{2.20}$$

setting $W_n(z) = 0$ determines $W_0(z), W_1(z), \dots, W_{n-1}(z)$. In particular, by setting $W_n(z) = 0$, we have $W_0(z) = f_n(z)$ and

$$W_k(z) = \frac{d_{k+1}z^{e_{k+1}}}{1} \underset{+}{} \cdots \underset{+}{} \frac{d_n z^{e_n}}{1}, \quad k = 1, 2, \dots, n-1. \tag{2.21}$$

Also note that direct substitutions yield the relations

$$R_0[W_0(z)] = \frac{-d_1 z^{e_1 - 1}}{(1 + W_1(z))^2} R_1[W_1(z)] \tag{2.22}$$

and

$$R_k[W_k(z)] = \frac{-d_{k+1}z^{e_{k+1}}}{(1 + W_{k+1}(z))^2} R_{k+1}[W_{k+1}(z)], \quad k = 1, 2, \dots . \tag{2.23}$$

Thus, we have

$$R_0[W_0(z)] = (-1)^k z^{\sum_{j=1}^{k}(e_j)-1} \prod_{j=1}^{k} \frac{(d_j)}{(1 + W_j(z))^2} \cdot R_k[W_k(z)], \quad k = 1, 2, \dots . \tag{2.24}$$

Therefore,

$$R_0[f_n(z)] = (-1)^n z^{\sum_{j=1}^{n} e_j} \prod_{j=1}^{n} \frac{(d_j)}{(1 + W_j(z))^2} \cdot A_n(z), \quad n = 1, 2, \dots . \tag{2.25}$$

Since $\Lambda_0[W_k(z)] = O(z)$, we have $\Lambda_0[1/(1+W_k(z))] = O(1)$, and hence,

$$\Lambda_0[R_0(f_n(z))] = O(z^{\sum_{j=1}^{n} e_j}). \tag{2.26}$$

Taking note that $e_j \geq 1$ for all j, equation (2.17) now follows.

The proof of (B) is completely analogous so we omit it. □

The next Theorem is a direct consequence of Lemma 7.

Theorem 8. (A) *Let* (2.3) *be an admissible Riccati equation. Then the C-fraction solution whose existence is guaranteed by Theorem 6 is a formal continued fraction solution to* (2.3).

(B) *Let* (2.2) *be an admissible Riccati equation. Then the C-fraction solution whose existence is guaranteed by Corollary 4 is a formal continued fraction solution to* (2.2).

It is well known that each C-fraction corresponds to a unique power series. Merkes and Scott proved the following theorem relating the C-fraction solution to the formal power series (fps) solution.

Theorem 9. *If the Riccati equation* (2.2) *is admissible, then the formal continued fraction solution whose existence is guaranteed by Corollary 4 corresponds to the unique formal power series solution that vanishes at $z = 0$.*

We will prove the analogous theorem for the generalization.

Theorem 10. *If the Riccati equation* (2.3) *is admissible, then the formal continued fraction solution whose existence is guaranteed by Theorem 6 corresponds to the unique formal power series solution that vanishes at $z = 0$.*

Proof. It is easy to show that $R[W(z)] = 0$ has a unique fps solution $L(z)$ that vanishes at $z = 0$. Let $L^*(z)$ be the fps to which the formal C-fraction corresponds. The relationship between the C-fraction solution and its corresponding fps is characterized by

$$L^*(z) - \Lambda_0(f_n) = O(z^{\sum_{j=1}^{n+1} e_j}). \tag{2.27}$$

Therefore,

$$L^*(z) - \Lambda_0(f_n) = O(z^{n+1}). \tag{2.28}$$

Now compute

$$\begin{aligned} R[L^*(z)] &= R[f_n(z) + L^*(z) - f_n(z)] \\ &= A(z) + B(z)[f_n(z) + (L^*(z) - f_n(z))] \\ &\quad + C(z)[f_n(z) + (L^*(z) - f_n(z))]^2 \\ &\quad - [f_n(z) + (L^*(z) - f_n(z))]' \\ &= R[f_n(z)] + B(z)[L^*(z) - f_n(z)] \\ &\quad + C(z)[(L^*(z))^2 - f_n^2(z)] - [(L^*(z))' - f_n'](z). \end{aligned}$$

Thus,

$$\begin{aligned} \Lambda_0(R[L^*(z)]) &= O(z^n) + O(z^n) + O(z^n) - O(z^n) \\ &= O(z^n), \end{aligned}$$

where the derivative of $L^*(z)$ is taken formally. Since this result holds for all $n \in \mathbf{Z}^+$, it follows that $\Lambda_0(R[L^*(z)]) \equiv 0$, and, hence, $L(z) = L^*(z)$. □

The next theorem states that the formal C-fraction solution to a Riccati equation (2.2) or (2.3) is unique.

Theorem 11. (A) *If the Riccati equation* (2.3) *is admissible, then it has a unique formal C-fraction solution of the form* (2.1).

(B) *If the Riccati equation* (2.2) *is admissible, then it has a unique formal C-fraction solution of the form* (2.1).

Proof. (A). Existence of a formal C-fraction solution to (2.3) of the form (2.1) is established by Theorems 6 and 8. Let

$$\frac{d_1 z^{e_1}}{1} + \frac{d_2 z^{e_2}}{1} + \cdots + \frac{d_n z^{e_n}}{1} + \cdots \tag{2.29}$$

be an arbitrary formal C-fraction solution to (2.3). Let $f_n(z)$ be the n^{th} approximant of (2.29). By Definition 1,

$$\Lambda_0(R[f_n(z)]) = O(z^{k_n}), \tag{2.30}$$

where $\lim_{n\to\infty} k_n = \infty$. Let $L^*(z)$ be the unique fps to which the formal C-fraction solution corresponds. By an argument analogous to that in the proof of Theorem 10,

$$\Lambda_0(R[L^*(z)]) = O(z^{k_n}) + O(z^n) = O(z^{\min\{k_n,n\}}). \tag{2.31}$$

This holds for all n, which implies that $\Lambda_0(R[L^*(z)]) \equiv 0$. Thus, L^* must be the fps solution to $R[W(z)] = 0$ which vanishes at $z = 0$. This fps is unique and corresponds to a unique C-fraction, hence the formal C-fraction solution is unique.

The proof of (B) is completely analogous. □

Merkes and Scott [**12**] also proved part (B) of the following theorem.

Theorem 12. (A) *If the Riccati equation* (2.3) *is admissible, and if its formal C-fraction solution converges uniformly in a neighborhood of $z = 0$ to a function $W(z)$, then $W(z)$ is the unique solution of $R[W(z)] = 0$ that is analytic at $z = 0$ satisfying $W(0) = 0$.*

(B) *If the admissible Riccati equation* (2.2) *has a formal C-fraction solution that converges uniformly in a neighborhood of $z = 0$ to a function $W(z)$, then $W(z)$ is the unique solution of $R[W(z)] = 0$ that is analytic at $z = 0$ satisfying $W(0) = 0$.*

Proof of (A). Let $f_n(z)$ be the n^{th} approximant of the C-fraction solution. By Theorem 5.13 in [**9**], $W(z) = \lim_{n\to\infty} f_n(z)$ is analytic in a neighborhood of $z = 0$, and the Taylor series expansion of $W(z)$ is the power series $L^*(z)$ to which the C-fraction corresponds at $z = 0$. By Theorem 9, $L^*(z)$ is a fps solution of $R[W(z)] = 0$ at $z = 0$. It is therefore a solution in the neighborhood of $z = 0$ in which it converges. The assertion follows from the fact that $W(z) = L^*(z)$ for z in this neighborhood. □

3. Regular C-fraction solutions. In Section 2 we saw that, given an admissible Riccati equation of either form (2.2) or (2.3), we are guaranteed a formal C-fraction solution

$$(3.1)\quad \frac{d_1 z^{e_1}}{1} + \frac{d_2 z^{e_2}}{1} + \cdots + \frac{d_n z^{e_n}}{1} + \cdots, \quad d_n \in \mathbf{C},\ n = 1, 2, \ldots .$$

If $e_n = 1$ for $n = 1, 2, \ldots$, then we have a *regular* C-fraction solution. There are certain advantages computationally to having a regular C-fraction solution, and it is the objective of this section to highlight these. We start with a theorem that tells us exactly when a Riccati equation has a regular C-fraction solution.

Theorem 13. *The admissible Riccati equation $R[W(z)] = 0$, of the form* (2.2) *or* (2.3), *has a regular C-fraction solution*

$$\frac{d_1 z}{1} + \frac{d_2 z}{1} + \cdots + \frac{d_n z}{1} + \cdots, \quad d_n \in \mathbf{C}, \tag{3.2}$$

if and only if $A_n(0) \neq 0$ for $n = 0, 1, \dots$.

Proof. Recall that, for $n = 0, 1, \dots,$

$$A_n(z) = a_{\alpha_n - 1, n} z^{\alpha_n - 1} + a_{\alpha_n, n} z^{\alpha_n} + \cdots, \quad a_{\alpha_n - 1, n} \neq 0, \tag{3.3}$$

and

$$e_{n+1} = \alpha_n. \tag{3.4}$$

The result follows directly. □

The next theorem is the key to the advantages inherent in a regular C-fraction solution.

Theorem 14. (A) *Let $R[W(z)] = 0$ be an admissible Riccati equation of the form* (2.2) *that has a regular C-fraction solution. Then d_n is a function of the first n coefficients of $A(z)$ and $B(z)$ and the first $n - 1$ coefficients of $C(z)$ (Note that d_1 does not depend on any coefficients of $C(z)$).*

(B) *Let $R[W(z)] = 0$ be an admissible Riccati equation of the form* (2.3) *that has a regular C-fraction solution. Then d_n is a function of the first n coefficients of $A(z)$, the first $n - 1$ coefficients of $B(z)$ and the first $n - 2$ coefficients of $C(z)$ (In case $n - 2$ is equal to zero, d_n does not depend on $C(z)$; if $n - 1 = 0$, then d_n depends on neither $B(z)$ nor $C(z)$).*

Proof. (A). Let

$$A_n(z) = a_{0,n} + a_{1,n} z + \cdots + a_{k,n} z^k + \cdots,$$
$$B_n(z) = b_{0,n} + b_{1,n} z + \cdots + b_{k,n} z^k + \cdots,$$

and

$$C_n(z) = c_{0,n} + c_{1,n}z + \cdots + c_{k,n}z^k + \cdots .$$

An inductive argument gives us that

(i) $a_{k,n}$ is a function of $a_{0,0}, a_{1,0}, \ldots , a_{k+n,0}, b_{0,0}, \ldots , b_{k+n,0}$, and $c_{0,0}, \ldots , c_{k+n-1,0}$;

(ii) if $n \geq 2$, $b_{k,n}$ is a function of $a_{0,0}, \ldots , a_{k+n-1,0}, b_{0,0}, \ldots,$ $b_{k+n-1,0}$, and $c_{0,0}, \ldots,$ then $c_{k+n-2,0}$; otherwise, $b_{k,n}$ is a function of $a_{0,0}, \ldots, a_{k+n-1,0}$, and $b_{0,0}, \ldots , b_{k+n-1,0}$;

(iii) if $n \geq 2$, $c_{k,n}$ is a function of $a_{0,0}, \ldots , a_{k+n-1,0}, b_{0,0}, \ldots,$ $b_{k+n-1,0}$, and $c_{0,0}, \ldots, c_{k+n-2,0}$; otherwise, $c_{k,n}$ is a function of $a_{0,0}$, $\ldots, a_{k+n-1,0}$ and $b_{0,0}$.

Since $d_{n+1} = -a_{0,n}/b_{0,n}$ if $k = 1$ or $d_{n+1} = a_{0,n}/(1 - b_{0,n})$ if $k > 1$, for $n = 0, 1, \ldots,$ the result now follows.

(B). This follows from the proof of Theorem 6 and from part (A) of this theorem. □

Perhaps the most significant consequence of this theorem is that when computing we may approximate the coefficient functions by polynomials without introducing any error into the computations due to these approximations. As an example, suppose an admissible Riccati equation of the form (2.2) has a regular C-fraction solution. In order to calculate accurately the first n elements of the continued fraction, all we need to use are the $(n-1)^{\text{st}}$ degree Taylor polynomials (at $z = 0$) for $A(z)$ and $B(z)$ and the $(n-2)^{\text{nd}}$ degree Taylor polynomial (at $z = 0$) for $C(z)$.

Another consequence is that one can considerably reduce the computation by calculating fewer terms of the coefficient functions at each step. Returning to the example above, to calculate $d_1, d_2, \ldots , d_n$, we start with the $(n-1)^{\text{st}}$ degree Taylor polynomials for $A(z)$ and $B(z)$ and the $(n-2)^{\text{nd}}$ degree Taylor polynomial for $C(z)$. At each step, the degree of the polynomials is reduced by one unit until, at the last step, just the constant terms of $A(z)$ and $B(z)$ (and nothing for $C(z)$) are calculated. These two terms are all that are needed to calculate d_n. Thus, the program is quite efficient.

Unfortunately, not every Riccati equation has a regular C-fraction solution. In the next section, we discuss what the ramifications are when we do not have a regular C-fraction.

4. C-fraction solutions. Given an admissible Riccati equation of either form (2.2) or form (2.3), we are guaranteed a formal C-fraction solution (2.1). By Theorem 13, we know that if $A_n(0) = 0$ for some n, then the equation does not have an infinite regular C-fraction solution. There are two possibilities. Either $A_n(z) \equiv 0$, or $A_n(0) = 0$ but $A_n(z) \not\equiv 0$. In the first case, the Riccati equation has a finite regular C-fraction solution, and hence, the solution is a rational function. In the second case, the equation has a C-fraction solution that is not regular. There is one important difference between regular C-fraction solutions and C-fraction solutions in general. This is partially characterized by the following theorem.

Theorem 15. (A) *The admissible Riccati equation $R[W(z)] = 0$, of the form* (2.2), *has a C-fraction solution of the form*

$$\frac{d_1 z^{e_1}}{1} + \frac{d_2 z^{e_2}}{1} + \cdots + \frac{d_n z^{e_n}}{1} + \cdots, \quad d_n \in \mathbf{C} \text{ and } e_n \in \mathbf{Z}^+. \tag{4.1}$$

Let

$$\delta_n := \sum_{j=1}^{n} e_j \text{ and } \gamma_n := \sum_{j=2}^{n} e_j. \tag{4.2}$$

Then d_n is a function of the first δ_n coefficients of $A(z)$, the first $\gamma_n + 1$ coefficients of $B(z)$ and the first $\gamma_n + 1 - \delta_1$ coefficients of $C(z)$.

(B) *The admissible Riccati equation $R[W(z)] = 0$, of the form* (2.3), *has a C-fraction solution of the form* (4.1) *where d_n is a function of the first δ_n coefficients of $A(z)$, the first γ_n coefficients of $B(z)$ and the first $\gamma_n - \delta_1$ coefficients of $C(z)$.* (Note that we are using the convention that if any of the quantities describing the number of coefficients is nonpositive, then d_n does not depend on any of the coefficients of the respective function.)

Proof. (A). Let

$$A_n(z) = a_{\alpha_n - 1, n} z^{\alpha_n - 1} + a_{\alpha_n, n} z^{\alpha_n} + \cdots, \quad a_{\alpha_n - 1, n} \neq 0,$$
$$B_n(z) = b_{0,n} + b_{1,n} z + \cdots + b_{k,n} z^k + \cdots,$$

and

$$C_n(z) = c_{0,n} + c_{1,n}z + \cdots + c_{k,n}z^k + \cdots .$$

An inductive argument gives us

(i) $a_{k,n}$ is a function of

$$a_{\alpha_0-1,0}, \ldots , a_{\sum_{j=1}^{n-1} \alpha_j,0}, b_{0,0}, \ldots , b_{(\sum_{j=1}^{n-1} \alpha_j)+1,0},$$
$$c_{0,0}, \ldots , c_{(\sum_{j=1}^{n-1} \alpha_j)-\alpha_0+k+1,0}, \text{ for } k \geq \alpha_n - 1,$$

(ii) $b_{k,n}$ and $c_{k,n}$ are functions of

$$a_{\alpha_0-1,0}, \ldots , a_{(\sum_{j=0}^{n-1} \alpha_j)+k-1,0}, b_{0,0}, \ldots , b_{(\sum_{j=1}^{n-1} \alpha_j)+k,0},$$
$$\text{and } c_{0,0}, \ldots , c_{(\sum_{j=1}^{n-1} \alpha_j)-\alpha_0+k,0}.$$

Since $e_{n+1} = \alpha_n$ and $d_{n+1} = a_{\alpha_n-1,n}/(\alpha_n - b_{0,n})$ if $k = 1$ or $d_{n+1} = -a_{\alpha_n-1,n}/b_{0,n}$ if $k > 1$ for $n = 0, 1, \ldots$, the result now follows.

(B) This follows from the proof of Theorem 6 and from part (A) of this theorem. □

Note that if the continued fraction solution is a regular C-fraction, then $e_n = 1$ for $n = 1, 2, \ldots$ and Theorem 14 is just a special case of Theorem 15. If the C-fraction solution is not a regular C-fraction, then the numbers of coefficients needed to calculate d_n accurately are functions of $e_1, e_2, \ldots , e_n$. Therefore, one cannot be sure of the degree polynomial to use in approximating the coefficient functions $A(z)$, $B(z)$ and $C(z)$ without a priori knowledge of the exponents in the continued fraction. As one of the examples in Section 5 illustrates, this is not of concern in every problem. If the coefficient functions are polynomials, then, of course, there is no need to approximate.

5. Numerical illustrations. The algorithms discussed in Section 2 can be viewed both as computational tools to find solutions to specific Riccati equations and as ways to generate continued fraction expansions for classes of functions. Merkes and Scott emphasized the latter approach. They used the algorithm to find a continued fraction

expansion for a ratio of ${}_2F_1$ hypergeometric functions. They did this by considering the Riccati equation
(5.1)
$$\frac{1}{(1-z)}\left(\frac{a}{c}(b-c)z+[(b-a)z-c]W(z)-cW^2(z)\right)-zW'(z)=0.$$

The equation thus defined is clearly admissible when c is not a negative integer, hence the algorithm may be applied to yield a convergent regular C-fraction (3.2) with coefficients

$$d_n=\begin{cases}\frac{(a+n)(b-c-n)}{(c+2n)(c+2n+1)}, & n=1,3,5,\dots,\\ \frac{(b+n)(a-c-n)}{(c+2n-1)(c+2n)}, & n=2,4,6,\dots.\end{cases} \tag{5.2}$$

The function to which this continued fraction converges is a solution to the Riccati equation (5.1), as is the ratio

$$W(z)=\frac{{}_2F_1(a,b,c;z)}{{}_2F_1(a,b+1,c+1;z)}-1. \tag{5.3}$$

Since the solution to the equation that satisfies $W(0)=0$ is unique, the two must be the same. A number of well known functions can be written as such a ratio of ${}_2F_1$ hypergeometric functions, for example,

$$\frac{\sin^{-1}z}{z\sqrt{1-z^2}}. \tag{5.4}$$

Another example of this use of the algorithm is to find the C-fraction expansion of $\tan z$. This function is clearly the solution of the Riccati equation

$$W'=1+W^2 \tag{5.5}$$

that satisfies $W(0)=0$. The algorithm applied to this problem yields the well-known expansion

$$\tan z=\frac{z}{1}+\frac{\frac{-1}{3}z^2}{1}+\frac{\frac{-1}{3\cdot 5}z^2}{1}+\frac{\frac{-1}{5\cdot 7}z^2}{1}+\frac{\frac{-1}{7\cdot 9}z^2}{1}+\cdots. \tag{5.6}$$

This last example is also useful as an illustration of the other use of the algorithms, namely that of numerical computation. The algorithms may be implemented on a computer using real or complex arithmetic as described in Sections 3 and 4. In our examples we used real single precision arithmetic in four computer programs, one for each algorithm described in Section 2 using C-fractions and regular C-fractions. These programs ran on several machines, among them a micro-VAX II and an IBM PC/AT compatible.

In example (5.5) the program for the regular C-fraction expansion predictably fails, while the C-fraction program gives an arbitrary number of coefficients d_n for the expansion correctly to machine precision. This illustrates the fact that the algorithm gives exact results for the coefficients of the continued fraction when the coefficients are polynomials. A more difficult problem in this vein is

$$W' = \sec^2 x + \sec^2 x W^2. \tag{5.7}$$

The coefficients here are analytic at $x = 0$ with infinite power series. The solution to this problem is $W(x) = \tan(\tan x)$. Note that this function has a singularity at $x_s \doteq 1.0038848$, and, hence, we would expect a truncated continued fraction to have an increasing error as $x \to x_s$. The first few terms of the C-fraction expansion for W are given approximately by

$$\begin{aligned} W(x) = \frac{x}{1} + \frac{-.66667x^2}{1} + \frac{-.23333x^2}{1} + \frac{-.22245x^2}{1} + \\ \frac{-.17113x^2}{1} + \frac{-.13442x^2}{1} + \frac{-.18695x^2}{1} + \frac{-.13809x^2}{1} + \cdots . \end{aligned} \tag{5.8}$$

Some results are summarized in Table 5.1. The errors given are relative errors between the true solution and the approximate solution given by the algorithm. The notation N_{coeff} is the number of terms of the power series expansion for the coefficient function given, while $N_{\text{c.f.}}$ is the number of correct terms in the continued fraction expansion for the solution. Note in particular that two terms of the Taylor series for $\sec^2 x$ (one of which is zero) are needed to calculate each additional nonzero term of the C-fraction. In other words, we must take two

N_{coeff}	$N_{\text{c.f.}}$	Error$(\frac{1}{2})$	Error(1)
3	2	.01041	.94468
5	3	.00064	.87188
7	4	.000069	.73328
9	5	.000009	.52593
19	10	$< 10^{-9}$	.01732

TABLE 5.1. Results for $W' = \sec^2 x + \sec^2 xW^2$.

derivatives of the coefficient functions in order to get one term of the continued fraction solution.

On the other hand, many Riccati equations give rise to regular C-fraction solutions. The equation

$$W' = \frac{1}{1 - xe^{-x}} - W^2 \tag{5.9}$$

is one such example. It is worth noting that similar problems arise frequently in physical applications; see [**2, 10**]. In this case we need only calculate one new term of the Taylor series expansion for $A(x)$ for each correct term of the continued fraction solution

$$\begin{aligned} W(x) = \frac{x}{1} + \frac{-.5x}{1} + \frac{1.1667x}{1} + \frac{-1.0238x}{1} + \\ \frac{.02259x}{1} + \frac{-.075565x}{1} + \frac{.13577x}{1} + \cdots . \end{aligned} \tag{5.10}$$

These results are summarized in Table 5.2.

N_{coeff}	$N_{\text{c.f.}}$	Error$(\frac{1}{2})$	Error(1)
2	2	.18049	1
3	3	.05138	.3
4	4	.00084	.01031
5	5	.00001	.00065
6	6	.000002	.00011

TABLE 5.2. Results for $W' = 1/(1 - xe^{-x}) - W^2$.

Finally, we should point out that when the solution to the Riccati equation is a rational function, then the associated continued fraction terminates, giving the exact solution with a finite number of terms.

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Department of Pure and Applied Mathematics, Washington State University, Pullman, WA 99163-3113

Department of Pure and Applied Mathematics, Washington State University, Pullman, WA 99163-3113

Department of Mathematics, University of Colorado, Boulder, CO 80309-0426